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Stability properties of a class of viscoelastic beams of the hereditary type

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Meccanica dei solidi. — *Stability properties of a class of viscoelastic beams of the hereditary type.* Nota (*) di FRANCESCO RUSSO SPENA, presentata dal Socio E. Giangreco.

ABSTRACT. — The paper deals with the problem of equilibrium stability of prismatic, homogeneous, intrinsically isotropic, viscoelastic beams subjected to the action of constant compressive axial force in the light of Lyapounov's stability theory. For a class of functional expressions of creeping kernels characteristic of no-aging viscoelastic materials of the hereditary type, solution of the governing integro-differential equations is given. Referring to polymeric materials of the PMMA type, numerical results are obtained showing the influence of the loading duration on system stability.

KEY WORDS: Viscoelasticity; Hereditary theory; Creep-stability.

RIASSUNTO. — *Proprietà di stabilità di una classe di travi in materiale viscoelastico-ereditario.* Si studia il problema della stabilità dell'equilibrio di travi prismatiche, omogenee, isotrope, appoggiate alle estremità, costituite di materiale linearmente elasto-viscoso, caricate in punta da forza assiale costante, alla luce del criterio dinamico di Lyapounov. Per una classe di espressioni funzionali di nuclei di scorrimento caratteristici dei materiali elasto-viscosi, non invecchianti, di tipo ereditario, si fornisce la soluzione dell'equazione integro-differenziale che descrive lo stato dinamico del sistema in prossimità della configurazione di equilibrio iniziale. Con riferimento a materiale polimerico del tipo PMMA, si ottengono risultati numerici che pongono quantitativamente in rilievo l'influenza della durata di applicazione del carico sulla stabilità del sistema.

INTRODUCTION

The problem of the equilibrium stability of homogeneous, isotropic, linearly viscoelastic prismatical beams subjected to a compressive constant axial force, has been widely studied since long ago [1-3].

As it is well known the so-called Eulerian buckling theory (the elastica) does not apply to those physical situations in which, even in the presence of conservative external loads, the material constituting the beam behaves as linearly viscoelastic. In such situations the analysis is generally performed under the assumption of no-perfect straigthness of the geometrical beam axis and studying the quasi-static evolution of the transverse displacements of the beam axis.

By using this problem approach we may well define the «critical» or buckling load as that value of load at which the deflections of the beam axis become unbounded so that we can define the structural collapse as caused by displacement excess.

In this context D'Onofrio and Franciosi [4] studied the stability of non-prismatical reinforced concrete beams within a constitutive model of viscoelastic behaviour ruled by an aging kernel of Withney's type.

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Using the same problem approach in [5] a comparison is performed between a pure hereditary viscoelastic model and a pure aging model.

The Author shows that the quasi-static evolution of beam deflections is asymptotically stable for the aging model up to the Eulerian buckling load; while the evolution is divergent for the hereditary model even for intensities of axial load less than Eulerian load.

In this paper, referring to a constitutive viscoelastic model of the hereditary kind, the problem is analyzed in the light of the dynamical Lyapounov's stability theory [6].

Numerical results referred to a homogeneous isotropic polymeric material are also given.

1. Relaxation and creep characterization

Let $\mathcal{B}$ be a prismatic, isotropic, homogeneous beam with a linearly viscoelastic behaviour of relaxation type and let $0\, x_1\, x_2\, x_3$ be a Cartesian reference frame whose origin is placed at the centroid of the section A (fig. 1) and whose x_3-axis is taken coincident with the beam's rectilinear axis in the reference configuration $\mathcal{B}^*$.

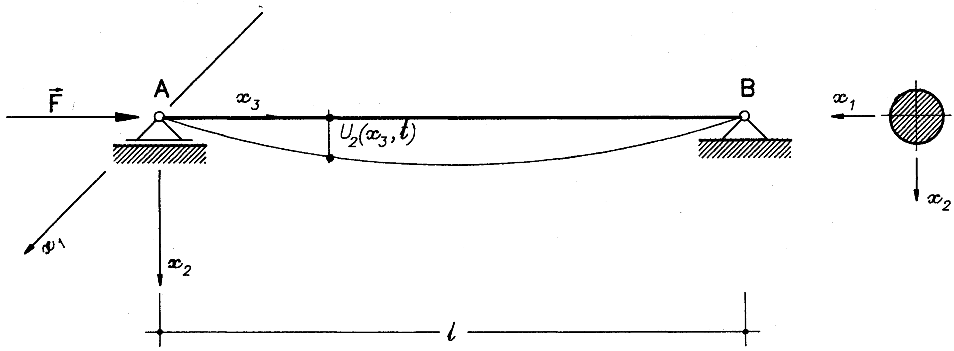

Fig. 1.

For the material constituting the beam behaves as linearly viscoelastic, then with reference to a uni-axial state of stress, let us denote with $\sigma_{33}(x_3, t)$ the single non-vanishing Cartesian component of the Cauchy stress tensor and with $\varepsilon_{33}(x_3, t)$ the corresponding Cartesian component of the infinitesimal strain tensor.

Then, considering the scalar valued relaxation function

$$(1.1)\qquad E(t) = E_\infty + (E_0 - E_\infty)\exp(-t/\eta),$$
$$E_0 = \lim_{t\to 0} E(t); \qquad E_\infty = \lim_{t\to+\infty} E(t); \qquad E_0 > E_\infty > 0; \qquad \eta > 0$$

typical for the standard linear model, the constitutive relation between the stress and strain Cartesian components, reads:

$$(1.2)\qquad \sigma_{33}(t) = E_0\,\varepsilon_{33}(t) - \eta^{-1}(E_0 - E_\infty)\int_{-\infty}^{t} \exp[-(t-\tau)/\eta]\,\varepsilon_{33}(\tau)\,d\tau.$$

Of course, according to Volterra's work on hereditary theory (*e.g.* [8]), in the scalar integral law (1.2) it is assumed that the past history of strain ε_{33} is prescribed on $\boldsymbol{R} \times \times]-\infty, 0]$ and is such that the integral is meaningful. For our future purposes it is useful to refer shortly also the creep representation of material behaviour.

As $E(t)$ is a «strong» relaxation function, then the viscoelastic material is also of the creep type [7]. In fact, some theorems proved by Volterra on the inversion of strong response function, imply the existence of strong creep-compliance $J(t)$ such that:

$$\varepsilon_{33}(t) = J(0)\sigma_{33}(t) + \int_{-\infty}^{t} \dot{J}(t-\tau)\sigma_{33}(\tau)\,d\tau \tag{1.3}$$

where a superimposed dot denotes time differentiation.

Furthermore if $*$ denotes the convolution on the interval $[0, t]$, and if we set

$$E_1(t) = E(t)/E(0); \qquad E_n(t) = E_1 * E_{n-1} \qquad (n \geq 2) \tag{1.4}$$

by means of known theorems [8], it results:

$$\dot{J}(t)/J(0) = \sum_{n=1}^{\infty} (-1)^n E_n(\tau) \qquad (J(0) = E(0)^{-1}).$$

Consequently, for the standard model, from (1.2), (1.4), we obtain:

$$E_n(t) = [-(1-k)/\eta]^n\, t^{n-1} \exp(-t/\eta)/(n-1)! \tag{1.5}$$

where $k = E_\infty / E_0 < 1$ and hence the creep-compliance conjugate with the relaxation function $E(t)$ is given by:

$$\dot{J}(t)/J(0) = \eta^{-1}(1-k)\exp(-kt/\eta). \tag{1.6}$$

So, the scalar creep integral law results:

$$\varepsilon_{33}(t) = E_0^{-1}\sigma_{33}(t) + [\eta E_0]^{-1}(1-k) \int_{-\infty}^{t} \exp(-k(t-\tau)/\eta)\,\sigma_{33}(\tau)\,d\tau. \tag{1.7}$$

For the sake of coinciseness in the sequel, the constitutive creep-relation (1.7) will be referred to the symbolic notation:

$$\varepsilon_{33}(t) = \mathcal{F}[\sigma_{33}(t)] \tag{1.8}$$

where we denoted by $\mathcal{F}[(\cdot)]$ the following linear operator

$$\mathcal{F}[(\cdot)] = \left[\frac{(\cdot)}{E_0} + \int_0^t (\cdot)\,\Phi(t, \tau)\,d\tau \right] \tag{1.9}$$

and by:

$$\Phi(t, \tau) = (\alpha/E_0)\exp(-\beta(t-\tau)) \tag{1.10}$$

with:

$$\alpha = (1-k)/\eta > 0; \qquad \beta = k/\eta > 0 \tag{1.11}$$

and we further assumed, according to the principle of fading memory, the material undisturbed up to the instant of time 0. We further remark that the introduction in (1.9) of the creeping kernel (1.10) can also characterize material behaviours of the aging type.

2. Problem formulation

The beam $\mathcal{B}$, depicted in fig. 1, is assumed to be simply supported at the end sections A, B, in the reference configuration $\mathcal{B}^*$ and is loaded, starting from a prescribed instant of time 0, by a compressive axial force $\vec{F}$, constant in time.

The planar deformed dynamical configuration of the beam at any instant of time is described by the displacement of a point C on its centroidal axis, with longitudinal and transverse components $u_3(s,t)$ and $u_2(s,t)$ respectively, s being the curvilinear abscissa on the deformed center-line, and by the rotation of the beam cross-section about C of amplitude $\varphi(s,t)$.

However in the classical conceptual framework of a linearized dynamical buckling analysis it is quite generally assumed [9] that the axial deformation is negligible and the restriction to the «elastica» (*i.e.* no extension of neutral fiber) is used.

From a vibrational point of view such an assumption implies uncoupling between longitudinal and transverse oscillations. On the other hand, restricting the analysis to small amplitude transverse oscillations in the neighbourhood of the initial rectilinear configuration $\mathcal{B}^*$, the displacement and rotation fields can be considered as being functions of the Cartesian abscissa x_3, and the D'Alembert's equilibrium dynamical equations can be written:

$$-\frac{\partial}{\partial x_3}(N\varphi) + \frac{\partial Q}{\partial x_3} - \mu\frac{\partial^2 u_2}{\partial t^2} = 0\,, \tag{2.1}$$

$$Q - \frac{\partial M}{\partial x_3} = 0\,. \tag{2.2}$$

In the preceding equations $Q(s,t)$, $N(s,t)$ and $M(s,t)$ represent respectively the transverse shear-force, the normal force and the bending moment, assumed to be positive according to the usual beam-theory convention; μ denotes the constant mass density per unit length of beam.

It is therefore possible to eliminate the transverse shear-force Q between (2.1) and (2.2), obtaining:

$$-\frac{\partial}{\partial x_3}(N\varphi) + \frac{\partial^2 M}{\partial x_3^2} - \mu\frac{\partial^2 u_2}{\partial t^2} = 0\,. \tag{2.3}$$

Finally from the equilibrium condition at $x_3 = l$

$$N = -|\vec{F}| \equiv -F \tag{2.4}$$

and from the kinematical linear condition:

$$\varphi = -\frac{\partial u_2}{\partial x_3} \tag{2.5}$$

we are led to the following linearized dynamical equilibrium equation:

$$-F\frac{\partial^2 u_2}{\partial x_3^2}+\frac{\partial^2 M}{\partial x_3^2}-\mu\frac{\partial^2 u_2}{\partial t^2}=0\,. \tag{2.6}$$

Equation (2.6) must be supplemented by the constitutive equation, relating the linearized curvature of the beam axis to the bending moment.

Under the assumed hypothesis of shear-indeformability, we get further

$$\frac{\partial^2 u_2}{\partial x_3^2}=-\frac{1}{I_1}\,\mathcal{F}[M(x_3,t)] \tag{2.7}$$

where I_1 denotes the moment of inertia of cross-section about the neutral axis-x_1. Differentiating eq. (2.7) twice with respect to x_3, we obtain:

$$\frac{\partial^4 u_2}{\partial x_3^4}=-\frac{1}{I_1}\,\mathcal{F}\left[\frac{\partial^2 M(x_3,t)}{\partial x_3^2}\right]$$

from which, accounting for the linearity of $\mathcal{F}[(\cdot)]$ and using (2.4), we get

$$\frac{\partial^4 u_2}{\partial x_3^4}=-\frac{F_3}{I_1}\,\mathcal{F}\left[\frac{\partial^2 u_2}{\partial x_3^2}\right]-\frac{1}{I_1}\,\mathcal{F}\left[\frac{\partial^2 u_2}{\partial t^2}\right].$$

By the explicitation of the operator $\mathcal{F}[(\cdot)]$ and denoting with a superimposed comma the spatial derivative and by a dot the time derivative, we are led to the following partial integro-differential equation:

$$u_2^{IV}+\frac{F}{E_0I_1}\left(u_2''+\int_0^t E_0u_2''\Phi(t,\tau)\,d\tau\right)+\frac{\mu}{E_0I_1}\left(\ddot{u}_2+\int_0^t E_0\ddot{u}_2\Phi(t,\tau)\,d\tau\right)=0\,. \tag{2.8}$$

Equation (2.8) governs the infinitesimal vibrations of the beam axis in the neighbourhood of the rectilinear initial configuration. It can be written in a different form by means of the following eigenfunction series expansion:

$$u_2(x_3,t)=\sum_{n=1}^{+\infty}v_n(t)\sin\frac{n\pi x_3}{l}\qquad n\in\mathbf{N} \tag{2.9}$$

where the amplitudes $v_n(t)$, being functions of the time only, have to satisfy the initial conditions:

$$v_n(0)=v_{n0}\,,\qquad \dot{v}_n(0)=\dot{v}_{n0} \tag{2.10}$$

while the trigonometric part fulfils the kinematic and static conditions

$$\begin{aligned} &u_2(0,t)=u_2(l,t)=0\,,\\ &u_2''(0,t)=u_2''(l,t)=0\,. \end{aligned} \tag{2.11}$$

Performing the necessary differentiations on (2.9) and substituting the result into the eq. (2.8), we obtain:

$$\sum_{n=1}^{+\infty}\frac{n^4\pi^4}{l^4}v_n(t)\sin\frac{n\pi x_3}{l}-\frac{F}{E_0I_1}\sum_{n=1}^{+\infty}\frac{n^2\pi^2}{l^2}\sin\frac{n\pi x_3}{l}\,\cdot \tag{2.12}$$

$$\cdot\left[v_n(t)+\int_0^t E_0\Phi(t,\tau)v_n(\tau)\,d\tau\right]+\frac{\mu}{E_0I_1}\sum_{n=1}^{+\infty}\sin\frac{n\pi x_3}{l}\left[\ddot{v}_n(t)+\int_0^t E_0\Phi(t,\tau)\ddot{v}_n(\tau)\,d\tau\right]=0\,.$$

The multiplication of both sides of eq. (2.12) by function $\sin m\pi x_3/l$, the integration with respect to x_3 between 0 and l, and taking into account the orthogonality property of trigonometric functions, leads to the following homogeneous set of integro-differential equations

$$(2.13)\qquad \frac{n^4\pi^4}{l^4}v_n(t) - \frac{F}{E_0 I_1}\frac{n^2\pi^2}{l^2}\left[v_n(t) + \int_0^t E_0\Phi(t,\tau)v_n(\tau)\,d\tau\right] + \\ + \frac{\mu}{E_0 I_1}\left[\ddot{v}_n(t) + \int_0^t E_0\Phi(t,\tau)\ddot{v}_n(\tau)\,d\tau\right] = 0\,.$$

We further remark that quantities

$$(2.14)\qquad F_{cn} = n^2\pi^2 E_0 I_1/l^2$$

$$(2.15)\qquad \omega_{on}^2 = n^4\pi^4 E_0 I_1/(l^4\mu)$$

define respectively the eigenvalues for the two Sturm-Liouville's problems related to the buckling and to the natural vibrational shapes of a linearly-elastic homogeneous beam. By means of position (2.14)-(2.15), eqs. (2.13), take the following equivalent form:

$$(2.16)\qquad \ddot{v}_n(t) + \omega_{on}^2\left(1 - \frac{F}{F_{cn}}\right)v_n(t) - \omega_{on}^2\frac{F}{F_{cn}}\int_0^t E_0\Phi(t,\tau)v_n(\tau)\,d\tau + \\ + \int_0^t E_0\Phi(t,\tau)\ddot{v}_n(\tau)\,d\tau = 0\,.$$

It is worth of pointing out that the well-posedness of the problem is assured by noticing that, in the absence of any memory effect, that is for $\Phi(t,\tau) = 0$, eq. (2.16) reduces to the following standard Sturm-Liouville problem:

$$\ddot{v}_n(t) + \omega_{on}^2(1 - F/F_{cn})v_n(t) = 0$$

whose eigenfunctions are:

$$v_n(t) = C_{1n}\exp(i\omega_n t) + C_{2n}\exp(-i\omega_n t)$$

where we set:

$$\omega_n = \omega_{on}\sqrt{1 - F/F_{cn}}\,; \qquad i = \sqrt{-1}$$

$C_{in}\,(i = 1, 2)$ being constants to be determined from initial conditions (2.15).

In the presence of memory effect, and using the linear viscoelastic representation (1.10), that is assuming no-aging material behaviour, eq. (2.16) can be written in the following explicit form:

$$(2.17)\quad \ddot{v}_n(t) + \omega_{0n}^2(1 - m_n)v_n(t) - \int_0^t [m_n\omega_{0n}^2 v_n(\tau) - \ddot{v}_n(\tau)]\,\alpha\exp(-\beta(t-\tau))\,d\tau = 0$$

with

$$(2.18)\qquad m_n = F/F_{cn}\,.$$

3. The first-order solution procedure

The integro-differential homogeneous system (2.17), with $n = 1$ yields:

$$(3.1)\quad \ddot{v}_1 + \omega_{01}^2(1 - m_1)v_1 - \alpha \int_0^t (m_1 \omega_{01}^2 v_1 - \ddot{v}_1) \exp(-\beta(t-\tau))\, d\tau = 0 .$$

Equation (3.1) can be easily transformed into a linear homogeneous differential equation. In fact, differentiating eq. (3.1) once with respect to t, we obtain

$$(3.2)\quad \dddot{v}_1 + \omega_{01}^2(1 - m_1)\dot{v}_1 +$$

$$+ \alpha\beta \int_0^t (m_1 \omega_{01}^2 v_1 - \ddot{v}_1) \exp(-\beta(t-\tau))\, d\tau - \alpha(m_1 \omega_{01}^2 v_1 - \ddot{v}_1) = 0 .$$

Multiplying eq. (3.1) by β and adding the result to (3.2) we have

$$(3.3)\quad \dddot{v}_1 + (\alpha + \beta)\ddot{v}_1 + \omega_{01}^2(1 - m)\dot{v}_1 + \omega_{01}^2 \beta(1 - ((\alpha+\beta)/\beta)m_1)v_1 = 0 .$$

This last parametric equation admits solution of the form

$$(3.4)\quad v_1(t) = C_1 \exp(\lambda_1 t) + C_2 \exp(\lambda_2 t) + C_3 \exp(\lambda_3 t)$$

$\lambda_i (i = 1, 2, 3)$ being the simple roots of the algebraic characteristic equation

$$(3.5)\quad \lambda^3 + (\alpha+\beta)\lambda^2 + \omega_{01}^2(1 - m)\lambda + \omega_{01}^2 \beta(1 - ((\alpha+\beta)/\beta)m_1)v_1 = 0$$

associated to (3.4) and $C_i (i = 1, 2, 3)$ are constants to be determined from initial conditions (2.13) and from the condition that the integro-differential equation (3.1) has to be fulfilled for $t = 0$.

For system stability the solution $v_1(t)$ has to be bounded and thus each root λ_i must have negative real part.

This condition, for the well-known theorem of Routh-Hurwitz [10], is satisfied if and only if the coefficients of (3.5) fulfil the following inequalities: $\alpha + \beta > 0$, $\omega_{01}^2(1 - m_1) > 0$, $\omega_{01}\beta(1 - ((\alpha+\beta)/\beta)m_1) > 0$, $\beta(1 - ((\alpha+\beta)/\beta)m_1) < (\alpha+\beta) \cdot \cdot (1 - m_1)$ that is, if and only if

$$(3.6)\quad \alpha > 0 ; \quad \beta > 0$$

$$(3.7)\quad 0 < m_1 < \beta/(\alpha + \beta) .$$

Inequalities (3.6)-(3.7) show that the amplitude of the asymptotic stability interval – which is related to the values of mechanical parameters α and β defining the creeping kernel (1.10) – is always less than 1, so that the initial configuration becomes unstable for values of the axial compressive force not greater than that corresponding to the Eulerian elastic buckling load.

The amplitude of the stability interval agrees perfectly with that determined in [5] using a quasi-static approach, and in [2], by means of dynamical approach using operational algorithms based on Laplace-transform properties.

It must be once again remarked the special character of the time-dependent buckling-load defined by the preceding analysis. As a matter of fact the static critical-load, as well as the dynamic one, for a linearly elastic isotropic structure is essentially instantaneous in nature, whereas for a viscoelastic structure the term buckling-load has to be correctly applied in a strong time-dependent meaning.

The same conceptual aspect is shown by results of an evolutive quasi-static analysis of an initially slightly curved beam, subjected to a constant axial compressive force, where to the unboundedness of transverse deflections is attached the meaning of a mathematical limiting process.

On the other hand Lyapounov stability theory, applied to a viscoelastic hereditary material, can easily lead to the determination of the loading duration T_c for which the beam vibrating motion becomes unbounded.

Referring to a polymeric homogeneous isotropic material of the PMMA type and considering the following relaxation characterization: $E_0 = 4000\,\mathrm{N/mm^2}$; $E_\infty = 1000\,\mathrm{N/mm^2}$; $\eta = 1800\,\mathrm{sec}^{-1}$, the stability interval is defined by $m \in [0, 0.25[$.

Therefore for this case no axial compressive load less than 0.25 F_c can lead to structural instability. On the other hand axial compressive load greater than 0.25 F_c will cause instability provided that it acts on the structure for a suitable time duration. The extent of this critical time duration depends on retardation-time η as well as on ratio $k = E_\infty / E_0$.

In order to give a deeper quantitative character to the above qualitative discussion, reference has been made to the case of a cylindrical PMMA sample-beam with the following mechanical parameters: $E_0 = 4 \times 10^3\,\mathrm{N/mm^2}$; $\eta = 1.8 \times 10^3\,\mathrm{sec}^{-1}$; slenderness-ratio $\lambda = 280$; cross-sectional radius $R = 1$ mm; volumetric mass-density $\rho = 1.2 \times 10^3\,\mathrm{kg/m^3}$; angular frequency of the first elastic natural vibrating mode $\omega_{01} = 630.7\,\mathrm{sec}^{-1}$.

Moreover the system evolution has been analyzed referring to the following initial conditions: $v_1(0) = 0$; $\dot{v}_1(0) = 1$ mm/sec.

The analysis has been carried out for an increasing sequence of values of ratio $k = E_\infty / E_0$ and namely: $k_1 = 0.10$; $k_2 = 0.25$; $k_3 = 0.50$; $k_4 = 0.75$. For each value of k_i ($i \in \{1, 2, 3, 4\}$) in fig. 2 the critical loading-duration T_c *vs.* ratio m of axial compres-

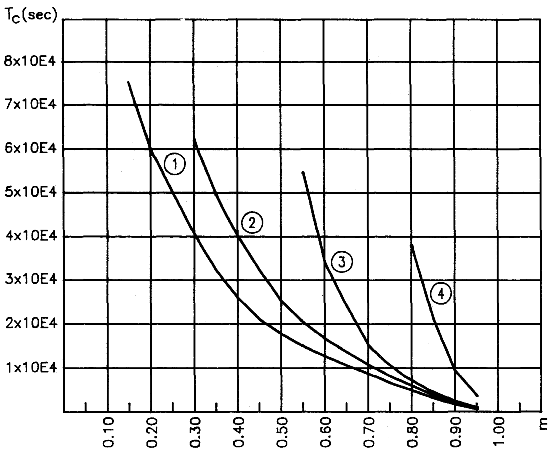

Fig. 2.

m	① T_c*10^{-4}	② T_c*10^{-4}	③ T_c*10^{-4}	④ T_c*10^{-4}
0.10				
0.15	7.453			
0.20	5.996			
0.25	5.000			
0.30	4.038	6.222		
0.35	3.232	4.969		
0.40	2.598	4.016		
0.45	2.113	3.236		
0.50	1.786	2.532		
0.55	1.516	2.048	5.479	
0.60	1.289	1.680	3.402	
0.65	1.077	1.386	2.418	
0.70	0.885	1.092	1.536	
0.75	0.673	0.822	1.081	
0.80	0.503	0.606	0.732	3.816
0.85	0.332	0.414	0.468	2.124
0.90	0.184	0.204	0.258	0.954
0.95	0.072	0.009	0.120	0.384

Fig. 3.

sive force F to the first Eulerian elastic buckling-load F_{c_1}, is plotted. The same results are recorded in tabular form in fig. 3. Each curve of fig. 2 is monotonically decreasing for increasing values of $m = F/F_{c_1}$ and, as $m \to 1$ each tends to zero in perfect agreement with the conceptual meaning of the Eulerian elastic buckling-load.

As a final consideration it seems to be worth of pointing out that, neglecting in (3.5) inertia term, we obtain the following Volterra's homogeneous integral equation

$$v_1(t) - \frac{\alpha m_1}{1 - m_1} \int_0^t \exp(-\beta(t-\tau))\, v_1(\tau)\, d\tau = 0 .$$

This last equation, admitting only the trivial solution $v_1(t) = 0$, confirms the statement that a quasi-static approach cannot be applied in studying physical situations involving materials endowed with memory.

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