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Linear approximation in Continuum Mechanics

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Meccanica dei continui. — *Linear approximation in Continuum Mechanics.* Nota (*) del Socio Giuseppe Grioli.

Abstract. — Some critical remarks are made about the theory of Linear Elasticity, questioning on its validity in general. An alternative linear approximation of the exact theory is proposed.

Key words: Continuum mechanics; Mathematical theory of elasticity; Mathematical physics.

Riassunto. — *Approssimazioni lineari in Meccanica dei Continui.* Si fanno talune osservazioni sulla teoria dell'Elasticità lineare, discutendo sulla sua validità in generale. Si formula una teoria lineare alternativa.

Any linear approximation of the general equations of Continuum Mechanics lies in the comparison of linear expressions of the displacement and its derivatives with the sources – loads, initial velocities, etc. – whatever the process may be for deducing the linear theory from the non linear exact one.

Generally, one deduces the linear approximation supposing that the magnitudes of the external sources are proportional to a parameter, h, and estimating the first derivative of certain quantities with respect to h, for $h = 0$. In this way, the linear approximation is often interpreted as the first term of a Taylor series, although that is not necessary and needs some analytical allowances.

Many Authors have studied the problem in the case of tri-dimensional elastic bodies [1-9].

A linear theory may be established directly, without being deduced from the exact one. Nevertheless a comparison with the exact theory seems to be useful for a right evaluation of its limits.

Thus, the customary linear approximation is generally doubt. That happens particularly for the linear theory of Elasticity.

The aim of this paper is to suggest a convenient modification of the customary process such that some inconveniences of linear theory are avoided.

In the following, I will consider tri-dimensional elastic bodies. Nevertheless, many conclusions are valid for more general bodies, like strain rate dependent materials and materials with memory.

In Continuum Mechanics the splitting up of external forces in «dead loads» and «alive loads» is fundamental. The first are dependent only on the material elements, the second depend also on the their place. These have the higher physical meaning. Nevertheless, the difference between dead and alive loads is generally absent in the linear theories because they are evaluated in the reference configuration from which displacements and deformations are measured. Therefore, one has a correct linear approxima-

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tion for the stress but not for the external forces. Consequently some difficulties arise, as I will demonstrate. This statement is obvious in the equilibrium problem, where the external loads must be balanced in the distorted state. Instead, in the customary linear approximation this condition is imposed on the reference configuration and requires a necessary but artificial device which generally cannot be satisfied.

Instead, it would seem preferable that the global balance of external loads be required in the distorted configuration in the linear theory as well. This requires an accurate evaluation of the linear approximation of the external forces.

1. Premises

Let C be a reference configuration for an elastic tri-dimensional elastic body. I maintain that C is a natural stable configuration of equilibrium, that is a stable equilibrium configuration without stress when the external loads are absent.

Let C' be the actual configuration of the body and σ and σ' the boundaries of C and C' respectively.

I denote by P and P' two corresponding points of $C + \sigma$ and $C' + \sigma'$ and by X_i, x_i their coordinates with respect to a rectangular cartesian coordinate system $(0; 1, 2, 3)$. Furthermore, let N_i, n_i be the components of the unit vectors perpendicular to σ and σ' respectively oriented towards the interior of the body and $d\sigma$ and $d\sigma'$ their surface elements.

Denoting by D the jacobian of the transformation from C to C' and by ε_{rs} the strain, one has, as well known,

$$\varepsilon_{rs} = (u_{r,s} + u_{s,r} + u_{i,r}\, u_{i,s})/2\,, \tag{1}$$

$$D = \det |x_{r,s}| > 0\,, \qquad dC' = D\, dC\,, \tag{2}$$

$$n_r = A_{rs}\, N_s / q\,, \qquad q = d\sigma/d\sigma' = \sqrt{A_{rs} A_{rl}\, N_s\, N_l} > 0\,, \tag{3}$$

where $u_r = x_r - X_r$ are the displacement components; the comma indicates differentiation with respect to X_i and A_{rs} is the cofactor of $x_{r,s}$ in the matrix $|x_{r,s}|$.

In the linear theory the preceding quantities are to be evaluated till the first order, neglecting the squares of the displacement and its derivatives. That is, the linear approximations of (1), (2), (3) are

$$\varepsilon_{rs} = (u_{r,s} + u_{s,r})/2\,, \qquad D = 1 + u_{i,i}\,, \qquad dC' = (1 + u_{i,i})\, dC\,, \tag{4}$$

$$q = 1 + (\delta_{im} - N_i\, N_m)\, u_{i,m}\,, \qquad n_r = N_r + (N_r\, N_m - \delta_{rm})\, N_i\, u_{i,m}\,, \tag{5}$$

where δ_{rs} are the Kronecker symbols.

As is well known, the extension of linear elements in the direction of the axis of index i is $\delta_i = \sqrt{1 + 2\varepsilon_{ii}} - 1$ and satisfies the condition $\delta_i > -1$. The condition $\delta_i \geqslant -1$ is always satisfied by any displacement u_r in the exact theory but in general is not in its linear approximation. Therefore it is necessary to verify that the solution of a linear problem satisfies the conditions

$$\delta_i = \varepsilon_{\underline{ii}} > -1\,; \qquad u_{i,i} > -1\,, \tag{6}$$

where the second inequality makes reference to $(4)_2$. The subscript means that there is no sum with respect to the index.

The preceding observation is valid also for more general Continua such as materials with memory for which the existence of a strain from the initial instant to a remote past is admitted; but this circumstance always is forgotten.

In the linear static theory one may give certain necessary conditions to the loads so that (6) may be satisfied, applying some known integral properties of stress [10].

2. Some remarks regarding linear elasticity

Let $t_{rs} = t_{sr}$ and $T_{rs} = T_{sr}$ be the Cauchy and Piola stress components, related by

$$t_{rs} = (1/D) x_{r,l} x_{s,m} T_{lm} . \tag{7}$$

Moreover, let $\gamma(X)$ and $\gamma'(x)$ be the material densities in C and C'.

Let us denote by $\gamma' F(x)\, dC'$ the body force acting on the element P' of mass $\gamma' dC'$ and by $f(X, x, n)\, d\sigma'$ the surface force acting through the surface element $d\sigma'$ of σ'. Vector F may depend on X_i and x_i. Vector f, in general, depends on X_i, x_i and n_i. I note that f is supposed to be a known function of x_i in a whole tri-dimensional region. In fact, the force in general depends on the actual configuration of the body. For example, in the case of a body submersed in a fluid the pressure on the surface elements $(P', d\sigma')$ is equal to the fluid pressure in P' in absence of the body.

In the case of «dead loads» F and f are independent of P'.

For clarification I recall some basic concepts.

Denoting by $W(\varepsilon)$ the density in C of the elastic potential energy, one has

$$T_{rs} = -(\partial W / \partial \varepsilon_{rs}) \tag{8}$$

and because of the hypothesis that C is a natural state of equilibrium

$$(\partial W / \partial \varepsilon_{rs})^0 = 0 \tag{9}$$

follows, where the index (0) denotes that the quantity is evaluated in C (that is, for $u_r = 0$).

Putting

$$a_{rspq} = (\partial^2 W / \partial \varepsilon_{rs} \partial \varepsilon_{pq})^0 \tag{10}$$

the condition of stability in C is expressed by the inequality

$$a_{rspq} z_{rs} z_{pq} > 0 \tag{11}$$

for any $z_{rs} = z_{sr} \neq 0$.

The equilibrium configurations of the body are characterized by the equations

$$\begin{cases} (x_{r,l} T_{ls})_{,s} = \gamma F_r [X, x(X, u)], & (\text{in } C), \\ (x_{r,l} T_{ls}) N_s = q f_r [X, x(X, u), n(u_{i,l})], & (\text{on } \sigma). \end{cases} \tag{12}$$

From (12) the necessary integral conditions of equilibrium

$$(13) \qquad \begin{cases} R_r = \int\limits_C \gamma F_r dC + \int\limits_\sigma q f_r d\sigma = 0\,, \\ M_r = e_{rlm}\left[\int\limits_C \gamma x_l F_m dC + \int\limits_\sigma q x_l\, f_m d\sigma\right] = 0 \end{cases}$$

follow.

The equations corresponding to (12) in the customary linear theory are, as is well known:

$$(14) \qquad \begin{cases} T^0_{rs,s} = -(a_{rspq}\,\varepsilon_{pq})_{,s} = \gamma F^0_r\,, & (\text{in } C)\,, \\ T^0_{rs} N_s = -a_{rspq}\,\varepsilon_{pq}\, N_s = f^0_r(X,\, N_r)\,, & (\text{on } \sigma)\,, \end{cases}$$

where ε_{rs} are given by $(4)_1$.

Equations (14) give place to the necessary conditions

$$(15) \qquad \begin{cases} R^0_r = \int\limits_C \gamma F^0_r dC + \int\limits_\sigma f^0_r\, d\sigma = 0\,, \\ M^0_r = e_{rlm}\left[\int\limits_C \gamma X_l F^0_m\; dC + \int\limits_\sigma X_l\, f^0_m\, d\sigma\right] = 0\,. \end{cases}$$

This is a very important point. While the conditions (13) have a physical meaning, on the contrary the (15) give other mathematical conditions without a physical meaning. In fact these conditions impose a restriction on the reference configuration which is not required by physical reasons and is difficult to satisfy even when f does not depend on P'. For example, let us consider the equilibrium problem of a heavy elastic body inside an incompressible liquid, whose constant density is η. I maintain that the coordinate plane (0; 1, 2) coincides with the horizontal limit plane of the liquid and the axis (0, 3) is parallel to the vertical and down oriented.

Denoting by g the acceleration of gravity, one has

$$(16) \qquad F_r = g\delta_{r3}\,, \qquad f_r = \eta g\, x_3 n_r$$

and equations (14) become:

$$(17) \qquad \begin{cases} (1/2)\,[a_{rspq}(u_{p,q} + u_{q,p})]_{,s} = -\gamma g\,\delta_{r3}, & (\text{in } C)\,, \\ (1/2)\,a_{rspq}(u_{p,q} + u_{q,p})\,N_s = -\eta g X_3\, N_r\,, & (\text{on } \sigma)\,. \end{cases}$$

Let us denote by m the mass of the body, by C its volume and by X^G_i the coordinates of the centre of gravity of the body in C. The equalities (15), which are necessary consequence of (14), (16) [that is of (17)], become

$$(18) \qquad R^0_1 = R^0_2 = 0\,, \qquad M^0_3 = 0\,,$$

$$
(19) \qquad \begin{cases} R_3^0 = g(m - \eta C) = 0\,, \\ M_1^0 = g\left[mX_2^G - \eta \int\limits_C X_2 dC\right] = 0\,, \\ M_2^0 = -g\left[mX_1^G - \eta \int\limits_C X_1 dC\right] = 0\,. \end{cases}
$$

In general it is impossible to satisfy (19) because they impose a restriction on the reference configuration. In particular, $(19)_1$ means that the volume of the body in C is equal to m/η in contrast with the hypothesis that C is an equilibrium configuration without stress. For example, if the body is homogeneous in C, from $(19)_1$ follows $\gamma = \eta$! Further, adding a rigid displacement of any type to a solution one obtains another solution. That is in contrast with the laws of Physics.

3. An alternative formulation of linear approximation

The condition that the external forces are balanced in the reference configuration is the basis of referred to drawback. The condition is artificial and must be replaced by the physically significative that the set of external loads is balanced in the distorted equilibrium configuration, as in the case of finite deformations.

To gain this end a more accurate evaluation of the linear approximation of the external loads is necessary.

I must state beforehand the following:

a) equations (14) are a comparison between certain linear terms in the displacement derivatives and the external loads;

b) in the dynamic case the inertia's forces $-\gamma \ddot{u}_{rs}-$ are added to the previous terms;

c) the first derivatives of displacement are added to finite quantities when one computes the volumes and lengths in the distorted configuration, C' (see (4)).

Therefore, in the linear approximation of external forces one must take into account linear terms in the displacement and its derivatives. Otherwise some quantities which are comparable with others that are present are neglected. Therefore, the linear approximation of external loads is

$$
(20) \qquad \begin{cases} \gamma F_r = \gamma[F_r^0 + (\partial F_r / \partial x_i)^0\, u_i]\,, \\ f_r = f_r^0 + (\partial f_r / \partial x_i)^0\, u_i + [(\partial f_r / \partial n_i)(\partial n_i / \partial u_{l,m})]^0\, u_{l,m}\,, \end{cases}
$$

while the quantity q must be approximated according to $(5)_1$.

Keeping in mind (4), (5), linear equations of elasticity are no longer (14), but the following:

$$(21)\qquad \begin{cases} -(1/2)[a_{rspq}(u_{p,q}+u_{q,p})]_{,s} = \gamma[F_r^0 + (\partial F_r/\partial x_i)^0\, u_i]\,, & (\text{in } C)\,, \\ -(1/2)[a_{rspq}(u_{p,q}+u_{q,p})]\,N_s = f_r^0 + (\partial f_r/\partial x_i)^0\, u_i + \\ \qquad + [\delta_{li} f_r^0 - (\partial f_r/\partial n_l)^0\, N_i](\delta_{lm} - N_l N_m)\, u_{i,m}\,, & (\text{on } \sigma)\,. \end{cases}$$

In particular when

$$(22)\qquad f_r = \psi_{rs}(x)\, n_s$$

is, as happens in the case of a pressure $p\,(\psi_{rs} = p\,\delta_{rs})$ and in that of a heavy body inside a liquid, the right members of $(12)_2$ has the simpler expression

$$(23)\qquad q f_r = [\psi_{rs}^0 + (\partial\psi_{rs}/\partial x_i)^0 u_i]\, N_s + \psi_{rs}^0\,[N_s u_{i,i} - N_i u_{i,s}]\,.$$

From (21) follow the necessary integral equations

$$(24)\qquad \begin{cases} R_r' = \displaystyle\int_C \gamma[F_r^0 + (\partial F_r/\partial x_i)^0 u_i]\, dC + \int_\sigma \{f_r^0 + (\partial f_r/\partial x_i)^0) u_i + \\ \qquad + [\delta_{li} f_r^0 - (\partial f_r/\partial n_l)^0\, N_i](\delta_{lm} - N_l N_m)\, u_{i,m}\}\, d\sigma = 0\,, \\ M_r' = e_{rpt}\left\{ \displaystyle\int_C \gamma X_p\,[F_t^0 + (\partial F_t/\partial x_i)^0\, u_i]\, dC + \int_\sigma X_p\,[f_t^0 + (\partial f_t/\partial x_i)^0\, u_i + \right. \\ \qquad \left. + [\delta_{li} f_t^0 - (\partial f_t/\partial n_l)^0\, N_i](\delta_{lm} - N_l N_m)\, u_{i,m}]\, d\sigma \right\} = 0\,. \end{cases}$$

The equations (24) are not conditions on the external loads or on the reference configuration, as happens in the case of (15), because of the dependence of R_r', M_r' on u_i, $u_{i,l}$. They are integral consequences of eqs. (21) and are satisfied by all their solutions and coincide with the linear approximation of (13), according to (20). Therefore, the eqs. (24) are the global equilibrium conditions in the distorted configuration in the linear approximation, necessary for the existence of a solution. This happens also in the case of dead loads, owing to the presence of the factor q. In fact, in this case F_r and f_r depend only on X_i, N_i and one has

$$(25)\qquad \begin{cases} R_r' = \bar{R}_r' = \displaystyle\int_C \gamma F_r^0\, dC + \int_\sigma [f_r^0 \delta_{li} - (\partial f_r/\partial n_l)\, N_i](\delta_{lm} - N_l N_m)\, u_{i,m}\, d\sigma\,, \\ M_r' = \bar{M}_r' = \\ = e_{rpt}\left\{ \displaystyle\int_C \gamma X_p F_t^0\, dC + \int_\sigma X_p\,[f_t^0 \delta_{li} - (\partial f_t/\partial n_l)^0\, N_i](\delta_{lm} - N_l N_m)\, u_{i,m}\, d\sigma \right\} = 0\,. \end{cases}$$

I observe that the solution of the eqs. (21) in general are not undeterminate for an arbitrary infinitesimal rigid displacement, as happens for the solutions of the eqs. (14). In fact, a rigid displacement in general influences the right members of (21). Therefore, the linear elasticity in general does not contain this indetermination, although it may be present in some particular cases.

Let be $C'(\mathcal{R})$ the configuration obtained from C' by means of the rigid displacement characterized by a rotation $\mathcal{R}$ and $u_i(\mathcal{R})$, $u_{i,l}(\mathcal{R})$, $n_i(\mathcal{R})$ the values of u_i, $u_{i,l}\, n_i$ corresponding to the configuration $C'(\mathcal{R})$. Further, let us denote by $Z(\ldots)$ a function of $u_i, \ldots,$ and $Z(\mathcal{R})$ the function obtained from $Z(\ldots)$ by the substitution of $u_i(\mathcal{R}), \ldots,$ to $u_i, \ldots$.

According to (12), necessary and sufficient conditions for $C'(\mathcal{R})$ to be a new equilibrium position are the equalities

$$(26)\qquad \begin{cases} \mathcal{R}_{rp}(x_{p,l}T_{ls})_{,s} = \gamma F_r[X, x(\mathcal{R})], & (\text{in } C), \\ \mathcal{R}_{rp}\, x_{p,l}\, T_{ls}\, N_s = q(\mathcal{R}) f_r[X, x(\mathcal{R}), n(\mathcal{R})], & (\text{on } \sigma). \end{cases}$$

Given $q = q(\mathcal{R})$, from (26) one deduces the necessary and sufficient conditions for equilibrium of $C'(\mathcal{R})$:

$$(27)\qquad F_r(\mathcal{R}) = \mathcal{R}_{rs} F_s(\ldots), \qquad f_r(\mathcal{R}) = \mathcal{R}_{rs}\, f_s(\ldots).$$

In the linear case one has

$$(28)\qquad \mathcal{R}_{rs} = \delta_{rs} + e_{rps}\,\omega_p\,,$$

where the constants ω_p are of the same order of size of $u_{r,s}$. Because of the equalities

$$(29)\qquad \begin{cases} u_i(\mathcal{R}) = u_i(\omega) = u_i + e_{ilm}\,\omega_l X_m\,, \\ u_{i,m}(\mathcal{R}) = u_{i,m}(\omega) = u_{i,m} + e_{ilm}\,\omega_l\,, \end{cases}$$

(27) become

$$(30)\qquad \begin{cases} F_r(\omega) = F_r(\ldots) + e_{rls} F_s(\ldots)\,\omega_l\,, \\ f_r(\omega) = f_r(\ldots) + e_{rls}\, f_s(\ldots)\,\omega_l\,. \end{cases}$$

After some calculations, taking into account (20), (21), one deduces

$$(31)\qquad \begin{cases} [(\partial F_r/\partial x_i)^0\, e_{ilm} X_m - e_{rlm} F_m^0]\,\omega_l = 0\,, \\ [(\partial f_r/\partial x_i)^0\, e_{ilm} X_m - (\partial f_r/\partial n_m)^0\, N_i e_{ilm} - e_{rlm} f_m^0]\,\omega_l = 0\,. \end{cases}$$

The equalities (31) are necessary and sufficient conditions such that the linearized equilibrium problem admits the indetermination of a linear rigid rotation, ω_p.

Of course, an uniqueness question occurs for equations (21) but the absence of uniqueness is not in contrast with physical reality. This is present in the exact theory of finite deformations but it may be present also in the linear approximation if that is obtained according to an accurate approximation of external loads.

I will consider again the problem of a heavy body inside an incompressible liquid. According to (16), one has

$$(32)\qquad \begin{cases} F_r^0 = g\delta_{r3}\,, \qquad (\partial F_r/\partial x_i)^0 = 0\,, \\ q f_r^0 = \eta g[(X_3 + u_3) N_r + X_3(\delta_{im} N_r - \delta_{rm} N_i) u_{i,m}]\,. \end{cases}$$

From (32) follows

$$
(33)\qquad \begin{cases} R_r' = mg\,\delta_{r3} - \eta g \int\limits_C (1 + u_{i,i})\,\delta_{r3}\,dC\,, \\ M_r' = e_{rpt}\left\{ m x_p^G\,\delta_{3t} - \eta g \int\limits_C [X_p(1+u_{i,i})\,\delta_{3t} - X_3 u_{p,t}]\right\} dC. \end{cases}
$$

R_r', M_r' depend on $u_{r,s}$. Therefore, their vanishing is a consequence of eqs. (21), satisfied by every solution.

I observe that:

a) keeping in mind the equality $dC' = (1 + u_{i,i})\,dC$, the vanishing of R_3' means that the weight of the body is equal to that of the liquid with the same volume in the distorted configuration but not in that of reference one as follows by $(19)_1$;

b) the equality $R_3' = 0$ is equivalent to

$$
(34)\qquad \eta \int\limits_C u_{i,i}\,dC = m - \eta C\,.
$$

Therefore, the inequality $(6)_2$ is satisfied on average. In fact, from (34) follows

$$
(35)\qquad \eta \int\limits_C u_{i,i}\,dC = m - \eta C > -\eta C\,.
$$

Finally, I observe that from $(5)_2$, (16) follows:

$$
(36)\qquad \begin{cases} F_r^0 = g\delta_{r3}\,, & (\partial F_r/\partial x_i)^0 = 0\,, \\ f_r^0 = \eta g X_3 N_r\,, & (\partial f_r/\partial x_i)^0 = \eta g\,\delta_{3i} N_r\,, (\partial f_r/\partial n_i)^0 = \eta g X_3\,\delta_{ri}\,. \end{cases}
$$

Therefore, from (31) one deduces

$$
(37)\qquad e_{rl3}\,\omega_l = 0\,, \qquad e_{3li}\,\omega_l X_i N_r = 0\,.
$$

From (37) follows the necessary and sufficient conditions $\omega_1 = \omega_2 = 0$.

That means that the solution contains an indeterminate rigid rotation around a vertical axis (and an arbitrary horizontal translation), as it was to be expected.

Conclusions

In linear Elasticity the evaluation of lengths and volumes in the distorted state, the adding of inertia's forces in the dynamic case and the same analytic structure of the left sides of the general equations means the comparison of linear terms in the displacement and its derivatives with external forces. Therefore, it seems necessary to evaluate the influence of external loads with the same degree of approximation. In this manner one avoids having an-analytical problem which in general has no solution because of the difficulty in satisfying the known global conditions for load in a natural reference configuration.

If the displacement is prefixed on a part of the boundary, as well in dynamical case many difficulties of the static one are absent because of the presence of reactions due to the constraints and forces of inertia. Nevertheless, the reflections made in the static case when loads are known on the entire boundary impose a more accurate evaluation of the linear approximation of external forces. Otherwise one neglects terms the greatness of which is comparable with that of terms which are present.

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