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Angela Iannelli, Giovanni Prouse, Alessandro Veneziani

On a nonlinear equation of the vibrating string

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Analisi matematica. — *On a nonlinear equation of the vibrating string.* Nota di ANGELA IANNELLI, GIOVANNI PROUSE e ALESSANDRO VENEZIANI, presentata (*) dal Corrisp. G. Prouse.

ABSTRACT. — A nonlinear model of the vibrating string is studied and global existence and uniqueness theorems for the solution of the Cauchy-Dirichlet problem are given. The model is then compared to the classical D'Alembert model and to a nonlinear model due to Kirchhoff.

KEY WORDS: Vibrating string; Weak solution; Approximable solution.

RIASSUNTO. — *Su un'equazione non lineare della corda vibrante.* Si studia un modello non lineare della corda vibrante e si enunciano teoremi di esistenza ed unicità in grande della soluzione del problema di Cauchy-Dirichlet. Si esegue poi un confronto tra questo modello ed i modelli di D'Alembert e di Kirchhoff.

1. INTRODUCTION

In this *Note* we wish to present some results relating to a nonlinear model of the vibrating string introduced in [1, 2]; details of proofs, together with some results of numerical computations, can be found in [3].

Let us consider a string of length l_0 and mass M, stretched on the x-axis and fixed at the points $x = 0$ and $x = l \geq l_0$, subject to an external force $f(t, x)$ normal to the x-axis. Assuming that the motion is transversal and denoting by $u(t, x)$ the displacement of the point x at the time t, the equation we consider is

$$(1.1) \qquad (M/l)\,u_{tt} - \partial(\varphi(l(1+u_x^2)^{1/2} - l_0)\,u_x(1+u_x^2)^{-1/2})/\partial x - f = \\ = (M/l)\,u_{tt} - Db(Du) - f = 0$$

having set $D = \partial/\partial x$, $b(\xi) = \varphi(l(1+\xi^2)^{1/2} - l_0)\,\xi(1+\xi^2)^{-1/2}$.

The function φ defines the *stress-strain law, i.e.* the elastic properties of the string:

$$(1.2) \qquad S = \varphi(\tau), \qquad \tau \geq 0 \ (^1).$$

Equation (1.1) is obtained by considering a discrete model of the string, constituted by n springs of length l/n, linked by frictionless hinges of mass $m = M/(n+1)$. The motion of this system is governed by a system of ordinary differential equations in the variable t; letting $n \to \infty$, this system «converges» to (1.1). Details of the procedure are given in [1, 2].

We shall make on the function $\varphi(\tau)$ the following assumptions, of obvious physical interpretation:

a) $\varphi(\tau) \in C^1[0, \infty)$, $\varphi(0) = 0$, $0 < \varphi'(\tau) \leq M < +\infty$;

b) $\varphi(\tau)$ *grows asyptotically like* τ^γ, with $0 < \gamma \leq 1$.

(*) Nella seduta del 23 aprile 1994.

(1) For the sake of simplicity, we assume that the function φ does not depend explicitly on x and that the density is constant. The general case could however be treated in a similar way.

It should be noted that the classical Hooke's law (perfectly elastic material, $\varphi(\tau) = \sigma\tau$, $\sigma > 0$) corresponds to $\gamma = 1$, while, in general, the behaviour of elastic materials corresponds to values of $\gamma \leqslant 1$ (*i.e.* the stress grows asymptotically at most linearly with the strain); the case $\gamma = 0$ (which we do not consider here), would correspond to a material which tends to become perfectly plastic as $\tau \to \infty$.

We shall consider, for eq. (1.1), the following Cauchy-Dirichlet problem, corresponding to the string fixed at both ends and with given initial position and velocity

$$\begin{cases} u(0,x) = \alpha(x), \quad u_t(0,x) = \beta(x) & (0 \leqslant x \leqslant l), \\ u(t,0) = u(t,l) = 0, & (0 \leqslant t \leqslant T), \end{cases} \tag{1.3}$$

with the obvious compatibility conditions $\alpha(0) = \alpha(l) = \beta(0) = \beta(l) = 0$.

A fundamental rôle in the study of (1.1), (1.3) is played by the «approximate» equation

$$(M/l)\, v_{tt} - Db(Dv) + \varepsilon D^4 v - f = 0 \qquad (\varepsilon > 0) \tag{1.4}$$

with the initial ad boundary conditions

$$\begin{cases} v(0,x) = \alpha(x), \quad v_t(0,x) = \beta(x) & (0 \leqslant x \leqslant l), \\ v(t,0) = v(t,l) = D^2 v(t,0) = D^2 v(t,l) = 0, & (0 \leqslant t \leqslant T), \end{cases} \tag{1.5}$$

corresponding to the motion of a rod with flexional rigidity $\varepsilon > 0$ and hinged at both ends; the solutions of (1.1), (1.3) will, in fact, be obtained as limits, when $\varepsilon \to 0$, of solutions of (1.4), (1.5), by a procedure of obvious physical interpretation.

2. Basic notations and definitions

In the sequel, we shall denote by $H^{s,p}$ the classical Sobolev space of functions $\in L^p$ together with their derivatives of order $\leqslant s$ and by $\langle , \rangle$ the duality between $L^{(\gamma+1)/\gamma}$ and $L^{\gamma+1}$ $(0 < \gamma \leqslant 1)$.

Assuming that $f \in L^2(Q)(Q = (0,T) \times (0,l))$, $\alpha \in H_0^1$, $\beta \in L^2$ [2], *we shall say that* u *is a weak solution in* Q *of* (1.1), (1.3) *if*:

i) $u(t) \in L^\infty(0,T; H_0^{1,\gamma+1}) \cap H^{1,\infty}(0,T;L^2)$, $u(o) = \alpha$;

ii) $u(t)$ *satisfies almost everywhere on* $(0,T)$ *the equation*

$$\int_0^t \{-(u',h')_{L^2} + \langle b(Du), Dh\rangle - (f,h)_{L^2}\}\, d\eta + (u'(t),h(t))_{L^2} - (\beta, h(0))_{L^2} = 0 \quad \forall h(t) \in L^2(0,T;H_0^{1,\gamma+1}) \cap H^1(0,T;L^2). \tag{2.1}$$

(2) For the sake of simplicity, we set $L^p(0,l) = L^p$, $H^{s,p}(0,l) = H^{s,p}$, $H^{s,2} = H^s$ and assume that $M/l = 1$. $H_0^{s,p}$ is the closure of $\mathcal{D}(0,l)$ in $H^{s,p}$.

We shall, on the other hand, *say that v is a weak solution in Q of* (1.4), (1.5) *if*:

i_ε) *$v(t) \in L^\infty(0, T; H_0^1 \cap H^2) \cap H^{1,\infty}(0, T; L^2)$, $v(0) = \alpha \in H_0^1 \cap H^2$;*

ii_ε) *$v(t)$ satisfies, a.e. on* (0, T), *the equation*

$$(2.2) \qquad \int_0^t \{-(v', k')_{L^2} + (b(Dv), Dk)_{L^2} + \varepsilon(D^2 u, D^2 k)_{L^2} - (f, k)_{L^2}\} d\eta +$$

$$+ (v'(t), k(t))_{L^2} - (\beta, k(0))_{L^2} = 0 \qquad \forall k(t) \in L^2(0, T; H_0^1 \cap H^2) \cap H^1(0, T; L^2).$$

Finally, *we shall say that u is an approximable solution in Q of* (1.1), (1.3) *if it satisfies conditions i*), *ii*) *and, moreover,*

iii) *There exists a sequence $\{v_{\varepsilon_n}\}$ of weak solutions of* (1.4), (1.5) *such that*

$$(2.3) \qquad \lim_{\varepsilon_n \to 0} v_{\varepsilon_n} = u$$

in the weak-star topology of $L^\infty(0, T; H_0^{1,\gamma+1}) \cap H^{1,\infty}(0, T; L^2)$ and in the strong topology of $C^0(0, T; L^2)$.

3. Existence and uniqueness theorems

Consider, to begin with, the case of the hinged rod. The following existence and uniqueness theorem holds:

THEOREM 1. *Assume that $\varphi(\tau)$ satisfies a*), *b*) *of section* 1 *and that $\alpha \in H_0^1 \cap H^2$, $\beta \in L^2$, $f \in L^2(Q)$. There exists then, $\forall \varepsilon > 0$, a weak solution of* (1.4), (1.5) *in Q; moreover, this solution is unique.*

Letting now $\varepsilon \to 0$, it is possible to show that there exists a sequence of weak solutions of (1.4), (1.5) which converges to a weak solution u of (1.1), (1.3). We have, precisely,

THEOREM 2. *If $\varphi(\tau)$ satisfies a*), *b*) *and $\alpha \in H_0^1$, $\beta \in L^2$, $f \in L^2(Q)$, there exists in Q an approximable solution of* (1.1), (1.3).

The problem of the uniqueness of a weak solution is still open; a «weak» uniqueness result can however be given for the approximable solution, as expressed by

THEOREM 3. *Under the hypotheses of Theorem* 2 *and assuming, moreover, that $\varphi(\tau)$ is analytic, there exists, for nearly all $f \in L^2(Q)$, a unique approximable solution.*

This theorem does not, therefore, exclude the existence of other weak solutions, which however are not approximable by the weak solutions of the vibrating rod problem.

4. Comparison with order vibrating string models

Many models have been proposed for the study of the transversal motion of a string. Of special interest, among these, are the ones proposed by D'Alembert and by Kirchhoff, leading, respectively, to the equations

$$(M/l)\,u_{tt} - \sigma(l - l_0)\,u_{xx} = f \tag{4.1}$$

and

$$(M/l)\,u_{tt} - \left(\sigma(l - l_0) + (E/2l)\int_0^l u_x^2\,dx\right)u_{xx} = f. \tag{4.2}$$

Equation (4.1) is deduced under the assumption that either $l_0 \ll l$, or $|u_x| \ll 1$, while (4.2) assumes that the tension along the string is constant; both equations suppose that Hooke's law holds, in which case (1.1) becomes

$$(M/l)\,u_{tt} - \sigma l u_{xx} + \sigma l_0 u_{xx}(1 + u_x^2)^{-3/2} = f. \tag{4.3}$$

While the properties of D'Alembert's equation are classical and well known, eq. (4.2) has been the object of very recent studies (see, for instance, [4, 5] and the relative bibliography). It is therefore interesting to study, especially from a numerical point of view, the different behaviour of models (4.1), (4.2), (4.3); this study, of which we shall give here only a general indication, is carried out in [3].

First of all, it can easily be seen that the three models tend to coincide when $l_0 \ll l$, and this circumstance is fully confirmed by numerical experiments. If this condition is not met (as in most cases of physical interest) the results may vary considerably, especially if the initial data are not «small». In particular:

a) The frequency of the oscillations is appreciably higher for the two nonlinear models than for the linear one. This feature has been confirmed by measurements made on a «real» string at the Dipartimento di Meccanica of the Politecnico di Milano.

b) If the initial position is a sine wave and $f = \beta = 0$, the motion, according to the D'Alembert model, remains sinusoidal. A very similar behaviour is found for the Kirchhoff model, while the motion, according to model (4.3), loses very rapidly its original sinusoidal shape.

These features are clearly shown in the pictures which illustrate the motion, in the time interval 0-10 ms, of a steel string of length 500 mm, with initial position corresponding to a sine wave of amplitude 15 mm and no initial velocity and external force. Figure 1:1, corresponds to the motion according to the model of D'Alembert, fig. 1:2, to the model studied in the present *Note*, fig. 1:3, to the Kirchhoff model.

It is interesting to note that numerical experiments on the D'Alembert equation with an added nonlinear term have been performed by Fermi, Pasta and Ulam [6], who also found that the presence of a nonlinearity completely disrupted any initial regularity of the motion.

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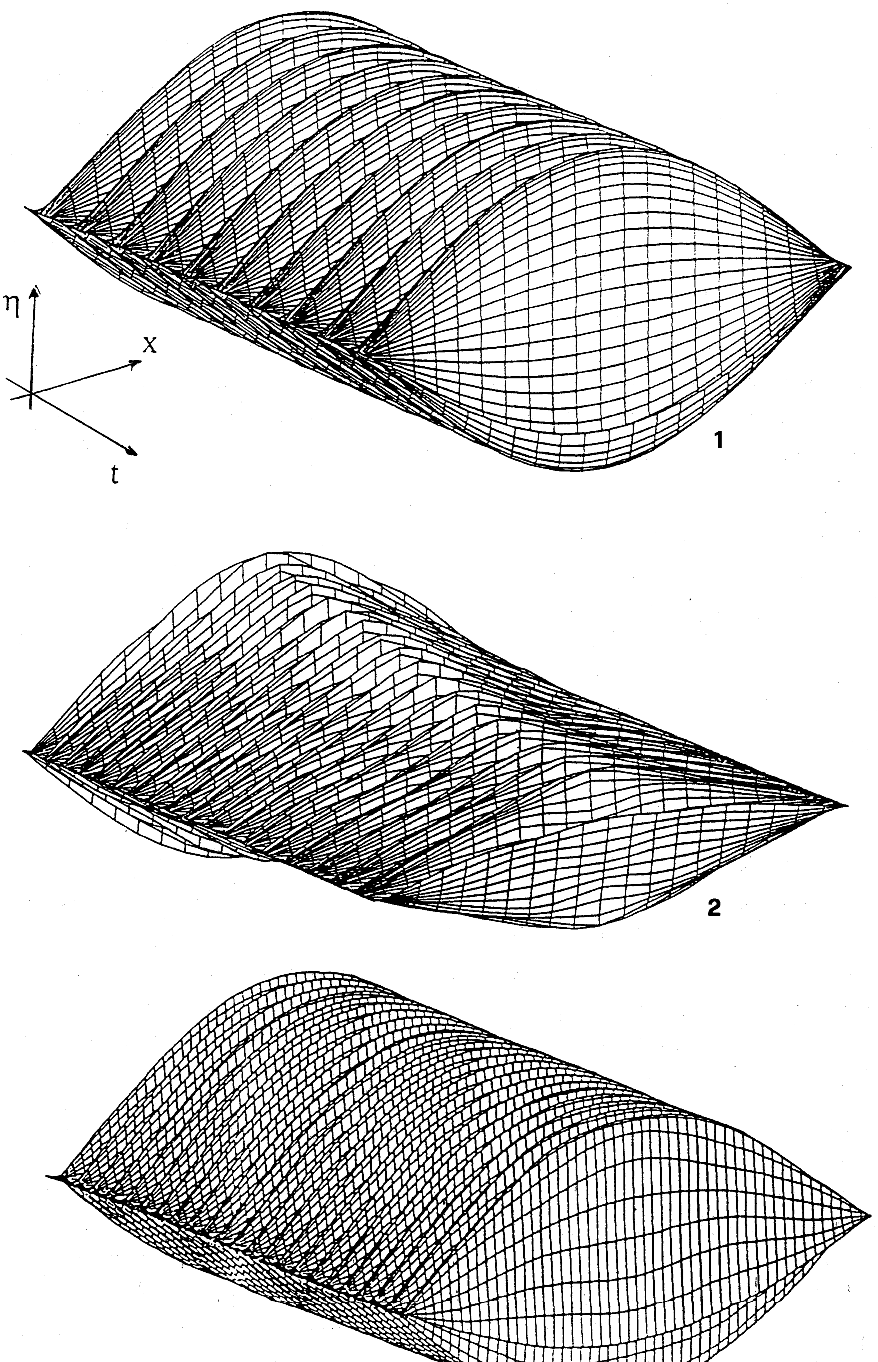

Fig. 1

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Dipartimento di Matematica
Politecnico di Milano
Piazza L. Da Vinci, 32 - 20133 Milano