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Meccanica dei solidi. — *Kinematic criteria of dynamic shakedown extended to nonassociative constitutive laws with saturation nonlinear hardening.* Nota di Alberto Corigliano, Giulio Maier e Slawomir Pycko, presentata (*) dal Corrisp. G. Maier.

Abstract. — The class of elastic-plastic material models considered allows for nonassociativity, nonlinear hardening and saturation in the sense that the static internal variables are constrained by a bounding surface described through convex bounding functions. With reference to finite element, generalized variables discretization in space, two dynamic shakedown criteria are established by a kinematic approach in Koiter's sense, based on weak constitutive restrictions and centered on two suitable definitions of admissible yield cycles.

Key words: Plasticity; Structural dynamics; Shakedown; Inadaptation.

Riassunto. — *Teoremi cinematici di adattamento dinamico estesi a leggi costitutive nonassociate con incrudimento nonlineare a saturazione.* La classe di materiali elastoplastici considerata tiene conto di nonassociatività, incrudimento nonlineare e saturazione (nel senso che le variabili interne statiche sono soggette a vincoli convessi che definiscono una superficie di delimitazione). Con riferimento ad una discretizzazione spaziale per elementi finiti in variabili generalizzate, si dimostrano due teoremi di adattamento («shakedown») dinamico in base ad un approccio cinematico nel senso di Koiter, fondato su ipotesi costitutive alquanto generali e su opportune definizioni di cicli plastici ammissibili.

1. Introduction

Considerable attention has been given in the structural mechanics literature to the extension of *shakedown* or *adaptation* theory *e.g.* (König [1]), in the direction of more and more realistic constitutive elastoplastic models. Such an extension is fostered by some important engineering situations.

For instance, gravity offshore platforms under severe waves transfer variable repeated loads to the soil beneath their foundations: shakedown theory of traditional associative plasticity has been employed in order to assess safety against inadaptation, see *e.g.* [2], but the geomaterials concerned are known to exhibit more or less significant internal friction and, hence, nonassociativity, which invalidates the classical shakedown theory. As a second example from engineering practice, thin cylindrical shells in the core of nuclear reactors are prone to incremental collapse (*ratchetting*) due to fluctuating thermal loading; the inelastic behaviour of their material (stainless steel) is fairly accurately described by constitutive laws such as Chaboche models (see *e.g.* [3]) endowed with nonlinear kinematic and isotropic hardening with saturation bounds, again outside the reach of the classical theory.

Representative recent contributions to the shakedown theory in the above mentioned direction are quoted herein [4-10]. Numerous earlier references to other

(*) Nella seduta del 16 giugno 1994.

key works to the same or related purposes are given in these citations, of which only the last two explicitly allowed for nonassociative flow rules.

The systematic treatment [9] of criteria for dynamic shakedown and of bounds on post-adaptation quantities, covered a broad class of nonassociative nonlinear-hardening material models. However no allowance was made there for the important constitutive feature of hardening *saturation*, namely for the possible presence of a bounding surface in the stress space such that the current yield surface which changes at hardening cannot pierce it, but can approach it from inside, possibly as some kinematic internal variables and plastic strains grow without limit (asymptotic saturation).

In the parallel paper [10] saturation was envisaged and covered by both theoretical developments and numerical examples, but concerning only the further generalization of *static* or *safe* criteria for shakedown (in the sense of Bleich-Melan, see *e.g.* [1]), and in the quasi-static regime alone (rather than as an extension of their dynamic counterparts established first by Ceradini (cp. [11, 12]).

The present paper is intended to provide the extension to saturation hardening in the dynamic regime of the *kinematic* or *unsafe* inadaptation criteria in the sense of Neal-Symonds-Koiter (see *e.g.* [1], or [13]) on the basis of their earlier generalization to dynamics developed by Corradi and Maier [14-15] and, more recently, by Comi and Corigliano [7].

2. Problem formulation

Reference is made in what follows to a structural model discretized in space by finite elements, so that its small-deformation dynamic evolution in time t under given external actions is governed by the following relation set:

(1) $$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{V}\dot{\boldsymbol{u}}(t) + \boldsymbol{C}^T\boldsymbol{\sigma}(t) = \boldsymbol{P}(t);$$

(2) $$\boldsymbol{C}\boldsymbol{u}(t) = \boldsymbol{\varepsilon}(t) = \boldsymbol{e}(t) + \boldsymbol{p}(t);$$

(3) $$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}_0;$$

(4) $$\boldsymbol{\sigma} = \frac{d\Psi_e}{d\boldsymbol{e}}(\boldsymbol{e}) = \boldsymbol{E}\boldsymbol{e}, \qquad \boldsymbol{\chi} = \frac{\partial\Psi}{\partial\boldsymbol{\eta}}(\boldsymbol{e}, \boldsymbol{\eta}) = \frac{d\Psi_s}{d\boldsymbol{\eta}}(\boldsymbol{\eta});$$

(5) $$\dot{\boldsymbol{p}} = \frac{\partial\boldsymbol{\Phi}^T}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\,\dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\eta}} = -\frac{\partial\boldsymbol{\Phi}^T}{\partial\boldsymbol{\chi}}(\boldsymbol{\sigma}, \boldsymbol{\chi})\,\dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} \geqslant \boldsymbol{0};$$

(6) $$\boldsymbol{\varphi}(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant \boldsymbol{0}, \qquad \boldsymbol{\varphi}^T\dot{\boldsymbol{\lambda}} = 0;$$

(7) $$\Psi(\boldsymbol{e}, \boldsymbol{\eta}) = \Psi_e(\boldsymbol{e}) + \Psi_s(\boldsymbol{\eta}), \qquad \dot{D} \equiv \boldsymbol{\sigma}^T\dot{\boldsymbol{\varepsilon}} - \dot{\Psi} = \boldsymbol{\sigma}^T\dot{\boldsymbol{p}} - \boldsymbol{\chi}^T\dot{\boldsymbol{\eta}} \geqslant 0\,.$$

While discussion and comments on the above governing relationships can be found in [9] and [10] and will not be duplicated here, the meaning of the above employed symbols can be briefly specified as follows. Dots indicate time derivatives, T transposition. Vectors of n components, n being the number of (nodal) degrees of freedom: $\boldsymbol{u} =$ = displacements; $\boldsymbol{P}$ = equivalent loads as input data. Vectors of m components, m denoting the number of generalized variables governing, by means of suitable interpolations, the strain and stress field: $\boldsymbol{\varepsilon}$, $\boldsymbol{e}$ and $\boldsymbol{p}$ = total, elastic and plastic strains, respectively (according to the *engineering definition*); $\boldsymbol{\sigma}$ = stresses. Vectors of y components, y being the number of yielding modes throughout the solid, resulting from the generalized variables semidiscretization: $\boldsymbol{\Phi}$ = plastic potentials; $\boldsymbol{\varphi}$ = yield functions; $\dot{\boldsymbol{\lambda}}$ = plastic multipliers. Vectors of l components, l being the number of generalized internal variables after modelling in space the relevant fields: $\boldsymbol{\eta}$ = kinematic internal variables; $\boldsymbol{\chi}$ = static internal variables. Matrices: $\boldsymbol{M}$ = inertia (assumed as positive definite and symmetric); $\boldsymbol{V}$ = viscous structural damping (symmetric positive semidefinite); $\boldsymbol{C}$ = = compatibility operator; $\boldsymbol{E}$ = element stiffnesses (block-diagonal symmetric, positive definite).

Scalars: Ψ_e = recoverable elastic energy; Ψ_s = energy stored at the microscale due to inelastic rearrangements; Ψ = Helmholtz free energy; D = dissipated energy (total, *i.e.* cumulative in space and time).

The discretization in space is meant to result from a multifield, variationally consistent modelling, such that all time-dependent vectors concern the whole aggregate of finite elements and contain generalized variables in Prager sense, occurring in work-conjugate pairs and preserving the dot product, (cf. *e.g.* [16-17]).

As for the mechanical meaning of the preceding relations, eqs. (1), (2), and (3) express dynamic equilibrium, geometric compatibility and initial conditions, respectively; eqs. (4) introduce the two constitutive energy potentials; eqs. (5) describe evolution laws; eqs. (6) define the changeable elastic domains in the stress space $\boldsymbol{\sigma}$ (or the fixed domain in the *augmented* space $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$) and the plastic flow rule; inequality (7*b*) expresses the usual thermodynamical requirement in the form of positive mechanical dissipation.

The constitutive setting materialized by eqs. (4) to (7) represents a broad class of constitutive laws which can be referred to, in the jargon of plasticity, as *generalized non-standard elastic-plastic material.*

An elastic-plastic dynamic system is said to *shakedown* under an assigned history of external actions if, and only if, a suitable overall cumulative measure of the yielding processes is bounded in time. In the present context shakedown can be characterized formally by the condition:

$$\lim_{t \to \infty} \left\{ D(t) = \int_0^t (\boldsymbol{\sigma}^T \dot{\boldsymbol{p}} - \boldsymbol{\chi}^T \dot{\boldsymbol{\eta}})\, d\tau \right\} < \infty . \tag{8}$$

The contrary event or *inadaptation*, to be avoided in most engineering situations, implies either unbounded ($\lim_{t \to \infty} \|\boldsymbol{u}(t)\| = \infty$) or bounded changes of configuration, namely: *i.e.* either incremental collapse (*ratchetting*) or alternating plasticity (*low cycle fatigue*), respectively.

3. Constitutive hypotheses and definitions

While only sake of brevity dictated to ignore imposed strains and displacements in the preceding formulation, the assumptions which follow are weak constitutive restrictions intended to make the derivation of shakedown criteria possible.

(a) All yield functions $\boldsymbol{\varphi}$ and plastic potentials $\boldsymbol{\Phi}$ are convex and differentiable.

(b) The locked-in strain energy $\Psi_s(\boldsymbol{\eta})$ is convex. This hypothesis can be shown to entail non softening behaviour (*i.e.* either positive hardening or perfect plasticity), but not necessarily material stability in Drucker's sense. In fact, nonassociativity can still make the second-order work negative for some incremental path (see *e.g.* [18, 19]).

(c) Having set $\boldsymbol{\Phi}(\mathbf{0}, \mathbf{0}) = \boldsymbol{\varphi}(\mathbf{0}, \mathbf{0})$, without loss of generality since the plastic potentials are defined to within additive constants, the following assumption is made [10].

For each α yield mode separately ($\alpha = 1, \ldots, n_\alpha$), the plastic potential Φ_α over the α^{th} portion of the yield locus is bounded from below, *i.e.* there is a finite constant B_α such that:

(9) $$B_\alpha = \min_{\boldsymbol{\sigma}, \boldsymbol{\chi}} \Phi_\alpha, \quad \text{subject to:}$$

(10) $$\varphi_\alpha(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0; \qquad \varphi_\beta(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0 \quad \forall \beta \neq \alpha; \qquad G_\alpha(\boldsymbol{\chi}) \leqslant 0.$$

In the above eq. (10) G_α are bounding functions which will be defined below.

As a geometrical interpretation of eqs. (9) and (10), vector $\boldsymbol{B}$ is such that the inequality

(11) $$\boldsymbol{\Phi}(\boldsymbol{\sigma}, \boldsymbol{\chi}) - \boldsymbol{B} \leqslant \mathbf{0}$$

defines, in the augmented space $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$, the domain Ω (*reduced domain*) which is, loosely speaking, the «largest» among the convex domains represented by eq. (11) with variable $\boldsymbol{B}$ and contained in the yield domain $\boldsymbol{\varphi} \leqslant \mathbf{0}$. The concept of reduced domain Ω, introduced in [20] for nonassociative perfect plasticity, has recently been adopted in [9, 10].

Finally, let us focus on the notion of *saturation* which is central to the present purposes.

The hardening behaviour is governed by the internal variable potential Ψ_s and plastic potentials $\boldsymbol{\Phi}$ through eqs. (4b) and (5b). Either as a consequence of this dependence and of the properties of Ψ_s and $\boldsymbol{\Phi}$ or as an independent additional constitutive requirement, it is assumed henceforth that the static internal variables are constrained to belong to an *admissible domain* described by the inequality:

(12) $$\boldsymbol{G}(\boldsymbol{\chi}) \leqslant \mathbf{0}$$

where vector $\boldsymbol{G}$ collects the *bounding functions* G_α ($\alpha = 1, \ldots, n_\alpha$) which are assumed to be convex and differentiable. Hardening saturation occurs when $\boldsymbol{\chi}$ approaches the boundary of the admissible domain (12) and is called *asymptotic* when, simultaneously, some kinematic internal variable tends to infinity. For instance, the saturation envisaged by Chaboche model adopted in [10] is asymptotic, the one in the overlay model proposed by Stein *et al.* [21] is not.

The main objective pursued in what follows is to introduce the new constitutive feature represented by saturation with respect to the convex admissible domain (12), in an extended shakedown theory based on the notion of *admissible cycles*. To this aim, in view of subsequent developments (sections 4-5), two kinds of cycles are defined below.

(*a*) We will call *admissible plastic cycle of kind* φ, in symbols $\dot{\tilde{\boldsymbol{p}}}_\varphi(t)$, $\dot{\tilde{\boldsymbol{\eta}}}_\varphi(t)$ a fictitious yielding process in which the flow rule is associated to the yield functions $\boldsymbol{\varphi}$ (not to the plastic potentials $\boldsymbol{\Phi}$ as in the case of real material), *i.e.*:

$$(13) \qquad \dot{\tilde{\boldsymbol{p}}}_\varphi = \frac{\partial \boldsymbol{\varphi}^T}{\partial \boldsymbol{\sigma}} \dot{\boldsymbol{\lambda}}, \qquad \dot{\tilde{\boldsymbol{\eta}}}_\varphi = -\frac{\partial \boldsymbol{\varphi}^T}{\partial \boldsymbol{\chi}} \dot{\boldsymbol{\lambda}}, \qquad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \qquad \boldsymbol{\varphi} \leqslant \mathbf{0}, \qquad \boldsymbol{\varphi}^T \dot{\boldsymbol{\lambda}} = 0$$

and also the following relations are complied with over a time interval $[t_1, t_2]$, $\tilde{\boldsymbol{\chi}}$ denoting a vector of time-independent static internal variables and $\Delta\boldsymbol{\mu}$ a vector of plastic multipliers concerning the internal variables:

$$(14) \qquad \Delta\tilde{\boldsymbol{p}}_\varphi \equiv \int_{t_1}^{t_2} \dot{\tilde{\boldsymbol{p}}}_\varphi(t)\, dt = \boldsymbol{C}\, \Delta\tilde{\boldsymbol{u}}_\varphi ;$$

$$(15) \qquad \Delta\tilde{\boldsymbol{\eta}}_\varphi \equiv \int_{t_1}^{t_2} \dot{\tilde{\boldsymbol{\eta}}}_\varphi(t)\, dt = \left.\frac{\partial \boldsymbol{G}^T}{\partial \boldsymbol{\chi}}\right|_{\tilde{\chi}} \Delta\boldsymbol{\mu} ;$$

$$(16) \qquad \Delta\boldsymbol{\mu} \geqslant \mathbf{0}, \qquad \boldsymbol{G}(\tilde{\boldsymbol{\chi}}) \leqslant \mathbf{0}, \qquad \boldsymbol{G}^T(\tilde{\boldsymbol{\chi}})\Delta\boldsymbol{\mu} = 0 .$$

(*b*) *Admissible plastic cycle of kind* Φ, in symbols $\dot{\tilde{\boldsymbol{p}}}_\Phi(t)$, $\dot{\tilde{\boldsymbol{\eta}}}_\Phi(t)$, will be the denomination of fictitious yielding processes characterized as in definition (*a*) but using subscript Φ instead of φ and replacing vector $\boldsymbol{\varphi}$ by $\boldsymbol{\Phi} - \boldsymbol{B}$ in eqs. (13)-(16).

For both kinds (*a*) and (*b*) of admissible cycles, we define the fictitious dissipations (dissipated powers), respectively, by the relations:

$$(17) \qquad \dot{D}_\varphi \equiv \tilde{\boldsymbol{\sigma}}_\varphi^T \dot{\tilde{\boldsymbol{p}}}_\varphi - \tilde{\boldsymbol{\chi}}_\varphi^T \dot{\tilde{\boldsymbol{\eta}}}_\varphi , \qquad \dot{D}_\Phi \equiv \tilde{\boldsymbol{\sigma}}_\Phi^T \dot{\tilde{\boldsymbol{p}}}_\Phi - \tilde{\boldsymbol{\chi}}_\Phi^T \dot{\tilde{\boldsymbol{\eta}}}_\Phi .$$

In each of these fictitious yielding processes obeying Drucker's postulate, the dissipation is uniquely defined by the rates of plastic strains and kinematic internal variables (even if stresses and static internal variables are not) by virtue of Hill's maximum principle.

In sections 4 and 5 use will be made of the following quantities concerning the structural behaviour of the structure supposed to be incapable of any plastic yielding:

– elasto-dynamic stress response $\boldsymbol{\sigma}^E$ to the given external actions $\boldsymbol{P}(t)$ with homogeneous initial conditions;

– free vibration stress response $\hat{\boldsymbol{\sigma}}^F$ due to suitably chosen, generally fictitious (capped symbols) initial conditions in the absence of external loads;

– self-equilibrated stresses $\hat{\boldsymbol{\rho}}$ constant in time, which can be interpreted as the elastostatic self-stress response to a time-independent plastic strain distribution $\hat{\boldsymbol{p}}_s$.

4. Sufficient condition for inadaptation

Proposition A. Inadaptation will occur under the give loading history $\boldsymbol{P}(t)$ (*i.e.* the system will not shakedown), if there exists an admissible plastic cycle of kind φ, say $(\dot{\tilde{\boldsymbol{p}}}_\varphi, \dot{\tilde{\boldsymbol{\eta}}}_\varphi)$ starting at a time instant $t_1 \geqslant \hat{t}$, such that for all $\hat{t}$ and fictitious initial conditions (and, hence, $\widehat{\boldsymbol{\sigma}}^F$):

$$\int_{t_1}^{t_2} (\boldsymbol{\sigma}^E + \widehat{\boldsymbol{\sigma}}^F)^T \dot{\widetilde{\boldsymbol{p}}}_\varphi \, dt > \int_{t_1}^{t_2} \dot{D}_\varphi(\dot{\widetilde{\boldsymbol{p}}}_\varphi, \dot{\widetilde{\boldsymbol{\eta}}}_\varphi) \, dt + \widetilde{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi . \tag{18}$$

PROOF. Suppose that shakedown occurs. The necessary condition for dynamic shakedown derived by a static approach in [9], after a straightforward generalization to the hardening saturation of concern here, requires that:

$$\boldsymbol{\varphi}(\boldsymbol{\sigma}^E(t) + \widehat{\boldsymbol{\sigma}}^F(t) + \widehat{\boldsymbol{\rho}}, \widehat{\boldsymbol{\chi}}) \leq \mathbf{0} \quad \forall t \geq \widehat{t}, \qquad \boldsymbol{G}(\widehat{\boldsymbol{\chi}}) \leq \mathbf{0} \tag{19}$$

for some instant $\widehat{t}$ and fictitious initial conditions (*i.e.* for some $\widehat{\boldsymbol{\sigma}}^F$) and for some time-independent vector of static internal variables $\widehat{\boldsymbol{\chi}}$ and self-stresses $\widehat{\boldsymbol{\rho}}$.

The convexity of the yield functions $\boldsymbol{\varphi}$ and the associativity of the fictitious flow rules (13) imply that, for all admissible plastic cycles of kind φ:

$$(\widetilde{\boldsymbol{\sigma}}_\varphi - \overline{\boldsymbol{\sigma}})^T \dot{\widetilde{\boldsymbol{p}}}_\varphi - (\widetilde{\boldsymbol{\chi}}_\varphi - \overline{\boldsymbol{\chi}})^T \dot{\widetilde{\boldsymbol{\eta}}}_\varphi \geq 0 , \qquad \forall (\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\chi}}) \colon \boldsymbol{\varphi}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\chi}}) \leq \mathbf{0} . \tag{20}$$

Because of (19*a*), the circumstance expressed by eq. (20) still holds if we set in (20):

$$\overline{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^E(t) + \widehat{\boldsymbol{\sigma}}^F(t) + \widehat{\boldsymbol{\rho}} \equiv \widehat{\boldsymbol{\sigma}}(t) , \qquad \overline{\boldsymbol{\chi}} \equiv \widehat{\boldsymbol{\chi}}. \tag{21}$$

After this substitution, integrate eq. (20) over the time interval $[t_1, t_2]$:

$$\int_{t_1}^{t_2} (\widehat{\boldsymbol{\sigma}}^T \dot{\widetilde{\boldsymbol{p}}}_\varphi - \widehat{\boldsymbol{\chi}}^T \dot{\widetilde{\boldsymbol{\eta}}}_\varphi) \, dt \leq \int_{t_1}^{t_2} \dot{D}_\varphi(\dot{\widetilde{\boldsymbol{p}}}_\varphi, \dot{\widetilde{\boldsymbol{\eta}}}_\varphi) \, dt . \tag{22}$$

Note that, in view of eqs. (14) and (15) and of the time independence of self-stresses $\widehat{\boldsymbol{\rho}}$ and static internal variables $\widehat{\boldsymbol{\chi}}$:

$$\int_{t_1}^{t_2} \widehat{\boldsymbol{\rho}}^T \dot{\widetilde{\boldsymbol{p}}}_\varphi \, dt = \widehat{\boldsymbol{\rho}}^T \Delta \widetilde{\boldsymbol{p}}_\varphi = \widehat{\boldsymbol{\rho}}^T \boldsymbol{C} \Delta \widetilde{\boldsymbol{u}}_\varphi = 0 ; \tag{23}$$

$$\int_{t_1}^{t_2} \widehat{\boldsymbol{\chi}}^T \dot{\widetilde{\boldsymbol{\eta}}}_\varphi \, dt = \widehat{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi . \tag{24}$$

Making use of eqs. (23) and (24) in inequality (22) and adding $\widetilde{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi$ to both sides of it, one obtains:

$$\int_{t_1}^{t_2} (\boldsymbol{\sigma}^E + \widehat{\boldsymbol{\sigma}}^F)^T \dot{\widetilde{\boldsymbol{p}}}_\varphi \, dt + (\widetilde{\boldsymbol{\chi}} - \widehat{\boldsymbol{\chi}})^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi \leq \int_{t_1}^{t_2} \dot{D}_\varphi(\dot{\widetilde{\boldsymbol{p}}}_\varphi, \dot{\widetilde{\boldsymbol{\eta}}}_\varphi) \, dt + \widetilde{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi . \tag{25}$$

Equations (15) and (16), by analogy to eqs. (13), can be interpreted as stating a fictitious associativity. This fact and the assumed convexity of the bounding functions $\boldsymbol{G}$, imply that, in analogy to eq. (20):

$$(\widetilde{\boldsymbol{\chi}} - \widehat{\boldsymbol{\chi}})^T \left. \frac{\partial \boldsymbol{G}^T}{\partial \boldsymbol{\chi}} \right|_{\widetilde{\boldsymbol{\chi}}} \Delta \boldsymbol{\mu} = (\widetilde{\boldsymbol{\chi}} - \widehat{\boldsymbol{\chi}})^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi \geq 0 \qquad \forall \widehat{\boldsymbol{\chi}} \colon \quad \boldsymbol{G}(\widehat{\boldsymbol{\chi}}) \leq \mathbf{0} . \tag{26}$$

In view of inequality (26), inequality (25) still holds true when the second addend on its l.h.s. is crossed out. Inequality (25) thus modified turns out to be in contradiction with the hypothesis (18). Note that inequality (25) holds true for all admissible cycles with $t_1 \geq \hat{t}$ and for the particular $\hat{\boldsymbol{\sigma}}^F$, $\hat{t}$ contemplated by the necessary shakedown condition. Therefore the above contradiction rules out shakedown, *i.e.* entails inadaptation (q.e.d.).

5. Sufficient condition for shakedown

Proposition B. Shakedown will occur under the given loading history $\boldsymbol{P}(t)$ if there exist fictitious initial conditions (and consequently a free vibration response $\bar{\boldsymbol{\sigma}}^F(t)$), a time instant $\bar{t}$ and a scalar $\bar{\xi} > 1$ such that:

$$(27) \qquad \bar{\xi} \int_{t_1}^{t_2} (\boldsymbol{\sigma}^E + \bar{\boldsymbol{\sigma}}^F)^T \dot{\tilde{\boldsymbol{p}}}_\Phi \, dt \leq \int_{t_1}^{t_2} D_\Phi (\dot{\tilde{\boldsymbol{p}}}_\Phi, \dot{\tilde{\boldsymbol{\eta}}}_\Phi) \, dt + \tilde{\boldsymbol{\chi}}^T \Delta \tilde{\boldsymbol{\eta}}_\Phi$$

for all admissible plastic cycles of kind Φ starting at $t_1 \geq \tilde{t}$.

Proof. In order to prove the above criterion, we will follow the path of reasoning adopted in [7] and [9] and based on the sufficient shakedown criterion derived by a static (Melan's type) approach. This criterion in the presence of hardening saturation can be formulated as follows by a slight generalization of those established in [9] and [10].

Shakedown occurs for a load factor α (*i.e.* $\alpha \leq \alpha_s$, α_s being the shakedown limit or *safety factor*), if exist a time instant $\hat{t}$, fictitious initial conditions (*i.e.* a $\hat{\boldsymbol{\sigma}}^F$), time-independent vectors of static internal variables $\hat{\boldsymbol{\chi}}$ and self-stresses $\hat{\boldsymbol{\rho}}$, a scalar $\xi > 1$, such that:

$$(28) \qquad \Phi(\xi(\alpha\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^F + \hat{\boldsymbol{\rho}}), \xi\hat{\boldsymbol{\chi}}) \leq \boldsymbol{B} \quad \forall t \geq \hat{t}, \qquad \boldsymbol{G}(\xi\hat{\boldsymbol{\chi}}) \leq \boldsymbol{0}, \qquad \boldsymbol{C}^T\hat{\boldsymbol{\rho}} = \boldsymbol{0}.$$

Let us denote by γ^- the maximum of all factors α which comply with the conditions (28) for some fixed $\xi > 1$ and $\hat{\boldsymbol{\sigma}}^F$ and by α_s^- the same maximum with $\xi > 1$ and $\hat{\boldsymbol{\sigma}}^F$ regarded as variables; thus we can write:

$$(29) \qquad \alpha_s \geq \alpha_s^- \geq \gamma^- = \max_{\alpha, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\chi}}} \{\alpha\} \quad \text{subject to (28).}$$

Consider the maximization problem (29) with $\hat{t}$, $\hat{\boldsymbol{\sigma}}^F$ and ξ fixed to values $\bar{t}$, $\bar{\boldsymbol{\sigma}}^F$ and $\bar{\xi}$ of the hypothesis of Proposition B. Defining for brevity $\tilde{\boldsymbol{\sigma}} \equiv \xi(\alpha\boldsymbol{\sigma}^E + \hat{\boldsymbol{\sigma}}^F + \hat{\boldsymbol{\rho}})$; $\tilde{\boldsymbol{\chi}} \equiv \xi\hat{\boldsymbol{\chi}}$, the Lagrangian function of the constrained optimization problem (29) reads:

$$(30) \qquad L = -\alpha + \int_{\bar{t}}^{t} \boldsymbol{\gamma}^T[\boldsymbol{\Phi}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}}) - \boldsymbol{B} + \boldsymbol{c}] d\tau + \int_{\bar{t}}^{t} \boldsymbol{v}^T[\boldsymbol{G}(\tilde{\boldsymbol{\chi}}) + \boldsymbol{d}] d\tau + \int_{\bar{t}}^{t} \boldsymbol{\omega}^T \boldsymbol{C}^T \hat{\boldsymbol{\rho}} d\tau,$$

where $\boldsymbol{\gamma}$, $\boldsymbol{v}$, $\boldsymbol{\omega}$ are vectors of Lagrange multipliers and $\boldsymbol{c}$, $\boldsymbol{d}$ vectors of slack variables c_i^2, d_j^2, with $i = 1 \dots y$, $j = 1 \dots n_\gamma$. The following Euler-Lagrange optimality conditions flow from eq. (30):

$$\int_{\bar{t}}^{t} \bar{\xi} \boldsymbol{\sigma}^{E^T} \frac{\partial \boldsymbol{\Phi}^T}{\partial \tilde{\boldsymbol{\sigma}}} (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma}\, d\tau = 1\,; \tag{31a}$$

$$\int_{\bar{t}}^{t} \left[\bar{\xi} \frac{\partial \boldsymbol{\Phi}^T}{\partial \tilde{\boldsymbol{\sigma}}} (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma} + \boldsymbol{C}\boldsymbol{\omega} \right] d\tau = \boldsymbol{0}\,; \tag{31b}$$

$$\int_{\bar{t}}^{t} \bar{\xi} \left[\frac{\partial \boldsymbol{\Phi}^T}{\partial \tilde{\boldsymbol{\chi}}} (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma} + \frac{\partial \boldsymbol{G}^T}{\partial \tilde{\boldsymbol{\chi}}} (\tilde{\boldsymbol{\chi}})\, \boldsymbol{v} \right] d\tau = \boldsymbol{0}\,; \tag{31c}$$

$$\boldsymbol{C}^T \hat{\boldsymbol{\rho}} = \boldsymbol{0}\,; \tag{31d}$$

$$\boldsymbol{\Phi}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}}) \leqslant \boldsymbol{B}\,, \qquad \boldsymbol{\gamma} \geqslant \boldsymbol{0}\,, \qquad \boldsymbol{\Phi}^T (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma} = 0\,; \tag{31e,f,g}$$

$$\boldsymbol{G}(\tilde{\boldsymbol{\chi}}) \leqslant \boldsymbol{0}\,, \qquad \boldsymbol{v} \geqslant \boldsymbol{0}\,, \qquad \boldsymbol{G}^T (\tilde{\boldsymbol{\chi}})\, \boldsymbol{v} = 0\,. \tag{31h,i,l}$$

Consider now the admissible cycle of kind Φ specified as follows:

$$\dot{\tilde{\boldsymbol{p}}}_\Phi \equiv \frac{\partial \boldsymbol{\Phi}^T}{\partial \tilde{\boldsymbol{\sigma}}} (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma}\,, \qquad \dot{\tilde{\boldsymbol{\eta}}}_\Phi \equiv -\frac{\partial \boldsymbol{\Phi}^T}{\partial \tilde{\boldsymbol{\chi}}} (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma}\,. \tag{32}$$

In view of eqs. (31), it is easily realized that the relations listed below hold true and guarantee that the definition (*b*), section 3, of admissible cycle is fulfilled:

$$\bar{\xi} \int_{\bar{t}}^{t} \boldsymbol{\sigma}^{E^T} \dot{\tilde{\boldsymbol{p}}}_\Phi\, d\tau = 1\,, \qquad \int_{\bar{t}}^{t} \dot{\tilde{\boldsymbol{p}}}_\Phi\, d\tau = \boldsymbol{C}\, \Delta \tilde{\boldsymbol{u}}_\Phi\,, \qquad \Delta \tilde{\boldsymbol{u}}_\Phi = -\frac{1}{\bar{\xi}} \int_{\bar{t}}^{t} \boldsymbol{\omega}\, d\tau\,; \tag{33a}$$

$$\int_{\bar{t}}^{t} \dot{\tilde{\boldsymbol{\eta}}}_\Phi\, d\tau = \frac{\partial \boldsymbol{G}^T}{\partial \tilde{\boldsymbol{\chi}}} (\tilde{\boldsymbol{\chi}})\, \Delta \boldsymbol{v}\,, \qquad \Delta \boldsymbol{v} = \int_{\bar{t}}^{t} \boldsymbol{v}\, d\tau\,; \tag{33b}$$

$$\boldsymbol{\Phi}(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}}) \leqslant \boldsymbol{B}\,, \qquad \boldsymbol{\gamma} \geqslant \boldsymbol{0}\,, \qquad \boldsymbol{\Phi}^T (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\chi}})\, \boldsymbol{\gamma} = 0\,; \tag{33c}$$

$$\boldsymbol{G}(\tilde{\boldsymbol{\chi}}) \leqslant \boldsymbol{0}\,, \qquad \Delta \boldsymbol{v} \geqslant \boldsymbol{0}\,, \qquad \boldsymbol{G}^T (\tilde{\boldsymbol{\chi}})\, \Delta \boldsymbol{v} = 0\,. \tag{33d}$$

Rewrite now the hypothesis inequality (27), which holds for all admissible plastic cycles of kind Φ, for the particular admissible plastic cycle of kind Φ introduce by eqs. (32):

$$\bar{\xi} \int_{\bar{t}}^{t} (\boldsymbol{\sigma}^E + \bar{\boldsymbol{\sigma}}^F)^T \dot{\tilde{\boldsymbol{p}}}_\Phi\, d\tau \leqslant \bar{\xi} \int_{\bar{t}}^{t} [(\alpha \boldsymbol{\sigma}^E + \bar{\boldsymbol{\sigma}}^F + \hat{\boldsymbol{\rho}})^T \dot{\tilde{\boldsymbol{p}}}_\Phi - \tilde{\boldsymbol{\chi}}^T \dot{\tilde{\boldsymbol{\eta}}}_\Phi]\, d\tau + \bar{\xi} \tilde{\boldsymbol{\chi}}^T \Delta \tilde{\boldsymbol{\eta}}_\Phi\,. \tag{34}$$

By taking into account relations (33*a*) and (33*b*), eq. (34) can be rewritten as follows:

$$1 \leqslant \alpha \tag{35}$$

which means that shakedown does occur, under the given loading history, when the hypothesis specified in the Proposition B holds true (q.e.d.).

6. Conclusions

The results achieved in what precedes can be elucidated as for their physical meaning and computational consequences by the following remarks.

(*a*) The nonlinear hardening saturation, contemplated by the constitutive laws assumed herein, is reflected by the new requirements (15), (16) in the definitions of admissible plastic cycles, and by the consequent additional terms $\widetilde{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\varphi$ and $\widetilde{\boldsymbol{\chi}}^T \Delta \widetilde{\boldsymbol{\eta}}_\Phi$ in the inequalities (18) and (27) in criteria A and B, respectively. The absence of saturation (*i.e.* of bounding surfaces in the space of static internal variables $\boldsymbol{\chi}$) means that always $\boldsymbol{G} < \mathbf{0}$ and, hence, through (15) and (16), that the above additional terms disappear, so that criteria A and B reduced to those established in [9].

(*b*) In both statements A and B fictitious initial conditions play the role of available parameters. The actual initial conditions have, as expected, no influence on the dynamic scenario asymptotically in time and, in particular, on the choice between shakedown and inadaptation. In the special case of period external actions, *i.e.* when $\boldsymbol{P}(t) = \boldsymbol{P}(t + T)$, T being a fixed finite time interval, it might be shown, like in [5] and [22] for narrower contexts, that the set of all fictitious initial conditions can be replaced by those initial conditions which would remove the transient motion in the linear elastic response; then the available time parameter can be fixed at the time origin by setting $\bar{t} = 0$.

(*c*) In the context of piece-wise-linear plasticity (linear yield functions and plastic potentials, linear hardening; hence polyhedral elastic domains), the kinematic theorem by Neal-Symonds-Koiter had been shown to provide a basis for giving the determination of shakedown limits a linear programming formulation [20]. This linear programming problem turned out to be the dual to the one achievable on the basis of Melan's static theorem [12, 20], and this duality pattern of shakedown analysis, with its obvious computational benefits, was extended to dynamics under periodic external actions [12, 14]. Under the same restriction of excitation periodicity and the consequent fixed choice of initial conditions, it is expected (and will be investigated elsewhere) that criteria A and B will lead to two minimization problems in convex nonlinear programming apt to generate an upper and a lower bound, respectively, on the critical value (*shakedown limit*) of a *load factor* amplifying the external actions.

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