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Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces

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Analisi matematica. — *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces.* Nota di MARCO BIROLI e UMBERTO MOSCO, presentata (*) dal Socio L. Amerio.

ABSTRACT. — We prove local embeddings of Sobolev and Morrey type for Dirichlet forms on spaces of homogeneous type. Our results apply to some general classes of selfadjoint subelliptic operators as well as to Dirichlet operators on certain self-similar fractals, like the Sierpinski gasket. We also define intrinsic BV spaces and perimeters and prove related isoperimetric inequalities.

KEY WORDS: Sobolev spaces; Dirichlet forms; Degenerate elliptic operators; BV spaces.

RIASSUNTO. — *Diseguaglianze isoperimetriche e di Sobolev per forme di Dirichlet su spazi di tipo omogeneo.* Si provano risultati di immersione locale del tipo Sobolev e Morrey per forme di Dirichlet su spazi di tipo omogeneo. I risultati si applicano a certe classi generali di operatori subellitici e a operatori di Dirichlet su certi frattali come il «Sierpinski gasket». Si definiscono inoltre spazi BV e perimetri intrinseci e si ottengono per essi diseguaglianze isoperimetriche.

1. SETTING AND NOTATIONS

We assume that X, d, m is a (space of homogeneous type, or simply a) *homogeneous space* with constants $K > 0$, $0 < R_0 \leqslant +\infty$, $c_0 \geqslant 2$, in the sense of R. R. Coifman and G. Weiss [5]. More precisely, X is a *topological space*; d is a *pseudo-distance* in X (with $d(x,z) \leqslant K[d(x,y) + d(y,z)]$ for every $x, y, z \in X$) whose *pseudo-balls* $B_r(x) = \{y \in X: d(x,y) < r\}$, $x \in X$, $r > 0$, form a basis of open neighbourhoods in X; m is a positive *Borel measure* satisfying the doubling condition $0 < m(B_{2r}(x)) \leqslant c_0 m(B_r(x))$ for every $x \in X$ and every $0 < r < R_0$. We then have the one-sided volume bound: $m(B_r(x)) \geqslant (1/2) m(B_R(x))(r/R)^\nu$, for every $x \in X$ and $0 < r \leqslant R < R_0$, where $\nu := \log_2 c_0$. Such a constant $\nu \geqslant 1$, will be referred to in the following as (a local upper bound of) the *homogeneous dimension* of X.

In addition X will be supposed to be *connected* and *locally compact.* Furthermore, we will assume that we are given a *strongly local, regular Dirichlet form* in the Hilbert space $L^2(X, m)$, in the sense of M. Fukushima [12], whose domain will be denoted by D[a]. Such a form a admits the following integral representation $a(u,v) = \int_X \alpha(u,v)$ for every $u, v \in D[a]$ where $\alpha(u,v)$ is a signed Radon measure on X, uniquely associated with the functions u,v. Moreover for any open subset Ω of X the restriction of $\alpha(u,v)$ to Ω depends only on the restrictions of u and v to Ω. This allows to extend unambiguously the definition of the measure $\alpha(u,v)$ in X to all m-measurable functions u, v in X, that coincide m-a.e. on every compact subset of Ω with some function of $D[a]$. The space of

(*) Nella seduta del 16 giugno 1994.

these functions will be denoted by $D_{\rm loc}[a, \Omega]$. For every real $1 \leqslant p < +\infty$, we then define the *Dirichlet-Sobolev spaces* $D_p[a, \Omega]$ as follows:

$$D_p[a, \Omega] := \left\{ u \in D_{\rm loc}[a, \Omega] : \int_\Omega \alpha(u, u) + \int_\Omega |u|^2 \, dm < +\infty \right\}, \quad \text{if } p = 2;$$

$$D_p[a, \Omega] := \left\{ u \in D_{\rm loc}[a, \Omega] : \int_\Omega \alpha(u, u)(x)^{p/2} \, dm + \int_\Omega |u|^p \, dm < +\infty \right\}, \quad \text{if } 1 \leqslant p \neq 2;$$

where we implicitely assume, if $p \neq 2$, that the Radon-Nikodym derivative $\alpha(u, u)(\cdot) := := d\alpha(u, u)/dm \in L^1_{\rm loc}(\Omega, m)$ exists. In the following we shall use the simplified notation $\alpha(u, u)^{p/2} = \alpha(u, u)$ if $p = 2$; $\alpha(u, u)^{p/2} = \alpha(u, u)(x)^{p/2} m$ if $1 \leqslant p \neq 2$.

The Dirichlet-Sobolev spaces just defined reduce to the classical Sobolev spaces $W^{1,p}(\Omega)$, $1 \leqslant p < +\infty$, when Ω is an open subset of R^n, m the n-dimensional Lebesgue measure and a is the form $a(u, u) = \int_\Omega |\nabla u|^2 \, dx$, with domain $D[a] = = W^{1,2}(\Omega)$ in $L^2(\Omega)$. For these classic spaces, Sobolev-, John-Nirenberg-, Morrey-imbeddings hold, according to the relation $p < n$, $p = n$, $p > n$ of the summability exponent p with the euclidean dimension n. The main purpose of this *Note* is to describe similar (local) imbeddings properties of the spaces $D_p[a, \Omega]$, Ω being a ball of X, according to the relation $p < \nu$, $p = \nu$, $p > \nu$ of p with the homogeneous dimension ν of X. To this end, we shall further assume that the space X and the form a are related by a Poincaré condition of some order p, $1 \leqslant p < +\infty$, and we shall derive from this the expected imbeddings of the spaces $D_q[a, \Omega]$ for every q with $p \leqslant q < +\infty$. Such a *Poincaré condition of order p* is formulated as follows: There exists some $1 \leqslant p < +\infty$ and two constants $c_1 > 0$, $k \geqslant 1$ such that for every $x \in X$ and $0 < r < R_0$, we have

$$\int_{B_r(x)} |u - u_{x;r}|^p \, dm \leqslant c_1 r^p \int_{B_{kr}(x)} \alpha(u, u)^{p/2}$$

for all $u \in D_p[a, B_{kr}(x)]$, where $u_{x;r} := m(B_r(x))^{-1} \int_{B_r(x)} u \, dm$.

As in the classic theory, we shall distinguish the three cases: (*i*) $1 \leqslant p \leqslant q < \nu$, leading to Sobolev type imbeddings of $D_q[a, B_{kr}(x)]$ into the space $L^{q^*}(B_r(x), m)$, with $q^* := q\nu/(q - \nu)$, Theorem 1; (*ii*) $1 \leqslant p \leqslant q = \nu$, leading to the embedding of $D_\nu[a, B_{kr}(x)]$ into the intrinsic BMO space on $B_r(x)$ and consequent exponential integrability of the functions $u \in D_\nu[a, B_{kr}(x)]$, Theorem 2; (*iii*) $1 \leqslant p \leqslant q$, $1 \leqslant \nu < q$ leading to Campanato-Morrey type imbeddings of $D_q[a, B_{kr}(x)]$ into the spaces of bounded, Hölder continuous functions on $B_r(x)$, Theorem 3. While the results of Theorem 2 and Theorem 3 are essentially a reformulation in our present local setting of analogue results concerning BMO and Campanato functions on homogeneous spaces, due to N. Burger [4], G. B. Folland - E. M. Stein [9], and to R. M. Macias - C. Segovia [20], respectively, the general Sobolev imbeddings of Theorem 1 and related intrinsic BV spaces and isoperimetric inequalities of section 3 are new, in the contest of homogeneous spaces and Dirichlet forms. The case $2 = p = q < \nu$, however, was previously considered by the authors in [3], in order to obtain a priori estimates of local weak solutions [1, 2]. The proof of Theorem 1 is carried on by iteration on the exponent p of the Poincaré condition through intermediate steps that involve, successively, Nash type inequalities (in a suitable weak form if $p = 1$), weak Sobolev inequalities and, finally, higher order Poincaré inequalities as in Theorem 4. We also point out the special case $1 < \nu < 2 = p = q$ of Theorem 3, which is related to a new interpretation of certain fractals, like the so-called Sierpinski gasket, as homogeneous spaces, and of their spectral dimension as the homogeneous dimension (see also [21]). Further applications and references will be mentioned in section 4.

The proof of the results will be given in the paper of the authors that will appear in the proceedings of the Conference on *Potential Theory and Partial Differential Operators with Nonegative Characteristic Form* (Parma, February 21-24, 1994), Kluwer, Amsterdam.

2. Imbeddings of $D_q[a, B_{kr}(x)]$

We assume throughout this section, that we are given a connected, locally compact, homogeneous space X, with constants K, R_0, c_0 and dimension ν, and a strongly local, regular Dirichlet form a in $L^2(X, m)$, which are related by a Poincaré condition of a given order p, $1 \leqslant p < +\infty$, with constants c_1 and k. By c we shall denote below suitable constants, that depend only on K, c_0, c_1, k, and possibly on q. We first consider the Sobolev case (i) of the previous section:

Theorem 1. *If $1 \leqslant p \leqslant q < \nu$, we have for every $x \in X$ and every $0 < r < R_0$,*

$$\left(\frac{1}{m(B_r(x))}\int_{B_r(x)}|u|^{q^*}\,dm\right)^{1/q^*} \leqslant c\left[\frac{r^q}{m(B_r(x))}\int_{B_{kr}(x)}\alpha(u,u)^{q/2}+\frac{1}{m(B_r(x))}\int_{B_r(x)}|u|^q\,dm\right]^{1/q}$$

for all $u \in D_q[a, B_{kr}(x)]$, where $q^* = q\nu/(\nu - q)$.

We now consider the John-Nirenberg case (ii):

Theorem 2. *If $1 \leqslant p \leqslant q = \nu$, we have for every $x \in X$ and every $0 < r < R_0$,*

$$\frac{1}{m(B_r(x))}\int_{B_r(x)}\left[\exp\left(\theta\frac{\tau}{\|u\|_*}|u|\right)-1\right]dm \leqslant c\theta/(1-\theta)$$

for all $u \in D_q[a, B_{kr}(x)] = D_\nu[a, B_{kr}(x)]$ and all constants θ, $0 < \theta < 1$, where

$$\|u\|_* = \left[\frac{r^\nu}{m(B_r(x))}\int_{B_{kr}(x)}\alpha(u,u)^{\nu/2}+\frac{1}{m(B_r(x))}\int_{B_r(x)}|u|^\nu\,dm\right]^{1/\nu}$$

and the constant τ depends only on K, c_0, c_1, k.

Finally, we consider the Campanato-Morrey case (iii):

THEOREM 3. *If* $1 \leqslant p \leqslant q$, $1 \leqslant \nu < q$, *we have for every* $x \in X$ *and every* $0 < r < R_0$,

$$|u(y) - u(z)| \leqslant c \left[\frac{r^\nu}{m(B_r(x))} \int_{B_{kr}(x)} \alpha(u, u)^{q/2} \right]^{1/q} d(y, z)^{1 - \nu/q}, \quad \forall y,\ z \in B_r(x),$$

$$|u(y)| \leqslant c \left[\frac{r^q}{m(B_r(x))} \int_{B_{kr}(x)} \alpha(u, u)^{q/2} + \frac{1}{m(B_r(x))} \int_{B_r(x)} |u|^q \, dm \right]^{1/q}, \quad \forall y \in B_r(x),$$

for all $u \in D_q[a, B_{kr}(x)]$.

Always under the assumption, common to all previous theorems, that a Poincaré condition of a given order $1 \leqslant p < +\infty$ is satisfied, from Theorems 1, 2 and 3 we get the following Poincaré inequalities of higher order:

THEOREM 4. *If* $1 \leqslant p \leqslant q < +\infty$, *we have for every* $x \in X$ *and every* $0 < r < R_0$,

$$\int_{B_r(x)} |u - u_{x;r}|^q \, dm \leqslant c r^q \int_{B_{kr}(x)} \alpha(u, u)^{q/2},$$

for all $u \in D_q[a, B_{kr}(x)]$.

REMARK 1. In the special case that the pseudo-metric d of the space X is given by the *intrinsic distance* d_a of the form a, as introduced in [1], *i.e.* $d_a(x, y) := \sup\{\phi(x) - \phi(y) \colon \phi \in C_0(X) \cap D[a], \alpha(\phi, \phi) \leqslant m \text{ in } X\}$, then suitable intrinsic cut-off functions on balls are available and the following Poincaré inequalities

$$\int_{B_r(x)} |u|^q \, dm \leqslant c r^q \int_{B_{kr}(x)} \alpha(u, u)^{q/2},$$

hold for all $u \in D_q[a, B_{kr}(x)]$ with $\operatorname{supp}(u) \subseteq B_r(x)$, $B_{2r}(x) \neq X$ and for every $1 \leqslant q < +\infty$, whenever a Poincaré condition of order $1 \leqslant p \leqslant q$ is satisfied, see [1, 2]. As a consequence, if $d = d_a$, the estimates of Theorems 1, 2 and 3, and of Corollary of Theorem 1 below, hold for every $u \in D_q[a, B_r(x)]$ with $\operatorname{supp}(u) \subseteq B_r(x)$, $B_{2r}(x) \neq X$, without the presence of the term

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |u|^q \, dm$$

at their right hand side.

3. Intrinsic BV spaces, perimeters and isoperimetric inequalities

From Theorem 1 we get in particular, when $1 = p = q < \nu$, the

COROLLARY OF THEOREM 1. *If a Poincaré condition of order* 1 *is satisfied and* $\nu > 1$, *then for every* $x \in X$ *and every* $0 < r < R_0$,

$$\left(\frac{1}{m(B_r(x))}\int_{B_r(x)} |u|^{\nu/(\nu-1)}\,dm\right)^{(\nu-1)/\nu} \leqslant$$

$$\leqslant c\left[\frac{r}{m(B_r(x))}\int_{B_{kr}(x)} \alpha(u,u)^{1/2} + \frac{1}{m(B_r(x))}\int_{B_r(x)} |u|\,dm\right]$$

for all $u \in D_1[a, B_{kr}(x)]$.

For every $0 < R < R_0$ we define the functional

$$\int_{B_R(x)} |\nabla_a u| : L^1(B_R(x), m) \to [0, +\infty],$$

by setting for every $u \in L^1(B_R(x), m)$

$$\int_{B_R(x)} |\nabla_a u| = \inf\left\{\liminf_{h\to+\infty}\int_{B_R(x)} \alpha(u_h, u_h)^{1/2} : \lim_{h\to+\infty} u_h = u \text{ in } L^1(B_R(x), m)\right\}$$

and $\int_{B_R(x)} \alpha(v,v)^{1/2}$ is extended to take the value $+\infty$ if $u \in L^1(B_R(x), m) - D_1[a, B_R(x)]$.

We call $\int_{B_R(x)} |\nabla_a u|$ the *intrinsic variation* of u in $B_R(x)$ with respect to the form a, and we then define the space $BV_a[B_R(x)] = BV_a[B_R(x), m]$ to be the space of all functions $u \in L^1(B_R(x), m)$ with finite total variation in $B_R(x)$. From the Corollary of Theorem 1, by taking also Remark 1 into account, we then get easily the following imbedding properties for the space $BV_a[B_{kr}(x)]$:

THEOREM 5. *If a Poincaré condition of order 1 is satisfied and* $1 < \nu$, *then for every* $x \in X$ *and every* $0 < r < R_0$, *we have*

$$\left(\frac{1}{m(B_r(x))}\int_{B_r(x)} |u|^{\nu/(\nu-1)}\,dm\right)^{(\nu-1)/\nu} \leqslant$$

$$\leqslant c\left[\frac{r}{m(B_r(x))}\int_{B_{kr}(x)} |\nabla_a u| + \frac{1}{m(B_r(x))}\int_{B_r(x)} |u|\,dm\right]$$

for all $u \in BV_a(B_{kr}(x))$; *moreover, if* $d = d_a$,

$$\left(\frac{1}{m(B_r(x))}\int_{B_r(x)} |u|^{\nu/(\nu-1)}\,dm\right)^{(\nu-1)/\nu} \leqslant c\left[\frac{r}{m(B_r(x))}\int_{B_{kr}(x)} |\nabla_a u|\right]$$

for all $u \in BV_a(B_{kr}(x))$, *with* $\operatorname{supp}(u) \subseteq B_r(x)$ *and* $B_{2r}(x) \neq X$.

If E is a subset of X such that $\mathbf{1}_E \in BV_a(B_R(x))$, where $\mathbf{1}_E$ denotes the characteristic function of E in X, we then define the (finite) *intrinsic perimeter* $P_a(E; B_R(x)) = P_a(E; B_R(x), m)$ of E in $B_R(x)$ with respect to a, by setting $P_a(E; B_R(x)) = \int_{B_R(x)} |\nabla_a \mathbf{1}_E|$.

COROLLARY OF THEOREM 5. *If a Poincaré condition of order* 1 *is satisfied and* $1 < \nu$, *then for every* $x \in X$ *and every* $0 < r < R_0$, *we have*

$$(m(B_r(x)))^{-1}(m(E \cap B_r(x)))^{(\nu-1)/\nu} \leq c[r(m(B_r(x)))^{-1} P_a(E; B_{kr}(x)) + (m(B_r(x)))^{-1}(m(E \cap B_r(x)))],$$

for every subset E *of* X; *moreover, if* $d = d_a$,

$$[m(E)(m(B_r(x)))^{-1}]^{(\nu-1)/\nu} \leq c\, r(m(B_r(x)))^{-1} P_a(E; B_{kr}(x)),$$

for every $E \subseteq \overline{E} \subseteq B_r(x)$.

4. APPLICATIONS

1. All results of section 3 apply to connected complete Riemannian manifolds X, with non-negative Ricci curvature (then $R_0 = +\infty$), or just with bounded Ricci curvature (then $R_0 < +\infty$). In this case $d = d_a$ is the geodesic distance, m is the Riemannian volume, a the form associated with the Laplace-Beltrami operator on X. More generally, they apply (with $p = 1$) to general selfadjoint second order Hörmander operators and (with $p = 2$) to more general (selfadjoint) subelliptic operators, in the sense of C. L. Fefferman, D. H. Phong [8]. Now the distance d is the so-called subelliptic distance associated with the operator as for instance [22, 8]. For these operators, the doubling condition and intrinsic Poincaré inequalities of order $p = 1$ or $p = 2$ are known to hold, see [14], the above mentioned references and, moreover [11]. We recall that there exists a wide recent literature concerning Sobolev and isoperimetric inequalities, for which we only refer here to the recent work [6, 23, 24, 10, 18, 11] as well as the many references therein.

2. The results apply also to weighted uniformly elliptic operators as in [7], and to weighted Hörmander operators, [9].

3. Let $\boldsymbol{K}$ be the unit Sierpinski gasket in R^{N-1}, $N \geq 3$. The Hausdorff dimension of $\boldsymbol{K}$ is $\log_2 N$ and there exists a unique invariant Hausdorff measure μ on $\boldsymbol{K}$. A local, regular Dirichlet form a in $L^2(\boldsymbol{K}, \mu)$ has been defined by M. Fukushima - T. Shima [13], whose self-adjoint generator Δ_K coincides with the so-called Kusuoka-Kigami's Laplacian in $\boldsymbol{K}$ (see [15, 17]). We define a *pseudo-distance* d in $\boldsymbol{K}$ by setting $d(x, y) := |x - y|^{\log_4 (N+2)}$, for $x, y \in \boldsymbol{K}$, where $|x - y|$ denotes the usual euclidean distance in R^{N-1}. According to [21], $\boldsymbol{K}$, d, μ is then a homogeneous space of dimension $\nu = 2 \log N / \log(N+2)$. Moreover, the form a verifies a Poincaré condition of order $p = 2$ on the intrinsic pseudoballs of $\boldsymbol{K}$, in the sense of section 1 (see [21]). Since $1 < \nu < 2$, Theorem 3 applies and gives in particular that the domain $D[a]$ of the form

a is imbedded in the space of bounded Hölder continuous functions on $\boldsymbol{K}$. Moreover, always according to Theorem 3, the Hölder exponent β with respect to the intrinsic distance d is $\beta = 1 - \nu/2 = (\log(N+2) - \log N)/\log(N+2)$ while the corresponding Hölder exponent with respect to the euclidean distance $|x - y|$ or R^{N-1} is $\beta_{\text{eucl}} = = \log_4 [(N+2)/N]$ in agreement for example, if $N = 3$, with [16]. Finally, let us remark that for the operator Δ_K an analogue of the Weyl's formula for the asymptotic distribution of the eigenvalues on the real line holds, which involves the so-called *spectral dimension* $d_s = 2\log N/\log(N+2)$ of $\boldsymbol{K}$ (see for example [13]). We then see that the spectral dimension coincides with the *homogeneous dimension* of $\boldsymbol{K}$. Such a «geometric» characterization of d_s may be of some interest in itself.

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