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Equazioni a derivate parziali. — *Two-weight Sobolev-Poincaré inequalities and Harnack inequality for a class of degenerate elliptic operators.* Nota di BRUNO FRANCHI, CRISTIAN E. GUTIÉRREZ e RICHARD L. WHEEDEN, presentata (*) dal Corrisp. B. Pini.

ABSTRACT. — In this *Note* we prove a two-weight Sobolev-Poincaré inequality for the function spaces associated with a Grushin type operator. Conditions on the weights are formulated in terms of a strong A_∞ weight with respect to the metric associated with the operator. Roughly speaking, the strong A_∞ condition provides relationships between line and solid integrals of the weight. Then, this result is applied in order to prove Harnack's inequality for positive weak solutions of some degenerate elliptic equations.

KEY WORDS: Weighted Sobolev-Poincaré inequalities; Degenerate elliptic equations; Harnack inequality.

RIASSUNTO. — *Disuguaglianze di Sobolev-Poincaré con due pesi e disuguaglianza di Harnack per una classe di operatori ellittici degeneri.* In questa *Nota* proviamo una disuguaglianza di Sobolev-Poincaré con due pesi per gli spazi funzionali associati ad un operatore tipo Grushin. Le condizioni sui pesi sono formulate in termini di un dato peso fortemente A_∞ rispetto a una metrica naturale per l'operatore, dove la condizione A_∞-forte richiede opportune relazioni tra gli integrali di linea e gli integrali solidi del peso. Successivamente, questo risultato è applicato per provare la disuguaglianza di Harnack per le soluzioni deboli positive di certe equazioni ellittiche degeneri.

1. The classical Sobolev-Poincaré inequality states that, if $B = B(x,r)$ is a Euclidean ball in $\boldsymbol{R}^N$ and f is (say) a continuously differentiable function in B, then if $1 \leq p < N$ we have:

$$(SP) \qquad \left(\fint_B |f - f_B|^q \, dx \right)^{1/q} \leq Cr \left(\fint_B |\nabla f|^p \, dx \right)^{1/p},$$

where, for any Lebesgue measurable set E, $|E|$ denotes the measure of E, $\fint_E v \, dx = (1/|E|) \int_E v(x)\, dx$ denotes the average of the function v over E, $f_B = \fint_B f \, dx$ and $0 < q \leq pN/(N-p)$. The constant C is independent of f and B. Moreover, it is known that, if supp $u \subset B$, we can drop the average f_B on the left hand side.

In fact, the inequality above involves deep properties both of the Euclidean structure and of Lebesgue measure. Indeed the limit exponent $q = pN/(N-p)$ is closely related to the exponent which appears in the isoperimetric inequality. It is well known that (SP), together with the analogous inequality for compactly supported functions, is a crucial tool in many important results for partial differential equations, such as the De Giorgi-Nash-Moser theorem, Harnack's inequality, properties of the Green function for linear second order elliptic equations and existence results for semilinear elliptic equations.

(*) Nella seduta del 12 febbraio 1994.

There are many generalizations of (SP) in the mathematical literature, mainly inspired by applications to degenerate elliptic equations. To introduce our results, let us briefly recall two different kinds of results proved recently. The first group can be typically illustrated by [8, 4, 5], where Sobolev-Poincaré inequalities (and hence regularity results for related degenerate elliptic equations) are proved when the Lebesgue averages in (SP) are replaced by the averages with respect to different measures: the measure $w(x)\,dx$ in the first paper (where the weight function w belongs to the Muckenhoupt class A_p: see below for precise definitions) and two different measures $u(x)\,dx$, $v(x)\,dx$ in the second paper, where $u, v \in L^1_{\mathrm{loc}}$ are nonnegative weights satisfying suitable integral conditions. In the second group of results, the norm of the gradient $|\nabla f|$ is replaced by an anisotropic gradient $\left(\sum_{j=1}^{p} |X_j f|^2\right)^{1/2}$ where $X_1, \dots, X_p$ are vector fields in $\boldsymbol{R}^N$ verifying the so-called Hörmander condition, *i.e.* the Lie algebra generated by $X_1, \dots, X_p$ has rank N at any point (or some generalization of this condition if the vector fields are not smooth). In [13, 21, 22, 3], Sobolev-Poincaré inequalities are proved with such degenerate gradients appearing on the right hand side of (SP); however, Euclidean balls in the averages must be replaced by the balls $B_\rho(x, r)$ given by a suitable metric ρ associated with $X_1, \dots, X_p$ by means of the so-called sub-unit curves (see, *e.g.* [10, 23]). Analogously, classical proofs of local regularity results must be adapted to the new geometry in order to prove the corresponding result for operators of the form $\sum_j X_j^2$. In the sequel, we will call such balls metric balls. A crucial feature of all these results is the fact that $\boldsymbol{R}^N$ with respect to the metric ρ and Lebesgue measure is a space of homogeneous type, *i.e.* $|B_\rho(x, 2r)| \leq c|B_\rho(x, r)|$ for all x, r. In fact, we can superimpose these two kinds of results by allowing both a degeneration of the metric and of the measure, assuming as in [14, 11, 24] that the measure dx is replaced by a measure $w(x)\,dx$, which belongs to the Muckenhoupt A_p class with respect to the balls B_ρ, *i.e.*,

$$\sup_{x,r} \fint_{B_\rho(x,r)} w\,dx \left(\fint_{B_\rho(x,r)} w^{-1/(p-1)}\,dx \right)^{p-1} < \infty .$$

However, in this approach, Lebesgue measure still plays a central role (we form the averages in the Muckenhoupt condition by using Lebesgue measure), whereas we can see in [7] that even in the Euclidean case we can choose different «base measures», enjoying good properties such as suitable isoperimetric inequalities. Our results will involve the A_p condition with respect to a measure which is different from Lebesgue measure; this condition is defined by replacing the Lebesgue averages above by the averages with respect to the measure (see [2] for facts about the A_p condition in a homogeneous space).

2. In this *Note* we will show how all these approaches can be unified, at least when we are dealing with a generalized Grushin operator like

$$\Delta_x + \lambda^2(x)\,\Delta_y$$

in $\boldsymbol{R}^N = \boldsymbol{R}^n_x \times \boldsymbol{R}^m_y$, where λ is a continuous nonnegative function. The «base measure» is associated with a weight function $w(z)$, $z = (x, y)$, which satisfies a condition like the strong A_∞ condition introduced by G. David and S. Semmes in [7]. Roughly speaking, this condition requires suitable relationships between line and solid integrals of the weight. Obviously, these conditions must be formulated in our case with respect to the geometry associated with the operator. However, we point out that our results are new even in the usual Euclidean case ($\lambda \equiv 1$).

The proof of (SP) can be divided into three main steps. First, a representation formula is proved for functions vanishing on a large part of a metric ball. This formula generalizes the usual representation formula by means of a Riesz potential of the gradient and involves both the geometry associated with the Grushin operator and the «base measure». The second step is essentially a weighted $L^p - L^q$ continuity result for a class of singular integrals in spaces of homogeneous type which extends previous analogous results of [25]. By applying this result to the above representation formula, a rough form of (SP) follows, where the integral on the right hand side is replaced by an integral over a larger ball $cB_\rho = B_\rho(z, cr)$, c being an absolute constant. Finally, by using a generalization to spaces of homogeneous type of a technique used in [1, 20, 6], we can prove (SP). On the other hand, the inequality (SP) provides us with one of the basic tools required to adapt classical proofs of Harnack's inequality to our situation.

A more detailed version of the first part of the present *Note*, together with complete proofs and a more exhaustive bibliography will appear elsewhere [15]. Moreover, our continuity results for singular kernels can be adapted to prove general isoperimetric inequalities in different settings [16, 17].

Let us now proceed more formally. We denote by $z = (x, y)$ a point in $\boldsymbol{R}^N$, with $x \in \boldsymbol{R}^N$, $y \in \boldsymbol{R}^m$, $m + n = N$ and we assume that:

(H1) $\lambda = \lambda(x)$ is a continuous nonnegative function vanishing only at a finite number of points;

(H2) λ^n is a strong A_∞ weight in the sense of [7] (see also our Definition 1 below);

(H3) λ satisfies an infinite order reverse Hölder inequality, *i.e.* for any $x_0 \in \boldsymbol{R}^n$, $r > 0$ we have:

$$\fint_{|x - x_0| < r} \lambda(x)\, dx \sim \max_{|x - x_0| < r} \lambda(x) .$$

If g is differentiable a.e., we put $|\nabla_\lambda g|^2 = |\nabla_x g|^2 + \lambda^2(x)|\nabla_y g|^2$. As in [10, 13], we can associate with ∇_λ a natural metric $\rho = \rho(z_1, z_2)$ by means of the so-called sub-unit curves. By our assumptions ρ is always finite. Arguing as in [11, Theorem 2.3] it easy to prove the following characterization of the metric ρ and of the ρ-balls.

PROPOSITION 1. *If* $z_0 = (x_0, y_0)$ *and* $t > 0$, *put*

(*i*) $\Lambda(z_0, t) = \max_{|x - x_0| < t} \lambda(x) \sim \fint_{|x - x_0| < t} \lambda(x)\, dx$;

(*ii*) $F(z_0, t) = t\Lambda(z_0, t)$;

(*iii*) $Q(z_0, t) = \{z = (x, y) \in \mathbf{R}^N$ *such that* $|x - x_0| < t,\ |y - y_0| < F(z_0, t)\}$.

Then there exists $b > 1$ *such that* $Q(z_0, t/b) \subseteq B(z_0, t) \subseteq Q(z_0, bt)$. *In particular,* $|B(z_0, t)| \sim t^N (\Lambda(z_0, t))^m$ *so that* $\mathbf{R}^N$ *together with the metric* ρ *and Lebesgue measure is a metric space of homogeneous type.*

We can now give the definition of strong A_∞ weights with respect to the metric ρ.

DEFINITION 1. *Let* w *be an* A_∞ *weight with respect to the metric* ρ *and Lebesgue measure,* i.e. *let* w *belong to* A_p *for some* $p \geq 1$. *If* z_1, z_2 *belong to* $\mathbf{R}^N$, *put*

$$\delta(z_1, z_2) = \inf_{B_\rho : z_1, z_2 \in B_\rho} \left(\int_{B_\rho} w(z)\, \lambda^{m/(N-1)}(z)\, dz \right)^{1/N}.$$

If $\gamma: [0, T] \to \mathbf{R}^N$ *is a continuous curve, we define the* w*-length of* γ *as*

$$l(\gamma) = \liminf_{|\sigma| \to 0} \sum_i \delta(\gamma(t_{i+1}), \gamma(t_i)),$$

where $\sigma = \{t_0, \ldots, t_p\}$ *is a partition of* $[0, T]$, *and we define a distance* $d(z_1, z_2)$ *as the infimum of the* w*-lengths of sub-unit curves connecting* z_1 *and* z_2. *If there exist constants* c_1, $c_2 > 0$ *such that*

$$c_1 \delta(z_1, z_2) \leq d(z_1, z_2) \leq c_2 \delta(z_1, z_2),$$

we say that w *is a strong* A_∞ *weight for the metric* ρ.

Moreover, in the sequel we will assume that w satisfies the following local boundedness condition near the zeros of λ:

(Z_λ) If $\lambda(x_1) = 0$, then $w(x, y)$ is bounded as $x \to x_1$ uniformly in y for y in any bounded set.

REMARK. *If* $\alpha \geq 0$ *and* z_0 *is a fixed point, then the function* $w(z) = \rho(z, z_0)^\alpha$ *is a strong* A_∞ *weight. In particular, constant functions are strong* A_∞ *weights.*

We note explicitly that the proof of the assertion above is definitely not trivial. Moreover, when $w \equiv 1$, it is possible to show that our definition of w-length has an intrinsic meaning related to the metric ρ. For example in the case $N = 2$, by Proposition 1 the ρ-balls are equivalent to the rectangles $Q(z, r)$ and it is easy to see that the definition of w-length makes both of the edges of $Q(z, r)$ (or, more precisely, of rectangles $Q^*(z, r)$ of comparable size) have equal length.

We can now state our main results. To this end, let us recall that a nonnegative function $u \in L^1_{\mathrm{loc}}$ will be said to be a doubling weight if the measure $d\mu = u(z)\, dz$ is doubling, *i.e.* if $\mu(B_\rho(z, 2r)) \leq C\mu(B_\rho(z, r))$ for any z and r.

THEOREM I. *Let* u, $v \in L^1_{\mathrm{loc}}$ *be nonnegative weights such that* u *is doubling and let* p, q *be such that* $1 \leq p < q < \infty$. *Assume that there exist positive constants* c_1, c_2 *such that, for all* *balls* $B_0 = B_\rho(z_0, r_0)$ *and* $B_r = B_\rho(z, r) \subseteq c_1 B_0$, *we have*

$$\frac{r}{r_0}\left(\frac{u(B_r)}{u(B_0)}\right)^{1/q} \leqslant c_2\left(\frac{v(B_r)}{v(B_0)}\right)^{1/p}, \tag{Ia}$$

where we have put, e.g., $u(B_0) = \int_{B_0} u(z)\,dz$.

If there exists a strong A_∞ *weight* w *satisfying* (Z_λ) *such that* $vw^{-(N-1)/N}$ *belongs to* A_p *with respect to the (doubling) measure* $w^{-(N-1)/N}\,dz$, *then the following Sobolev-Poincaré inequality holds:*

$$\left(\fint_{B_\rho(z_0, r)} |g-\mu|^q u(z)\,dz\right)^{1/q} \leqslant Cr\left(\fint_{B_\rho(z_0, r)} |\nabla_\lambda g|^p v(z)\,dz\right)^{1/p} \tag{Ib}$$

for any Lipschitz continuous function g, *where* μ *can be chosen to be the* u*-average of* g *over* $B_\rho(z_0, r)$.

In the case $p = q$ we can prove an analogous result, but the assumptions must be modified in a suitable way.

The first step in deriving Theorem I consists of proving a representation formula for functions vanishing on a large part of a metric ball $B_\rho = B_\rho(\bar{z}, r)$. More precisely, we prove the following lemma.

LEMMA 1. *Let* w *be a strong* A_∞ *weight satisfying* (Z_λ) *and let* g *be a continuously differentiable function. If there exists* $\beta \in (0, 1)$ *such that* $|\{z \in B_\rho; g(z) = 0\}| \geqslant \beta|B_\rho|$, *then there exist* $c(\beta)$ *and a constant* $\tau > 0$ *depending only on* λ, n *and* w *such that*

$$|g(z)| \leqslant c(\beta)\int_{\tau B_\rho} |\nabla_\lambda g(\xi)|\, w^{1-1/N}(\xi) K(z, \xi)\,d\xi \tag{1a}$$

for almost all $z \in B_\rho$ *(with respect to Lebesgue measure), where*

$$K(z, \xi) = \left(\int_{B_\rho(z, \rho(z, \xi))} w(\zeta)\,\lambda^{m/(N-1)}(\zeta)\,d\zeta\right)^{(1-N)/N}.$$

In order to obtain a rough form of (*SP*), we prove weighted norm inequalities for the operator that appear on the right hand side of the integral representation (1*a*). This argument can also be carried out for other integral representations. In particular, when $w = 1$ we can derive two-weight inequalities which generalize those in [4, 11, 24]. This is because in this case we can use the representation formulas in [12, 9] which are valid for larger classes of anisotropic gradients than the ones considered here.

3. Let us show now how our previous results can be used to prove pointwise inequalities for some classes of degenerate elliptic equations. Let Ω be an open bounded subset of $\boldsymbol{R}^N = \boldsymbol{R}^m \times \boldsymbol{R}^n$ and let us keep the notations of sect. 2. Let $\mathcal{L}$ denote the sec-

ond order differential operator in divergence form defined by

$$\mathcal{L} = \sum_{i,j} \frac{\partial}{\partial z_i}\left(a_{ij}(z)\,\frac{\partial}{\partial z_j}\right)$$

where the coefficients $a_{ij} = a_{ji}$ are measurable real-valued functions such that

$$v(z)(|\xi|^2 + \lambda^2(z)|\eta|^2) \leqslant \sum_{i,j} a_{ij}(z)\,\zeta_i\zeta_j \leqslant u(z)(|\xi|^2 + \lambda^2(z)|\eta|^2)$$

for any $\zeta = (\xi, \eta)$ and $z = (x, y) \in \boldsymbol{R}^n \times \boldsymbol{R}^m$, where the weight functions u and v satisfy the assumptions of Theorem I with $p = 2$.

We are interested in proving local regularity (in particular Harnack's inequality) of weak solutions of $\mathcal{L}u = 0$ in Ω. To this end, let us first give precise definitions of what a weak solution is (see [5, section 2]). We will denote by $H(\Omega)$ the completion of $\mathrm{Lip}\,(\overline{\Omega})$ with respect to the norm

$$\|f; H(\Omega)\| = \left(\int_\Omega \sum_{i,j} a_{ij}(z)\,\partial_i f(z)\,\partial_j f(z)\,dz + \int_\Omega f^2(z)\,v(z)\,dz\right)^{1/2}.$$

Arguing as in [5], we can canonically associate with any $f \in H(\Omega)$ a function $\tilde{f} \in L^2_v$ (the usual Lebesgue space with respect to the measure $v(z)\,dz$). Analogously, we can define a new space $H_0(\Omega)$ as the completion of $\mathrm{Lip}_0(\overline{\Omega})$ with respect to the norm

$$\|f; H_0(\Omega)\| = a_0(f, f) = \left(\int_\Omega \sum_{i,j} a_{ij}(z)\,\partial_i f(z)\,\partial_j f(z)\,dz\right)^{1/2}.$$

We note that, by (Ib), $a_0(f,f)^{1/2}$ is in fact a real norm on $H_0(\Omega)$ since it is not difficult to see that, if g is supported in $B_\rho(z_0, r)$, then, by integrating on the annulus $B_\rho(z_0, 2r) \backslash B_\rho(z_0, r)$ we can use (Ib) to estimate the constant μ and hence the usual Sobolev inequality follows.

The bilinear form $a_0(\cdot,\cdot)$ can be extended to $H(\Omega) \times H_0(\Omega)$, so that the following definition is well-posed.

Definition 2. *Let $f \in H(\Omega)$ be given; we will say that f is a weak solution of $\mathcal{L}u = 0$ if*

$$a_0(f, \phi) = 0 \quad \textit{for any } \phi \in H_0(\Omega),\ \ \phi \geqslant 0,$$

where $\phi \geqslant 0$ means that the associated function $\tilde{\phi} \in L^2_v$ is nonnegative a.e.

We can now state Harnack's inequality.

Theorem II. *Let u, v, be weight functions satisfying the assumptions of Theorem I with $p = 2$, and let $f \in H(\Omega)$ be a weak solution of $\mathcal{L}u = 0$. Then there exist $c_0, c_1 > 0$ (depending only on λ, u, v, n and m) such that*

$$\operatorname*{ess\,sup}_{B_\rho(x,t)} \tilde{f} \leqslant \exp\{c_1 \mu(B_\rho(x, t))\} \operatorname*{ess\,inf}_{B_\rho(x,t)} \tilde{f}$$

for any ball $B_\rho(x, t)$ such that $B_\rho(x, c_0 t) \subset \Omega$, where $\tilde{f} \in L^2_v$ is the function associated with f and $\mu(B_\rho(x, t)) = [v(B_\rho(x, t))/u(B_\rho(x, t))]^{1/2}$.

In particular, if $\lambda \equiv 1$, then Theorem II can be proved by applying Theorem B in [5] (see also the remarks in [19, section 7]). To this end we only need to verify that conditions (1.2-*i*)-(1.2-*iii*) therein are satisfied. Now conditions (*ii*) and (*iii*) follow from Theorem I, whereas condition (*i*) (doubling property of $u(z)\,dz$ and $v(z)\,dz$) are satisfied since u is a doubling weight by hypothesis and $vw^{(N-1)/N}$ belongs to A_p with respect to the measure $w(z)^{(N-1)/N}dz$, and consequently v can be shown to be a doubling measure. If $\lambda \not\equiv 1$, then the proof can be carried out by adapting the arguments of [5] to the geometry associated with the metric ρ as in [13, 14, 11]. To this end, we need to use suitable cut-off functions which fit the ρ-balls. The existence of such functions can be proved as in [11, Proposition 5.10].

Remark. If $u \approx v$, *i.e.*, u is a constant multiple of v, (or u and v are not too different in a suitable sense, see [5]), then we can derive from the Harnack inequality above that weak solutions of $\mathcal{L}u = 0$ are Hölder continuous or at least continuous if a further approximation hypothesis is satisafied (condition (5.1) in [5]: see below). This follows by repeating the arguments of [5, section 5] and keeping in mind that the distance ρ is Hölder continuous with respect to the Euclidean metric, see [13, 11]. Condition (5.1) in [5] requires that, if f is an element of $H(B_\rho)$ whose associated function $\tilde{f}$ satisfies $\tilde{f} \geq 0$ a.e. in B_ρ, then $f \geq 0$ in the sense of $H(B_\rho)$. In particular, if $\lambda \equiv 1$, and $w \equiv 1$ (see the following remark), then the approximation condition follows by a convolution argument.

Remark. We stress the fact that, if $u \equiv v$, then our results provide a unified approach to two different situations: the case when u belongs to the Muckenhoupt class A_2 (with respect ρ), and the case $\lambda \equiv 1$ and $u(z) = |\phi'(z)|^{1-2/N}$, where ϕ is a global quasiconformal mapping of $\boldsymbol{R}^N$, $N \geq 2$ and $|\phi'(z)|$ denotes the absolute value of its Jacobian determinant. Indeed, in the first case it is enough to choose $w \equiv 1$ in Theorem I (and then we reobtain the Harnack inequality of [14]), whereas in the second case we must choose $w(z) = |\phi'(z)|$, which is a strong A_∞ weight (see, for instance [7]). However, in this case we need some careful calculations in order to verify the assumptions of Theorem I. First, note that the measure $u(z)\,dz$ is doubling, since $|\phi'|$ belongs to A_∞. Thus, we have only to prove that $uw^{-1+1/N}$ belongs to A_2 with respect to the measure $w(z)^{1-1/N}dz$, *i.e.*

$$(*)\qquad \int_B |\phi'(z)|^{1-2/N}dz \cdot \int_B |\phi'(z)|\,dz \leq \mathrm{const}\left(\int_B |\phi'(z)|^{1-1/N}dz\right)^2$$

for all balls B. Now, by a result of Gehring [18, Lemma 4] and by Hölder inequality, we get

$$⨍_B |\phi'(z)|\,dz \leq \mathrm{const}\left(⨍_B |\phi'(z)|^{1/N}dz\right)^N \leq \mathrm{const}\left(⨍_B |\phi'(z)|^{1-1/N}dz\right)^{N/(N-1)}$$

for any ball B, and hence, again by Hölder inequality, the left side of (*) is bounded by a constant times

$$|B|^2 \left(\fint_B |\phi'(z)|^{1-1/N} dz \right)^{(N-2)/(N-1)+N/(N-1)} = \left(\int_B |\phi'(z)|^{1-1/N} dz \right)^2 ,$$

and we are done. Hence, in particular, the Harnack inequalities proved in both parts of [8] are particular cases of Theorem II.

REMARK. Typically, the following equation satisfies the assumptions of Theorem II but not those of previous results existing in the literature:

$$\mathcal{L}f = \operatorname{div}\left(\left(|x|^{\sigma+1} + |y| \right)^{\alpha/(\sigma+1)} (\partial_x f, |x|^\sigma \partial_y f) \right) = 0$$

where $\alpha, \sigma > 0$ and $z = (x, y) \in \Omega \subset \boldsymbol{R}^2$ with $(0, 0) \in \Omega$. In Theorem II, choose $u(x) = v(x) = (|x|^{\sigma+1} + |y|)^{\alpha/(\sigma+1)}$, and $\lambda(x) = |x|^\sigma$. Formula (*Ia*) follows by doubling. In [13] it is proved that $(|x|^{\sigma+1} + |y|)^{1/(\sigma+1)} \approx \rho(z, 0)$, where ρ is formed by using $\lambda(x) = |x|^\sigma$. Keeping in mind that any positive power of the distance is a strong A_∞ weight, we can choose $w(z) = \rho(z, 0)^\beta$, where β is such that $\beta - \alpha + \sigma > -2$, so that $u(z)^{-1} w(z)$ is locally summable. Then, it is possible to repeat the arguments of [14], Example 3.7, and then to prove that $uw^{-1/2}$ belongs to A_2 with respect to metric ρ and the measure $w(z)^{1/2} dz$. Thus, the hypotheses of Theorems I and II are satisfied.

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REFERENCES

[1] B. BOJARSKI, *Remarks on Sobolev imbedding inequalities.* Complex Analysis, Lecture Notes in Mathematics 1351, Springer, 1989, 52-68.

[2] A. P. CALDERON, *Inequalities for the maximal function relative to a metric.* Studia Math., 57, 1976, 297-306.

[3] L. CAPOGNA - D. DANIELLI - N. GAROFALO, *Embedding theorems and the Harnack inequality for solutions of nonlinear subelliptic equations.* C.R. Acad. Sci. Paris, 316, 1993, 809-814.

[4] S. CHANILLO - R. L. WHEEDEN, *Weighted Poincaré and Sobolev inequalities and estimates for the Peano maximal function.* Amer. J. Math., 107, 1985, 1191-1226.

[5] S. CHANILLO - R. L. WHEEDEN, *Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations.* Comm. Partial Differential Equations, 11, 1986, 1111-1134.

[6] S. CHUA, *Weighted Sobolev inequality on domains satisfying chain conditions.* Proc. Amer. Math. Soc., to appear.

[7] G. DAVID - S. SEMMES, *Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings* In: C. SADOSKY (ed.), *Analysis and Partial Differential Equations.* Marcel Dekker, 1990, 101-111.

[8] E. FABES - C. KENIG - R. SERAPIONI, *The local regularity of solutions of degenerate elliptic equations.* Comm. Partial Differential Equations, 7, 1982, 77-116.

[9] J. C. FERNANDES, *Mean value and Harnack inequalities for a certain class of degenerate parabolic equations.* Revista Matématica Iberoamericana, 7, 1991, 247-286.

[10] C. FEFFERMAN - D. H. PHONG, *Subelliptic eigenvalue problems.* In: W. BECKNER *et al.* (eds.), *Conference on Harmonic Analysis* (Chicago 1980). Wadsworth 1981, 590-606.

[11] B. FRANCHI, *Weighted Sobolev-Poincaré inequalities and pointwise estimates for a class of degenerate elliptic equations.* Trans. Amer. Math. Soc., 327, 1991, 125-158.

[12] B. FRANCHI, *Inégalités de Sobolev pour des champs de vecteurs lipschitziens.* C.R. Acad. Sci. Paris, 311, 1990, 329-332.

[13] B. FRANCHI - E. LANCONELLI, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients.* Ann. Scuola Norm. Sup. Pisa, 10, (4), 1983, 523-541.

[14] B. FRANCHI - R. SERAPIONI, *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach.* Ann. Scuola Norm. Sup. Pisa, 14, (4), 1987, 527-568.

[15] B. FRANCHI - C. GUTIÉRREZ - R. L. WHEEDEN, *Weighted Sobolev-Poincaré inequalities for Grushin type operators.* Comm. Partial Differential Equations, to appear.

[16] B. FRANCHI - S. GALLOT - R. L. WHEEDEN, *Sobolev and isoperimetric inequalities for degenerate metrics.* Math. Ann., to appear.

[17] B. FRANCHI - S. GALLOT - R. L. WHEEDEN, *Inégalités isopérimetriques pour des métriques dégénérées.* C.R. Acad. Sci. Paris, Ser. A, 317, 1993, 651-654.

[18] F. W. GEHRING, *The L^p-integrability of the partial derivatives of a quasi conformal mapping.* Acta Math., 130, 1973, 265-277.

[19] C. GUTIÉRREZ - R. L. WHEEDEN, *Mean-value and Harnack inequalities for degenerate parabolic equations.* Colloq. Math., 60/61, 1990, 157-194.

[20] T. IWANIEK - C. A. NOLDER, *Hardy-Littlewood inequality for quasiregular mappings in certain domains in $\boldsymbol{R}^n$.* Ann. Acad. Sci. Fenn. Series A I Math., 10, 1985, 267-282.

[21] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander condition.* Duke Math. J., 53, 1986, 503-523.

[22] G. LU, *The sharp Poincaré inequality for free vector fields: an endpoint result.* Revista Matématica Iberoamericana, to appear.

[23] A. NAGEL - E. M. STEIN - S. WAINGER, *Balls and metrics defined by vector fields I: basic properties.* Acta Math., 155, 1985, 103-147.

[24] O. SALINAS, *Harnack inequality and Green function for a certain class of degenerate elliptic differential operators.* Revista Matématica Iberoamericana, 7, 1991, 313-349.

[25] E. SAWYER - R. L. WHEEDEN, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces.* Amer. J. Math., 114, 1992, 813-874.

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