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On control problems of minimum time for Lagrangian systems similar to a swing. I. Convexity criteria for sets

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Meccanica. — *On control problems of minimum time for Lagrangian systems similar to a swing. I. Convexity criteria for sets.* Nota di Aldo Bressan e Monica Motta, presentata (*) dal Socio A. Bressan.

Abstract. — One establishes some convexity criteria for sets in $\boldsymbol{R}^2$. They will be applied in a further *Note* to treat the existence of solutions to minimum time problems for certain Lagrangian systems referred to two coordinates, one of which is used as a control. These problems regard the swing or the ski.

Key words: Analytical mechanics; Lagrangian systems; Control theory.

Riassunto. — *Su problemi di controllo di tempo minimo per sistemi Lagrangiani simili a un'altalena. I. Criteri di convessità per insiemi.* Si stabiliscono dei criteri di convessità per insiemi in $\boldsymbol{R}^2$. Essi verranno applicati in una prossima *Nota* per trattare l'esistenza di soluzioni di problemi di tempo minimo per certi insiemi meccanici riferiti a due coordinate, una delle quali è usata come controllo. Tali problemi riguardano l'altalena o lo sci.

1. Introduction

The main aim of the present work is to study the existence of solutions to minimum time problems for the Lagrangian holonomic system $\Sigma = \mathcal{A} \cup \mathcal{U}$ (with two degrees of freedom) introduced in [6], in case Σ's parts $\mathcal{A}$ and $\mathcal{U}$ schemetize a (possibly) revolutory swing or a pair of skis, and whom uses $\mathcal{A}$ respectively, frictions and air resistance being neglected.

In general, Σ can be referred to the coordinates s and u where, among other things, (*i*) s is an arclength on a line l belonging to a vertical (oriented) plane Oc_1c_2, (*ii*) $\mathcal{A}$ is a rigid body whose configurations are determined by s's values, (*iii*) l's (signed) curvature function $s \mapsto c(s)$ is in C^1, and (*iv*) u determines $\mathcal{U}$'s configuration relative to (a frame joint to) $\mathcal{A}$. One regards $\mathcal{U}$ as a man looking forward to using u as a control (in optimization problems). *E.g.*, for Σ's instance Σ_1 that schemetizes the above skis-skier system, l is the trajectory of $\mathcal{U}$'s skis (or feet) – see [6] (1).

We state a condition, say Γ, on Σ_1's structural data that implies the existence of a solution to any problem of minimum time for Σ_1. First we show that Γ implies a well known convexity condition, say *WCC*, sufficient for that purpose (2). In case l's curva-

(*) Nella seduta del 12 febbraio 1994.

(1) Briefly, in [3] to [5] Aldo Bressan started a systematic (non linear) application of control theory to Lagrangian mechanical systems, by using coordinates as controls. This is based on the purely mathematical paper [1] (extended by [2]). A. Bressan's aforementioned work has been further developed by himself and other researchers: F. Rampazzo, M. Favretti, and M. Motta – see [6-15, 17]. The present paper belongs to this research line.

(2) Since, as Galilei remarked, the period of small oscillations for a pendulum (independent of the amplitude) is proportional to $\sqrt{l}$ where l is the pendulum length (and the theory of mechanical similitude somehow extends this assertion), one might be pushed to conjecture that to minimize the time it is sufficient to minimize l. However there are simple counterexamples for very large trips of the revolutory swing. In [13], among other things, these counterexamples are extended to arbitrarily small trips.

ture $c(s)$ (with a sign relative to Oc_1c_2) is constant, Γ becomes rather simple and even equivalent to the *WCC*.

In the optimization problems considered on Σ only «monotone» motions occur – see [6, 9, 10]; therefore Σ's two scalar dynamic equations reduce to one. This simplification causes the presence of a phase constraint – see $(2.16)_2$. Furthermore it renders the functional to be minimized singular (at the «border» of this constraint). Therefore the existence of the solution to our problem under the *WCC* must be proved in a new version.

In our treatment, for $c'(\cdot) \equiv 0$ and $c = c(s) > 0$, Σ_1 in effect becomes the afore-mentioned system schemetizing (a natural) revolutory swing, say Σ^+. Its analogue Σ^- with $0 > c(s) =$ const can also be regarded to be such a swing, admittedly unnatural; in fact in Σ^- [Σ^+] the floor of the revolutory swing is nearer [farther from] $\mathcal{A}$'s revolution axis than $\mathcal{A}$'s ceiling; consequently in Σ^- the man $\mathcal{U}$ is below $\mathcal{A}$'s floor when his feet are in the infimum point of their trajectory (the circle l), while for Σ^+ the same occurs when $\mathcal{U}$'s feet are in l's upper point (3).

It is worth noting that to treat minimum time problems for the unnatural swing Σ^- is interesting also because this is helpful in connection with the same problem for Σ_1, in case the skis' trajectory l contains circle arcs that are concave down – see assertions ($\mathcal{A}$) and ($\mathcal{B}$) below (6.1).

In more details, our minimum time problem ($\mathcal{P}$) for Σ_1 and $\Sigma^\pm$ is specified in sect. 2. In sect. 3 the *WCC* is treated in some special cases; and for dealing with the other cases a necessary and sufficient condition for the convexity of any suitable regular set is stated as a preliminary – see Theor. 3.1. In sect. 4 the afore-mentioned structural condition Γ is worked out for $\Sigma^\pm$ and it is shown to be equivalent to the *WCC*. In sect. 5 one considers the case where $c' \neq 0$ and Σ's moment of inertia w.r.t. its center of mass is negligible; and for it, on the basis of Part 1, one establishes a new simple structural condition equivalent to the *WCC*.

We note here that (Γ as well as) the *WCC* fails to hold for ($\mathcal{P}$) and Σ^+ when $\mathcal{A}$'s mass is negligible, *i.e.* $\Sigma^+(=\Sigma^-)$ is a pendulum with variable mass – see Example 1 below (4.5); furthermore the validity of Γ for Σ_1 or $\Sigma^\pm$ implies that $\mathcal{A}$'s mass is large enough.

In sect. 6 some cases for Σ_1 or $\Sigma^\pm$ are considered, where the solution to problem ($\mathcal{P}$) exists even when the *WCC* fails. The main existence theorem for the solution to ($\mathcal{P}$) in our general case is proved in sect. 7.

2. The skis-skier system Σ_1 and the revolutory swings $\Sigma^\pm$. Minimum time problems

We consider a righthanded Cartesian frame $Oxyz$ joint to the earth, such that its axes have the respective unit vectors c_1 to c_3 with $c_r \cdot c_s = \delta_{rs}$ (4) $(r, s = 1, 2, 3)$ and the

(3) Thus in Σ^- $\mathcal{U}$'s head is below [above] $\mathcal{U}$'s feet when the centrifugal force increases [decreases] $\mathcal{U}$'s weight.

(4) The scalar [vector] product for vectors is denoted by «$\cdot$» [«$\times$»]. *E.g.* «T» denotes a vector and «$|T|$» its modulus.

gravity acceleration g has in it the expression $-gc_2$. As well as in [6, p.155], we consider a line l in the plane Oxy, represented by

(2.1) $$P = P(s) = O + x(s)c_1 + y(s)c_2 \quad \forall s \in \Delta \doteq [s_0, s_1] \quad (g = -gc_2)$$

where s is an arclength on it; and we set

(2.2) $$T = x'(s)c_1 + y'(s)c_2, \quad n = c_3 \times T, \quad cn = dT/ds \text{ (hence } n \times c_3 = T)$$

so that $|T| = 1$ and $c = c(s)$ is l's curvature at $P(s)$ with the sign relative to c_3.

The line l will be regarded as the trajectory of either a pair $\mathfrak{A}$ of skis or the floor of a revolutory swing. In the first case we assume that

(2.3) $$x'(s) > 0 \quad \forall s \in \Delta;$$

otherwise $c \gtrless 0 = c'(s)$ and l can be represented by

(2.4) $$P = P(s) \equiv O + |r| \sin\theta c_1 - r\cos\theta c_2 \quad \forall\theta \in [\theta_0, \theta_1]$$
$$\text{with } \theta = s/|r|, \quad r = 1/c, \quad \theta_i = s_i/|r| \quad (i = 0, 1).$$

In each of the instances Σ_1, Σ^+, and Σ^- of Σ introduced in sect. 1 we call m the mass of the system, G its center of mass, and I_G its moment of inertia w.r.t. the axis z_G through G, parallel with z. As well as in [6, sect. 2] one considers the control constraint

(2.5) $$u \in U \doteq [u_1, u_2] \quad \text{with } 0 < u_1 < u_2;$$

and one assumes that

(2.6) $$1 - cu_2 > 0, \quad \text{hence } \xi \doteq 1 - cu > 0 \quad \forall (s,u) \in \Delta \times U, \quad c(\cdot) \in C^2(\Delta), \quad I_G(\cdot) \in C^2(U);$$

and that

(2.7) $$dI_C/du \lesseqgtr 0 \quad \text{for } c \gtrless 0, \quad \text{where } I_C \doteq I_G(u) + m(r-u)^2, \quad r \doteq c^{-1}.$$

One defines

(2.8) $$\mathfrak{I}(s,u) \doteq c^2 I_G(u) + m\xi^2, \quad \text{hence } \mathfrak{I} = c^2 I_C \text{ for } c \neq 0 \text{ and } \mathfrak{I}(\cdot) \in C^2(\Delta \times U).$$

Then (2.16) in [6] holds:

(2.9) $$\mathfrak{I} = \mathfrak{I}(s,u) > 0, \quad \mathfrak{I}_u \lesseqqgtr 0 \text{ for } c \gtreqqless 0 \quad \forall u \in U.$$

As well as in [6, (2.17)], Σ's kinetic energy is assumed to have the form

(2.10) $$2\mathcal{T} = \mathfrak{I}(s,u)\dot{s}^2 + \beta(u)\dot{u}^2 \quad \text{for some } \beta(\cdot) \in C^1.$$

Now we denote by $\Sigma_{\tilde{u}(\cdot)}$ the generally time-dependent holonomic system obtained from Σ by using the function $u = \tilde{u}(\cdot)$, defined on some time-interval $[t_0, t_1]$, as a control physically implemented by means of some reaction forces internal to $\mathcal{U}$ – and hence by the addition of a frictionless constraint, see [6, sect. 3]. Thus for $\Sigma_{\tilde{u}(\cdot)}$ $(2.5)_1$ acts as a control constraint and $\Sigma_{\tilde{u}(\cdot)}$'s (*semi-hamiltonian*) dynamic equations read – see [6, (3.7)]

(2.11) $$\dot{s} = p/\mathfrak{I}(s,u), \quad \dot{p} = (\mathfrak{I}_s(s,u)/2\mathfrak{I}^2(s,u))p^2 - mg\xi y'(s) \quad (u = \tilde{u}(t))$$
$$\text{for a.e. } t \in [t_0, t_1].$$

For Σ_1's increasing motions in Δ these equations can be put in the form [6, (10.1)] – see also [9, Part 1] –, *i.e.*

(2.12) $$\frac{d\mathcal{P}}{ds} = G(s,\mathcal{P},u) \doteq -2\mathfrak{I}(s,u)\,mg\xi y'(s) + \frac{\mathfrak{I}_s(s,u)}{\mathfrak{I}(s,u)}\mathcal{P}, \qquad \frac{dt}{ds} = \frac{\mathfrak{I}(s,u)}{\sqrt{\mathcal{P}}},$$

where $\mathcal{P}(s) = p^2[t(s)]$ and $u = u(s) = \tilde{u}[t(s)]$.

We consider the following problem:

(α) *given* $u_0^- \in U$ *and* $\dot{s}_0^- \geq 0$, *to determine a behaviour*

(2.13) $$u = u(\cdot) \in \mathcal{U} \doteq \mathcal{L}^1(\Delta, U)$$

of $\mathcal{U}$ *which minimizes the time necessary for* Σ_1 *to reach* s_1 *along a monotone motion under the initial condition* $(s(t_0^-), \dot{s}(t_0^-), u(t_0^-), \dot{u}(t_0^-)) = (s_0, \dot{s}_0^-, u_0^-, 0)$.

Setting

(2.14) $$p_0 = \mathfrak{I}(s_0, u_0^-)\,\dot{s}_0^-, \quad \text{so that } \mathcal{P}(s_0) = \mathcal{P}_0 \doteq p_0^2,$$

problem (α) becomes the following optimization problem.

($\mathcal{P}$) *To minimize the above time, which is given by*

(2.15) $$T[u(\cdot)] = T[\mathcal{P}(\cdot), u(\cdot)] \doteq \int_{s_0}^{s_1} \mathfrak{I}[s, u(s)]\,\mathcal{P}(s)^{-1/2}\,ds,$$

under the differential and the initial constraints $(2.12)_1$, $(2.14)_2$, *and the control constraint*

(2.16) $$u(\cdot) \in \mathcal{U}^* \doteq \{u(\cdot) \in \mathcal{U}: T[u(\cdot)] < +\infty\} \quad (\mathcal{P}(s) \geq 0 \ \forall s \in \Delta),$$

which implies the phase constraint $(2.16)_2$.

For any $\xi' \in \boldsymbol{R}$ and $s' \in \Delta$ we denote by $\mathcal{P}(\cdot, \xi', s', u)$ the solution in Δ to $(2.12)_1$ corresponding to the control function $u \in \mathcal{U}$ and such that $\mathcal{P}(s') = \xi'$; and we set

(2.17) $$\mathcal{U}^*_{\xi', s'} \doteq \{u(\cdot) \in \mathcal{U}: T[\mathcal{P}(\cdot, \xi', s', u), u(\cdot)] < +\infty\}.$$

Furthermore, $\mathcal{P}^{\pm}(\cdot, \xi', s')$ is the solution (at least in Δ) to the Cauchy problem

(2.18) $$d\mathcal{P}/ds = \mathcal{G}^{\pm}(s, \mathcal{P}), \quad \mathcal{P}(s') = \xi'; \qquad \mathcal{G}^{\pm}(s,\mathcal{P}) \doteq \substack{\max\\ \min}\, G(s, \mathcal{P}, U).$$

We remember that there exists $u^{\pm}(\cdot) \in \mathcal{U}$ such that $\mathcal{P}^{\pm}(\cdot, \xi', s') \equiv \mathcal{P}(\cdot, \xi', s', u^{\pm})$ where the process $(\mathcal{P}(\cdot, \xi', s', u^{\pm}), u^{\pm}(\cdot))$ solves the optimization problem – see [6, Theor. 8.2, pp. 169-70] or [9, Theor. 7.1, p. 25]

(2.19) $$\mathcal{P}(s_1, u) \to \substack{\max\\ \min}.$$

We set $\mathcal{P}^{\pm}(\cdot) \doteq \mathcal{P}^{\pm}(\cdot, \mathcal{P}_0, s_0)$ and, to treat some special cases, also

(2.20) $$\mathcal{P}_\sigma(s) \doteq \begin{cases} \mathcal{P}^+(s) & \forall s \in [s_0, \sigma] \\ \mathcal{P}^-(s, \mathcal{P}^+(\sigma), \sigma) & \forall s \in [\sigma, s_1] \end{cases} \quad (\sigma \in \Delta).$$

Consider now the instances Σ^+ and Σ^- of the system $\mathcal{A} \cup \mathcal{U}$ where $\mathcal{A}$ schemetizes a revolutory swing with $c > 0 = c'(s)$ and $c < 0 = c'(s)$, respectively. In these cases the line l is represented by (2.4). Then, using $\theta = s/|r|$ instead of s and noting that by

$(2.8)_1$ $\Im(s,u) \equiv \Im(u) \forall (s,u) \in \Delta \times U$, in connection with Σ^+ or Σ^- problem (α) reads

$(\widehat{\mathscr{P}})$ *to render*

$$(2.21) \qquad T[\widehat{u}(\cdot)] = T[\widehat{\mathscr{P}}(\cdot), \widehat{u}(\cdot)] \doteq \int_{\theta_0}^{\theta_1} \Im[\widehat{u}(\theta)]\, \widehat{\mathscr{P}}(\theta)^{-1/2}\, d\theta \to \inf$$

under the differential constraint

$$(2.22) \qquad d\mathscr{P}/d\theta = rG(|r|\,\theta, \mathscr{P}, u) = -\,2mg\,|r|\,\Im(u)\,\xi \sin\theta\,,$$

as well as the initial and control constraints

$$(2.23) \qquad \widehat{\mathscr{P}}(\theta_0) = \mathscr{P}_0\,, \quad u = \widehat{u}(\cdot) \in \widehat{\mathcal{U}}^* \doteq \{\widehat{u} \in \widehat{\mathcal{U}}\colon T[\widehat{u}(\cdot)] < +\infty\},$$

$$\textit{where } \widehat{\mathcal{U}} \doteq \mathscr{L}^1([\theta_0, \theta_1], U)\,.$$

For any $\xi' \in \boldsymbol{R}$ and $\theta' \in [\theta_0, \theta_1]$ we call $\widehat{\mathscr{P}}(\cdot, \xi', \theta', \widehat{u})$ the solution $\widehat{\mathscr{P}}(\cdot)$ to the differential equation (2.22) such that $\widehat{\mathscr{P}}(\theta') = \xi'$, while the analogue $\widehat{\mathcal{U}}^*_{\xi',\theta'}$ of the set $\mathcal{U}^*_{\xi',s'}$ defined by (2.17) in terms of $\mathcal{U}$, denotes the set of the corresponding admissible controls. Sometimes we shall briefly write $\widehat{\mathscr{P}}(\cdot, \xi', \widehat{u})[\widehat{\mathscr{P}}(\cdot, \widehat{u})]$ for $\widehat{\mathscr{P}}(\cdot, \xi', \theta_0, \widehat{u})[\widehat{\mathscr{P}}(\cdot, \mathscr{P}_0, \theta_0, \widehat{u})]$ while $\widehat{\mathscr{P}}^{\pm}(\theta, \xi', \theta') \doteq \mathscr{P}^{\pm}(r\theta, \xi', r\theta')[\widehat{\mathscr{P}}^{\pm}(\theta, \xi') \doteq \widehat{\mathscr{P}}^{\pm}(\theta, \xi', \theta_0)]$ $\forall \theta \in [\theta_0, \theta_1]$.

3. On the convexity of the set $F(s, \mathscr{P})$ in very special cases. Preliminaries for the other cases

First we study the convexity of the closed set

$$(3.1) \qquad F(s, \mathscr{P}) \doteq \{(\eta, \zeta) \in \boldsymbol{R}^2\colon \eta = G(s, \mathscr{P}, u),\ \zeta \geqslant \Im(s,u)/\sqrt{\mathscr{P}} \text{ for some } u \in U\}$$

for $s \in \Delta$ and $\mathscr{P} > 0$. As it is well known, up to some regularity conditions (lacking in our case) this convexity property implies the existence of a solution to the (reduced) problem $(\mathscr{P})$ in sect. 2.

Let us preliminarily remember that

$(\mathcal{A})$ *if* $\widetilde{z}(\cdot), \widetilde{\eta}(\cdot) \in C^2(U)$ *and* $\widetilde{\eta}_u(\overline{u}) \neq 0$ *for some* $\overline{u} \in U$, *then there are some neighborhoods* $\mathcal{M} \subset U$ *and* $\mathcal{N} \subset \boldsymbol{R}$ *of* $\overline{u}$ *and* $\widetilde{\eta}(\overline{u})$ *respectively such that* $\widetilde{\eta}(\cdot)$*'s restriction to* $\mathcal{M}$ *has a* C^2*-inverse* $u = u(\eta)$ $\forall \eta \in \mathcal{N}$. *Furthermore, we can construct the* C^2*-function* $z = z(\eta) \doteq \widetilde{z}[u(\eta)]$ $\forall \eta \in \mathcal{N}$ *and we have* (5)

$$(3.2) \qquad u_{\eta\eta} = -\,\eta_{uu}/\eta_u^3\,, \qquad z_{\eta\eta} = (z_{uu}\,\eta_u - z_u\,\eta_{uu})/\eta_u^3\,.$$

We now fix any $(s, \mathscr{P}) \in \Delta \times (0, +\infty)$ and consider the functions (6)

$$(3.3) \qquad \beta = -\,\Im\xi\,, \qquad \gamma = \Im_s/\Im = 2(cI_G(u) - m\xi u)\,c'/\Im = 2c'c^{-1}(1 - m\xi\Im^{-1})$$

(5) Indeed, briefly, $\eta_u u_\eta \equiv 1$; hence $\eta_{uu} u_\eta^2 + \eta_u u_{\eta\eta} = 0$, which yields $(3.2)_1$. Furthermore $z_\eta = z_u u_\eta$, hence $z_{\eta\eta} = z_{uu} u_\eta^2 + z_u u_{\eta\eta} = z_{uu}\eta_u^{-2} - z_u \eta_{uu} \eta_u^{-3}$ by $(3.2)_1$. Then $(3.2)_2$ holds.

(6) By $(2.8)_1$ we have $(3.3)_3$ and also $c^2[I_G(u) - m\xi u] = \Im - m\xi^2 - m\xi c u = \Im - m\xi$, which yields $(3.3)_4$.

of u, as well as the functions $z = \tilde{z}(u)$ and $\eta = \tilde{\eta}(u)$ defined by

$$(3.4) \qquad z = \Im(s, u)/\sqrt{\mathcal{P}}, \qquad \eta = G(s, \mathcal{P}, u) = \gamma\mathcal{P} + 2mgy'(s)\beta \; - \text{ see } (2.6)_2 .$$

Note that, by $(3.3)_{2\text{-}4}$

($\mathcal{B}$) *the above expression of* $G(s, \mathcal{P}, u)$ *(with* $\mathcal{P} > 0$*) reduces to its last term,* i.e. $\gamma = 0$ *(or equivalently* $\Im_s = 0$*)* $\forall u \in U$, *iff* $c'(s) = 0$.

Case $c = 0$. In it $\Im = m$ by $(2.8)_1$; hence, by $(3.3)_1$ and $(3.4)_3$,

($\mathcal{C}$) *(for* $c = 0$*)* $F(s, \mathcal{P})$ *is the set* $\{2m^2 gy'(s)\} \times \{\zeta : \zeta \geq m/\sqrt{\mathcal{P}}\}$, *convex and closed.*

Case $c \neq 0 = \dot{c}' = y'(s)$. By $(3.3)_{2\text{-}3}$ and $(3.4)_3$,

($\mathcal{D}$) *in this case* $F(s, \mathcal{P})$ *is the set* $\{0\} \times \{\zeta : \sqrt{\mathcal{P}}\zeta \geq \min \Im(s, U)\}$, *convex and closed.*

Remark that, in case $c = c(s) \neq 0$ (with $s \in \Delta$, $\mathcal{P} > 0$), for Σ_1 we have $\tilde{z}_u(u) \neq 0$ $\forall u \in U$ – see $(3.4)_1$ and $(2.9)_2$; therefore it will be useful to set

$$(3.5) \qquad u' \doteq \begin{cases} u_1 \\ u_2 \end{cases}, \qquad u'' \doteq \begin{cases} u_2 \\ u_1 \end{cases} \qquad \text{for } \tilde{z}_u(\cdot) \gtrless 0, \quad \text{hence } z' \doteq \tilde{z}(u') < z'' \doteq \tilde{z}(u''),$$

and to prove the following preliminary theorem.

THEOREM 3.1. *Assume that (a) at least one of the functions* $\tilde{\eta}(\cdot)$, $\tilde{z}(\cdot) \in C^2(U)$ *has a non-vanishing first derivative on* $U \doteq [u_1, u_2]$. *Then the closed set*

$$(3.6) \qquad \mathcal{F} \doteq \{(\eta, \zeta) : \eta = \tilde{\eta}(u),\ \zeta \geq \tilde{z}(u) \text{ for some } u \in U\}$$

is convex iff one of the four cases $(3.7)^{\pm}$ *and* $(b)^{\pm}$ *below holds* (7).

$$(3.7) \qquad \tilde{\eta}_u(u) \gtrless 0, \quad \tilde{z}_{uu}(u)\tilde{\eta}_u(u) \begin{cases} \geq \\ \leq \end{cases} \tilde{\eta}_{uu}(u)\tilde{z}_u(u) \quad \forall u \in U .$$

$(b)^{\pm}$ *First,* $(3.5)_3^{\pm}$ *holds; second, there exists*

$$(3.8) \qquad \bar{u} \doteq {}^{\min}_{\max} \{u \in U : \tilde{\eta}_u(u) = 0\} \quad (\textit{hence}, \text{ e.g.}, \ \bar{z} \doteq \tilde{z}(\bar{u}) \textit{ and } \bar{\eta} \doteq \tilde{\eta}(\bar{u}) \textit{ exist});$$

third,

$$(3.9) \qquad \tilde{\eta}(u) \in [\bar{\eta}, \eta'] \quad \forall u \in [\bar{u}, u''] \qquad (\eta' \doteq \tilde{\eta}(u'),\ \eta'' \doteq \tilde{\eta}(u''));$$

and fourth,

$$(3.10) \qquad \tilde{\eta}_u(u)(\tilde{z}_{uu}\tilde{\eta}_u - \tilde{\eta}_{uu}\tilde{z}_u)(u) \geq 0 \quad \forall u \in [u', \bar{u}] .$$

This theorem is useful because, first, as noted above (3.5), for $c \neq 0$, $s \in \Delta$, $\mathcal{P} > 0$, and $\mathcal{F} = F(s, \mathcal{P})$ its unique assumption (a) holds for Σ_1 in that $(3.4)_1$ and $(2.9)_2$ imply $\tilde{z}_u(u) \neq 0$ $\forall u \in U$; hence its alternative $(b)^{\pm}$ can always be used for it; and, second, we shall also use the alternative $(3.7)^{\pm}$ for the special systems Σ^+ and Σ^- because its use is simpler.

(7) *E.g.* $(3.5)^+$ is the upper part of (3.5).

Before proving the theorem we write an easy corollary of it and the meaning, $(\mathcal{E})$ below, of condition (3.9).

COROLLARY 3.1. *If $\mathcal{F}$ is convex, condition $(3.5)_3^{\pm}$ holds, there exists $\bar{u}$ satisfying $(3.8)^{\pm}$, and it coincides with u', then*

$$\mathcal{F} = \{\eta'\} \times \{\zeta : \zeta \geqslant z'\}. \tag{3.11}$$

By (3.1) it is easy to check directly the following assertion.

$(\mathcal{E})$ *in case $\bar{u}$ exists and* (3.9) *holds, then $\tilde{\eta}(\cdot)$'s restriction $\hat{\eta}(\cdot)$ to $[u', \bar{u}]$ has a continuous inverse $u = \hat{u}(\eta)$, which is C^2 on $Y \doteq \tilde{\eta}([u', \bar{u}))$; the function $\eta \mapsto z = \hat{z}(\eta) \doteq \tilde{z}[\hat{u}(\eta)] \in C^0(\overline{Y})$ is C^2 on Y; and $\mathcal{F}$ is $\hat{z}(\cdot)$'s epigraph.*

PROOF OF THEOR. 3.1. Assume (*i*) $\tilde{\eta}_u(u) \neq 0 \ \forall u \in U$. Then $\tilde{\eta}(\cdot)$ has a C^2-inverse $u = u(\eta)$ and we can construct the C^2-function $z = z(\eta) \doteq \tilde{z}[u(\eta)]$ on $\tilde{\eta}(U)$; furthermore (*ii*) $\mathcal{F}$ is $z(\cdot)$'s epigraph. Hence (*iii*) $\mathcal{F}$ is convex iff (*iv*) $z_{\eta\eta}(\eta) \geqslant 0 \ \forall \eta \in \tilde{\eta}(U)$.

On the other hand (*i*) implies either of the cases $(3.7)_1^{\pm}$. Furthermore, by $(3.2)_2$ and $(3.7)_1^{\pm}$ condition (*iv*) is equivalent to $(3.7)_2^{\pm}$. We conclude that

$(\mathcal{F})$ *condition* (*i*) *implies that $\mathcal{F}$ is convex iff either of the cases $(3.7)^{\pm}$ holds.*

Now we assume (*i*)'s failure, *i.e.* that $(v)^{\pm}$ $\bar{u}$ defined by $(3.8)^{\pm}$ exists. Then, by assumption (*a*), one of the cases $(3.5)_3^{\pm}$ holds. First, besides $(3.5)_3^{\pm}$ we suppose (*iii*) ($\mathcal{F}$'s convexity and), as an hypothesis for reduction ad absurdum, the failure of (3.9). Then, for some $u^* \in (\bar{u}, u'')$, we have (*vi*) $\eta^* \notin [\eta', \bar{\eta}]$, $\tilde{\eta}(\cdot)$'s $[\tilde{z}(\cdot)$'s$]$ value at u^* being denoted by $\eta^*[z^*]$. Hence, either (*vii*) $\eta' \in (\bar{\eta}, \eta^*)$ and the (oriented) segment $[A, B] \doteq [(\eta', z'), (\eta^*, z^*)]$ has some point P, near A, outside $\mathcal{F}$ – see (3.6) –, or (*viii*) $\bar{\eta} \in (\eta^*, \eta')$ and the segment $[C, D] \doteq [(\bar{\eta}, \bar{z}), (\eta^*, z^*)]$ has some point Q, near C, outside $\mathcal{F}$. Thus both cases (*vii*) and (*viii*) contrast with (*iii*) ($\mathcal{F}$'s convexity). Hence (3.9) must hold.

By (3.9) and (*v*) – see (3.8) – assertion $(\mathcal{E})$ below (3.11) implies that (*ix*) $\mathcal{F}$ is the epigraph of $\hat{z}(\cdot)$ – see (*i*) to (*iii*) in $(\mathcal{A})$. Hence (*x*) $\mathcal{F}$'s convexity is equivalent to $0 \leqslant (\hat{z}_{\eta\eta}(\eta) =)z_{\eta\eta}(\eta) \ \forall \eta \in Y$; and by $(3.2)_2$ and the regularity assumptions this is in turn equivalent to (3.10). Thus $(v)^{\pm}$ and (*iii*) imply $(b)^{\pm}$.

Now we conversely assume $(b)^{\pm}$. Then $(v)^{\pm}$ and (3.9) hold, so that $(\mathcal{E})$ again implies (*ix*) and the equivalence of $\mathcal{F}$'s convexity to (3.10). Since (3.10) is included in both $(b)^+$ and $(b)^-$, $\mathcal{F}$ is convex. We conclude that

$(\mathcal{G})$ *either $(v)^+$ or $(v)^-$,* i.e. *(i)'s failure, implies that $\mathcal{F}$ is convex iff either $(b)^+$ or $(b)^-$ holds.*

Together with $(\mathcal{F})$, $(\mathcal{G})$ implies the thesis of Theor. 3.1. □

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