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Edoardo Ballico

On the automorphisms of surfaces of general type in positive characteristic, II

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Geometria algebrica. — *On the automorphisms of surfaces of general type in positive characteristic, II.* Nota (*) di EDOARDO BALLICO, presentata dal Corrisp. E. Arbarello.

ABSTRACT. — Here we give an upper polynomial bound (as function of K_{X^2} but independent on p) for the order of a p-subgroup of $\mathrm{Aut}(X)_{\mathrm{red}}$ with X minimal surface of general type defined over the field $\boldsymbol{K}$ with $\mathrm{char}(\boldsymbol{K}) = p > 0$. Then we discuss the non existence of similar bounds for the dimension as $\boldsymbol{K}$-vector space of the structural sheaf of the scheme $\mathrm{Aut}(X)$.

KEY WORDS: Surfaces of general type; Automorphism group; Group scheme; p-group.

RIASSUNTO. — *Sugli automorfismi delle superfici di tipo generale in caratteristica positiva, II.* In questa *Nota* si dimostra una stima polinomiale (come funzione di K_{X^2}) indipendente da p per l'ordine dei p-sottogruppi di $\mathrm{Aut}(X)_{\mathrm{red}}$, con X superficie minimale di tipo generale definita sul campo $\boldsymbol{K}$ con $\mathrm{char}(\boldsymbol{K}) = p > 0$. Si mostra anche la non esistenza di analoghe stime per la dimensione come $\boldsymbol{K}$-spazio vettoriale del fascio strutturale dello schema $\mathrm{Aut}(X)$.

In the last few years several mathematicians (see [4], announcement in the introduction after the statement of 3.14 [5, 9, 10, 20, 21]) considered the problem of bounding (in terms of suitable numerical invariants, *e.g.* the Chern numbers) the order of the automorphism group $\mathrm{Aut}(X)$ of a smooth projective manifold X of general type or with K_X ample. Here «bounding» means «find a good polynomial bound». Except for the work in progress mentioned in the introduction of [4], all the quoted papers considered the case in which X is a surface of general type. All the quoted papers used in an essential way the fact that the algebraically closed base field $\boldsymbol{K}$ has $\mathrm{char}(\boldsymbol{K}) = 0$. We think that the problem is interesting even if $p := \mathrm{char}(\boldsymbol{K}) > 0$. This paper is a continuation of [1]. In the first section we prove the following result.

THEOREM 0.1. *Let X be a minimal surface of general type defined over an algebraically closed field $\boldsymbol{K}$; set $c := K_{X^2}$. Then there is a universal constant D (which does not depend on $\mathrm{char}(\boldsymbol{K})$) such that for every p-subgroup G of $\mathrm{Aut}(X)$ we have $\mathrm{Card}(G) \leq Dc^6$.*

In [1, Th. 0.1], it was proved a result corresponding to Theorem 0.1 for every subgroup of $\mathrm{Aut}(X)$ with order prime to p (and with «45/2» instead of «6» as exponent). We stress that the exponent «6» is just for funny: the important fact is that it is independent of the prime p (as it is the universal constant) and that it is explicit. The union of the statements of Theorem 0.1 and [1, Th. 0.1], gives bounds on the existence of suitable subgroups of $\mathrm{Aut}(X)_{\mathrm{red}}$ (*e.g.* the solvable ones), but it seems to us not good enough for reasonable results on $\mathrm{card}(\mathrm{Aut}(X)_{\mathrm{red}})$; see the discussion at the end of section 1.

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Theorem 0.1 concludes (from our point of view) the p-power part of the «discrete» part (*i.e.* Aut $(X)_{\text{red}}$) of the research project on Aut (X) (with X minimal surface of general type) raised in the introduction of [1]. It remained also to gain informations on the connected 0-dimensional component of the identity of the group scheme Aut (X). Recall that its tangent space at the identity is $H^0(X, TX)$. It was proved [2, 3.12] that $h^0(X, TX) \leq 18(K_{X^2})$. Note that if X is defined over a field $\boldsymbol{K}$ of characteristic p and t denotes $h^0(X, TX)$, the scheme Aut (X) has dimension (as $\boldsymbol{K}$-vector space of its structural sheaf) at least p^t. Thus the following result shows that, even fixing the prime p, there is no polynomial bound for this vector space dimension (and shows that the bound «$h^0(X, TX) \leq 18(K_{X^2})$» given in [2, 3.12] is, up to the constant, the right bound).

Theorem 0.2. Fix an odd prime p congruent to 2 modulo 3 and an algebraically closed field $\boldsymbol{K}$ with char $(\boldsymbol{K}) = p$. Set $C(p)^{-1} = 2p^4$. Then there is a sequence $\{X(n)\}_{n \geq 1}$ of minimal surfaces of general type over $\boldsymbol{K}$ with $K_{X(n)^2}$ going to infinity with n and with $h^0(X(n), TX(n)) \geq C(p)(K_{X(n)^2})$ for every n.

Theorem 0.2 will be proved (just using the examples constructed in [14]) in the second (and last) section.

1. Proof of Theorem 0.1

In the first part of this section we collect a few remarks needed for the proof of Theorem 0.1. Then we give the proof of 0.1. At the end of this section we discuss the implications of 0.1 and of [1, Th. 0.1], for the structure of Aut $(X)_{\text{red}}$.

From now on in this section we fix a prime p and an algebraically closed base field $\boldsymbol{K}$ with char $(\boldsymbol{K}) = p$. We fix a minimal surface of general type X over $\boldsymbol{K}$, and set $K := K_X$ and $c := K^2$. For simplicity we will write Aut (X) instead of Aut $(X)_{\text{red}}$. The notation $\Phi \propto \Gamma$ means that there is a universal constant D (not depending on the characteristic of the base field) such that $\Phi \leq D\Gamma$; the notation $\propto \Gamma$ means that there is a universal constant D such that the object considered in that sentence has order at most $D\Gamma$; usually when we use this notation Γ will be an explicit power of c (the unique exception arising with Γ power of the genus of a suitable curve).

Remark 1.1. Let $W := \boldsymbol{P}(V)$ be a projective space and H a p-group contained in Aut (W). By [3, proof of 3.1.4, p. 409, lines 11-15], the action of H on W lifts to a linear action of H on V. Fix any such linear action of H. There is a basis of V in which every $h \in H$ is in triangular form with only 1 on the diagonal.

Remark 1.2. By a particular case of 1.1 every p-subgroup H of Aut $(\boldsymbol{P}^1)$ has a common fixed point. Taking any such fixed point as the point at infinity, we see that H acts as a group of translations. Hence H is abelian, every $h \in H$, $h \neq$ Id, has order p, and fixes only the point at infinity.

Remark 1.3. Let C be a singular rational curve C; set $t :=$ card (C_{sing}). First assume $t \geq 2$ and fix two point P, Q of C_{sing}. Taking the normalization, we see that C has no automorphism of order p fixing both P and Q; hence every p-subgroup of Aut (C) has order at most $t(t-1)$. Now assume $t = 1$ and call t' the number of branches of C at its singular point, P. If $t' \geqslant 2$ for the same reason every p-subgroup of Aut (C) has order at most $t'(t'-1)$. Now assume $t' = 1$. By the discussion in 1.2, the curve C may have a family of abelian elementary p-subgroups of Aut (C) with unbounded cardinality (the translations on the affine line). Fix $L \in \mathrm{Pic}(C)$, L ample. We claim that C has no automorphism of order p fixing the isomorphism class of L. Taking a partial normalization, to prove the claim we may assume that C has an ordinary cusp, *i.e.* that $\mathrm{Pic}^0(C)$ is isomorphic to the additive group, $\boldsymbol{K}$. The claim follows from the last part of 1.2.

REMARK 1.5. Fix a smooth the curve C of genus $g \geqslant 2$. Then card $(\mathrm{Aut}(C)) \propto g^3$ and every cyclic subgroup of Aut (C) has order $\propto g$ (use *e.g.* the lifting theorem in [15] to extends the classical characteristic 0 case given *e.g.* in [7]).

REMARK 1.5. Fix a singular curve T and let $C \to T$ be its normalization. Fix a p-subgroup H of Aut (T) (hence of Aut (C)). Let H' be the subgroup of H fixing every singular point of T. If $p_a(C) = 1$, H' acts on C with at least a common fixed point. Note that if H is contained in Aut (X), then it fixes the isomorphism class of $K_X \mid T$. Hence if H is contained in Aut (X) the group H' is trivial by 1.3.

REMARK 1.6. *1.6.1.* The number of irreducible components of C is $\propto c$ (this was proved in [1, part (b1) of the proof of 1.1]), using the fact (checked in [1, Remark 1.6]) that the number of smooth rational curves, Z, contained in X and with $K \cdot Z = 0$ is $\propto c$).

1.6.2. Every irreducible component T of C_{red} has $p_a(T) \propto c$, because $K \cdot T + T^2 = 2p_a(T) - 2$ and C is numerically connected (hence $T \cdot (K - T) \geqslant 0$, while $(K - T) \cdot K \geqslant 0$). The same computation shows that the sum of the arithmetic genera of all the irreducible components of C_{red} is $\propto c$.

1.6.3. Let H be a p-subgroup of Aut (C). Fix an irreducible component, T, of C_{red}. By 1.6.1 H has a subgroup H' of index $\propto c$ which stabilizes T. Since C is numerically connected, we see that for every elliptic curve $E \subsetneq C_{\mathrm{red}}$ there is $P \in E$ such that $h(P) = P$ for every $h \in H$. Hence by 1.4 there is a subgroup H'' of index $\propto c^2$ in H' and fixing every point of T if the normalization of T is not rational. By 1.3 we may find such a subgroup fixing pointwise T also if T is not smooth. By 1.3 we may find such a subgroup fixing also every smooth rational curve, R, intersecting $C_{\mathrm{red}} \setminus R$ in at least 2 points (note that card $((C_{\mathrm{red}} \setminus R) \cap R) \propto c$ because C is numerical connected and $p_a(C) \propto c$).

PROOF OF 0.1. The proof is divided into 5 parts.

(*a*) Fix a p-subgroup H of Aut (X) (*e.g.* a p-Sylow subgroup) and a small integer x, say $x = 12$, such that the linear system $|xK|$ has no base point and the associated morphism gives the canonical model of X. Set $V := H^0(X, K^{\otimes x})$. In this part we assume $\dim(V^H) \geqslant 2$ and prove card $(H) \propto c^4$. Fix a pencil generated by two invariant pluricanonical divisors; hence every curve in this pencil is sent into itself by H and H acts on the generic fiber of the pencil. Call B the base component of the pencil and $\boldsymbol{J}$ the generic fiber (over a suitable function field obtained by the Stein factorization of the rational map induced by the pencil) of the invariant pencil obtained deleting B. If the geometric genus of $\boldsymbol{J}$ is at least 1, we have card $(H) \propto c^2$ by 1.6.1 and 1.6.2. If $\boldsymbol{J}$ has geometric genus 0, it has at least a cusp and we find card $(H) \propto c$ by 1.6.1 and 1.5. Hence from now on we will assume $\dim(V^H) = 1$.

(*b*) Fix any H-invariant pencil. Let B be the sum of the base components of this pencil. Hence, after deleting B and making a few blow-ups (obtaining a surface X' on which H acts) we get an H-invariant morphism $\pi\colon X' \to \boldsymbol{P}^1$. Let $B + J$ the invariant fiber of the pencil. Assume the existence of a singular fiber different from J. In this part we will assume that π has only finitely many singular fibers. Thus by [6] π has $\propto c$ singular fibers. Hence there is a subgroup H' of H with index $\propto c$ and fixing two fibers of π. By the proof of part (*a*) we have card $(H') \leqslant \propto c$. Hence card $(H) \propto c^5$.

(*c*) Let A be the subgroup of H fixing every point of $T := J_{\text{red}}$. By the proof of part (*a*) to obtain an upper bound for card (A) we may (and will) assume that $|xK|^A = \{J\}$; by part (*b*) we may assume that every A-invariant pencil of $|xK|$ has either J as unique singular fiber or all fibers are singular; call ($) this property. Call U the image of X in $\boldsymbol{\Pi} := |xK|$ (hence its canonical model) and U^* its dual in the dual projective space $\boldsymbol{\Pi}^*$. Since we may take $x = 2y$ with $|yK|$ inducing the canonical model of X the following facts are known as general properties of Veronese embedding (see [11, Th. 2.5] or [12, Th. (20), p. 180]). U^* is a hypersurface and it is reflexive (hence biduality holds for U). Let $j^* \in \boldsymbol{\Pi}^*$ be the point corresponding to J; by assumption $j^* \in U^*$. Fix a general point $O \in T$ and take the A-invariant hyperplane H_O of $|xK|$ formed by divisors containing 0. By 1.1 H_O contains at least an invariant pencil, V_0; by assumption ($) either $V_0 \subset U^*$ or V_0 intersects U^* exactly at O. Since T is infinite, varying O we see that U^* has multiplicity $\deg(U^*)$ at j^*. Hence U^* is a cone with vertex j^*. By biduality we have $U = U^{**}$; hence U is contained in the hyperplane dual to j^* (the image of T), contradiction.

(*d*) Note that in part (*c*) to obtain that U^* is a cone we needed only that the p-group has as fixed points at least an irreducible component of T. Here we assume that T contains no smooth rational curve, Z, with $K \cdot Z = 0$, leaving the case with such Z for the next (and last) step. Hence by 1.1, 1.2 and 1.5 we conclude unless every irreducible component of T is a smooth rational curve and card $(\text{Sing}(T_{\text{red}})) \leqslant 1$. T_{red} cannot be smooth, because it is connected, $K^2 > 0$ and no smooth rational curve on X moves. Taking a partial normalization, we see that $\text{Pic}^0(\text{T}_{\text{red}})$ has a unipotent subgroup, unless T_{red} is the union of two smooth rational curves, J'' and T'', meeting transversally. If $\text{Pic}^0(T_{\text{red}})$ has a unipotent subgroup, use the proof given for a cuspidal rational curve. In the remaining case the contradiction comes from the following inequalities: $(J'' + T'')^2 > 0$, $J'' \cdot T'' = 1$, $J''^2 < 0$ and $T''^2 < 0$.

(*e*) Here we assume the existence of a smooth rational curve $Z \subsetneq (T + J)$ with $K \cdot Z = 0$. If the fundamental cycle corresponding to Z is contained in other curves of V_0, then it is in the base locus of V_0 and we may repeat the calculation of part (*d*) on the movable part of the pencil. If Z is contained only in $T + J$ (hence in T) we may assume by 1.6.1 (adding 1 to the exponent of the bound obtained) and part (*b*) that Z is the unique rational curve in the corresponding fundamental cycles, that the same is true for the other curves, Z', with $K \cdot Z' = 0$ and that $Z \cap (T_{red} \setminus Z)$ is the unique singular point of $J + T$ (hence the reduction of the base locus of V_0). Again, the numerical computations at the end of part (*d*) work and conclude the proof of 0.1. ♦

Suppose to have a bound (say $\propto c^a$) for the subgroups, G, of $\mathrm{Aut}(X)$ with card (G) prime to p, and a bound (say $\propto c^b$) for the subgroups with order a power of p; by [1, Th. 0.1] we may take $a = 45/2$, while by 0.1 we may take $b = 6$. We do not see how to obtain only from these informations a good bound for card $(\mathrm{Aut}(X))$. Of course, we must have $p \propto c^b$ and every prime $\neq p$ which divides card $(\mathrm{Aut}(X))$ is $\propto c^a$. However, in this way we obtain only card $(\mathrm{Aut}(X)) \propto c^{\log(c)}$. By [17, Ch. 4, Th. 5.6] every solvable subgroup of $\mathrm{Aut}(X)$ has order $\propto c^{a+b}$.

2. Proof of Theorem 0.2

In this section we prove 0.2 using the examples constructed in [14]. For other examples of surfaces of general type with non trivial vector fields, see [8] and [13]. The surfaces constructed in [14] depend on various integral invariants p (the characteristic), d and n. We need only the ones with $n = 1$. In this case one start with a smooth curve, C (which will be the Albanese variety) and X would be a smooth fibration over C. The integer d is the degree of a suitable line bundle L on C with $L^{\otimes p(p-1)} \cong \omega_C$. By [14, Th. 1] we have $h^0(X, TX) \geq h^0(C, L)$ and the lower bound claimed by 0.2 is satisfied for the corresponding surface X if we may find (C, L) with $h^0(C, L) \geq d/2$ (hence, since $d := \deg(L)$, with C hyperelliptic) (see [14, Th. 2]). To check that the examples given at the end of [14] are sufficient to prove Theorem 0.2 we will use the formula for the Hasse-Manin matrix and Cartier operator of hyperelliptic curves proved by Yui ([19] or see [16], bottom of page 55). We use the notations of [14, §3]; set $w := p(p-1)d + 3 = 2g + 1$ (with $g := p_a(C)$). With these notations in our situation the condition on the Cartier operator given in the discussion and formula at the bottom of [16, p. 55], is that the polynomial $(x^W - 1)^{(p-1)/2}$ has no monomial with non zero coefficient and with exponent $\beta p - 1$ with β integer, *i.e.* the non existence of an α with $1 \leq \alpha \leq (p-1)/2$ with $\beta w = \alpha p - 1$. Just note that if p is congruent to 2 modulo 3, then $(p-1)/3$ is not an integer, while $(2p-1)/3$ is an integer bigger than $(p-1)/2$. Hence we conclude the proof of 0.2.

Remark 2.1. Note that the surfaces, X, constructed in [14] and just considered answer a question raised in [18, end of p. 317], *i.e.* they are smooth projective varieties, X (with $p > 2$) having an ample line bundle, M, with $h^0(X, TX \otimes M^*) \neq 0$; indeed by the formulas in [14, pp. 171 and 172], the zero locus of any non trivial section of TX is an ample divisor.

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Dipartimento di Matematica
Università degli Studi di Trento
38050 Povo TN