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Vsevolod A. Solonnikov, Giuseppe Mulone

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Magnetofluidodinamica. — *On the solvability of some initial boundary value problems of magnetofluidmechanics with Hall and ion-slip effects.* Nota di VSEVOLOD A. SOLONNIKOV e GIUSEPPE MULONE, presentata (*) dal Corrisp. S. Rionero.

ABSTRACT. — The solvability of three linear initial-boundary value problems for the system of equations obtained by linearization of MHD equations is established. The equations contain terms corresponding to Hall and ion-slip currents. The solutions are found in the Sobolev spaces $W_p^{2,1}(Q_T)$ with $p > 5/2$ and in anisotropic Hölder spaces.

KEY WORDS: Magnetohydrodynamics; Anysotropic currents; Existence; Uniqueness.

RIASSUNTO. — *Sulla risolubilità di qualche problema ai valori iniziali e al contorno per la magnetofluidodinamica con effetto Hall e di ion-slip.* Si stabilisce la risolubilità di tre problemi ai valori iniziali e al contorno per il sistema ottenuto linearizzando le equazioni della MHD. Le equazioni contengono termini corrispondenti alle correnti di Hall e di ion-slip. Le soluzioni sono trovate negli spazi di Sobolev $W_p^{2,1}(Q_T)$ con $p > 5/2$ e negli spazi di Hölder anisotropi.

1. INTRODUCTION AND MAIN RESULTS

The motion of an incompressible fluid in a magnetic field in the non-relativistic case is governed by the system of equations

$$(1.1)\qquad \begin{cases} \boldsymbol{U}_t + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = \mu\rho_0^{-1}\boldsymbol{H}\cdot\nabla\boldsymbol{H} - \nabla(p_1\rho_0^{-1} + \mu H^2\rho_0^{-1}/2) + \nu\Delta\boldsymbol{U} + \boldsymbol{F}(x,t)\,, \\ \boldsymbol{H}_t = -\mu^{-1}\nabla\times\boldsymbol{E}\,, \\ \nabla\cdot\boldsymbol{U} = 0\,, \qquad \nabla\cdot\boldsymbol{H} = 0\,, \\ \boldsymbol{J} = \nabla\times\boldsymbol{H}\,, \\ \boldsymbol{J} + \mu\sigma\beta\boldsymbol{J}\times\boldsymbol{H} + \mu\sigma\beta_1\boldsymbol{H}\times(\boldsymbol{J}\times\boldsymbol{H}) = \sigma(\boldsymbol{E} + \mu\boldsymbol{U}\times\boldsymbol{H})\,, \end{cases}$$

where $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{U}$ is the velocity field, $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density vector, p_1 is the pressure and $\boldsymbol{F}$ is the external force. The last equation of (1.1) represents a generalized Ohm's law including the Hall and ion-slip currents [1-4]. The constant coefficients ρ_0, ν, μ, σ are the density, the kinematic viscosity, the magnetic permeability, the electrical conductivity. β and $\beta_1 > 0$ are the Hall and the ion-slip coefficients. As usually, we have neglected the dispacement current.

Excluding $\boldsymbol{E}$ and $\boldsymbol{J}$ we can write the system (1.1) in the form

$$(1.2)\qquad \begin{cases} \boldsymbol{U}_t + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = \mu\rho_0^{-1}\boldsymbol{H}\cdot\nabla\boldsymbol{H} - \nabla(p_1\rho_0^{-1} + \mu H^2\rho_0^{-1}/2) + \nu\Delta\boldsymbol{U} + \boldsymbol{F}(x,t)\,, \\ \boldsymbol{H}_t = \nabla\times(\boldsymbol{U}\times\boldsymbol{H}) + \eta_0\Delta\boldsymbol{H} + \beta\nabla\times[\boldsymbol{H}\times(\nabla\times\boldsymbol{H})] + \\ \qquad\qquad + \beta_1\nabla\times\{\boldsymbol{H}\times[\boldsymbol{H}\times(\nabla\times\boldsymbol{H})]\}\,, \\ \nabla\cdot\boldsymbol{U} = 0\,, \qquad \nabla\cdot\boldsymbol{H} = 0 \end{cases}$$

(*) Nella seduta dell'11 febbraio 1995.

where $\eta_0 = (\mu\sigma)^{-1}$ is the magnetic resistivity. We assume that the fluid is contained in a bounded domain $\Omega \subset \boldsymbol{R}^3$. We denote by Q_T the cylinder $\Omega \times (0, T)$ $(x \in \Omega,$ $t \in (0, T))$, $S = \partial\Omega$, $\Sigma_T = S \times (0, T)$ where T is a positive number.

To the system (1.2) we add the initial conditions

$$(1.3) \qquad \boldsymbol{U}(x, 0) = \boldsymbol{U}_0(x), \qquad \boldsymbol{H}(x, 0) = \boldsymbol{H}_0(x),$$

and the boundary conditions

$$(\mathrm{A}) \qquad \boldsymbol{U} = \boldsymbol{a}', \qquad \boldsymbol{H}_\tau = \boldsymbol{b}' \quad \text{on } \Sigma_T,$$

$$(\mathrm{B}) \qquad \boldsymbol{U}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{D}(\boldsymbol{U})\cdot\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{U})\cdot\boldsymbol{n})\,\boldsymbol{n} = \boldsymbol{c}', \qquad \boldsymbol{H}_\tau = \boldsymbol{b}' \quad \text{on } \Sigma_T,$$

$$(\mathrm{C}) \qquad \begin{cases} \boldsymbol{U} = \boldsymbol{a}, \qquad \boldsymbol{H}\cdot\boldsymbol{n} = 0, \\ \{\eta_0 \nabla \times \boldsymbol{H} - \beta[\boldsymbol{H} \times (\nabla \times \boldsymbol{H})] - \beta_1 \boldsymbol{H} \times [\boldsymbol{H} \times (\nabla \times \boldsymbol{H})]\}_\tau = \boldsymbol{d}', \end{cases} \quad \text{on } \Sigma_T,$$

where $\boldsymbol{a}', \boldsymbol{b}', \boldsymbol{c}', \boldsymbol{d}'$ are given fields with $\boldsymbol{a}'\cdot\boldsymbol{n} = \boldsymbol{b}'\cdot\boldsymbol{n} = \boldsymbol{c}'\cdot\boldsymbol{n} = \boldsymbol{d}'\cdot\boldsymbol{n} = 0$, $\boldsymbol{n}$ is the unit normal to S, and $\boldsymbol{D}(\boldsymbol{U})$ is the symmetric part of $\nabla\boldsymbol{U}$.

Conditions (A), (B) and (C) are appropriate for a rigid non-conducting, free non-conducting and rigid perfect-conducting boundary, respectively. Because of the importance of MHD problems both in mathematical and physical applications, many writers [4-10], have studied uniqueness, continuous dependence and stability of a basic motion of system (1.2) (with or without ion-slip currents) with initial conditions (1.3) and boundary conditions (A), (B), or (C). In the papers [11-14], the existence and uniqueness problem in suitable Hilbert spaces have been studied in the absence of Hall and ion-slip currents. The papers [15, 16] deal with existence theorems when the displacement currents are not neglected. Finally in the papers of [7, 8] existence theorems (in the linear and non-linear case) for a MHD flow with Hall current in a toroidal domain are proved in the Sobolev space $W_2^{4,2}(Q_T)$ and also a linearization principle is established.

In the present paper we consider eq. (1.2) linearized about certain given solenoidal vector fields $\boldsymbol{U}_0(x,t)$ and $\boldsymbol{H}_0(x,t)$ *i.e.*,

$$(1.4) \qquad L_1(\boldsymbol{u}, p, \boldsymbol{b}) = \boldsymbol{f}, \qquad \nabla\cdot\boldsymbol{u} = 0, \qquad L_2(\boldsymbol{u}, \boldsymbol{b}) = \boldsymbol{g}, \qquad \nabla\cdot\boldsymbol{b} = 0$$

where

$$(1.5) \qquad L_1(\boldsymbol{u}, p, \boldsymbol{b}) \equiv$$

$$\equiv \boldsymbol{u}_t + \boldsymbol{U}_0\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{U}_0 - \mu\rho_0^{-1}[\boldsymbol{H}_0\cdot\nabla\boldsymbol{b} + \boldsymbol{b}\cdot\nabla\boldsymbol{H}_0] - \nu\Delta\boldsymbol{u} + \rho_0^{-1}\nabla p,$$

$$(1.6) \qquad L_2(\boldsymbol{u}, \boldsymbol{b}) \equiv \boldsymbol{b}_t - \eta_0\Delta\boldsymbol{b} - \beta[\nabla \times (\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})) + \nabla \times (\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0))] -$$

$$-\beta_1 \nabla \times \{\boldsymbol{H}_0 \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] + \boldsymbol{H}_0 \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)] + \boldsymbol{b} \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{H}_0)]\} -$$

$$-\nabla \times (\boldsymbol{U}_0 \times \boldsymbol{b} + \boldsymbol{u} \times \boldsymbol{H}_0),$$

and $\boldsymbol{f}$, $\boldsymbol{g}$ are given vector-valued functions of x and t.

For this system, we consider initial-boundary value problems consisting in the determination of $\boldsymbol{u}$, p, $\boldsymbol{b}$, which satisfy (1.4), initial conditions

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0(x), \qquad \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0(x) \tag{1.7}$$

and one of the following conditions at the boundary S

(A′) $\boldsymbol{u} = \boldsymbol{a}$, $\quad \boldsymbol{b}_\tau = \boldsymbol{b}$ on Σ_T,

(B′) $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, $\quad \boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{n})\boldsymbol{n} = \boldsymbol{c}$, $\quad \boldsymbol{b}_\tau = \boldsymbol{b}$ on Σ_T,

(C′)
$$\begin{cases} \boldsymbol{u} = \boldsymbol{a}, \quad \boldsymbol{b}\cdot\boldsymbol{n} = 0, \\ B_\tau(\boldsymbol{b}) \equiv (\gamma_0 \nabla \times \boldsymbol{b} - \beta[\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0) + \boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] - \\ \quad -\beta_1\{\boldsymbol{b} \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{H}_0)] + \boldsymbol{H}_0 \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)] + \\ \quad\quad + \boldsymbol{H}_0 \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})]\})_\tau = \boldsymbol{d}, \qquad \text{on } \Sigma_T. \end{cases}$$

We establish the solvability of problems (1.4), (1.7)-(A′), (1.4), (1.7)-(B′), (1.4), (1.7)-(C′), in anisotropic Sobolev and Hölder spaces $W_p^{2,1}(Q_T)$, $C^{2+\alpha, 1+\alpha/2}(Q_T)$. We recall that the norm in the space $W_p^{2,1}(Q_T)$, is given by the formula

$$\|u\|^p_{W_p^{2,1}(Q_T)} = \sum_{0 \le |j| \le 2} \|D^j u\|^p_{L_p(Q_T)} + \|D_t u\|^p_{L_p(Q_T)},$$

where $\|v\|^p_{L_p(Q_T)} = \int_0^T \int_\Omega |v(x,t)|^p \, dx\, dt$ is a standard L_p-norm.

Functions $u \in W_p^{2,1}(Q_T)$ have traces on the cross-sections $(x \in \Omega, t = t_0 \in [0,T])$ of the cylinder Q_T and on the lateral surface $S \times (0,T) = \Sigma_T$. These traces belong to the fractional Sobolev spaces (or, which is the same, to the Besov spaces) $W_p^{2-2/p}(\Omega)$ and $W_p^{2-1/p,\, 1-1/2p}(\Sigma_T)$, respectively. The norms in $W_p^r(\Omega)$ and $W_p^{r, r/2}(\Sigma_T)$ with non-integer $r \in (0, 2)$ are defined by the formulas

$$\|u\|^p_{W_p^r(\Omega)} = \sum_{0 \le |j| < r} \|D^j u\|^p_{L_p(\Omega)} + \sum_{|j| = [r]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^p}{|x-y|^{3+p(r-[r])}}\, dx\, dy,$$

$$\|u\|^p_{W_p^{r,r/2}(\Sigma_T)} = \|u\|^p_{L_p(\Sigma_T)} + \|\nabla_S^{[r]} u\|^p_{L_p(\Sigma_T)} + \int_0^T \int_S \int_S |\nabla_S^{[r]} u(x,t) - \nabla_S^{[r]} u(y,t)|^p \cdot$$
$$\cdot \frac{dS_x\, dS_y\, dt}{|x-y|^{2+p(r-[r])}} + \int_0^T \int_0^T \int_S |u(x,t) - u(x,\tau)|^p \frac{dS_x\, dt\, d\tau}{|t-\tau|^{1+pr/2}}.$$

Here ∇_S is the surface gradient; in the case $r < 1$, $\nabla_S^{[r]} u$ should be replaced by u in this formula. We shall always assume that $p > 5/2$. Under this conditions, $p(2 - 2/p) > 3$ and the following estimate holds:

$$\sup_{Q_T} |u(x,t)| \le C \sup_{t \le T} \|u\|_{W_p^{2-2/p}(\Omega)} \le C_1 \|u\|_{W_p^{2,1}(Q_T)}. \tag{1.8}$$

This estimate will be especially useful for the investigation of the nonlinear problems which will be the subject of subsequent publications.

The norms in the Hölder spaces $C^{k+\alpha}(\Omega)$ and $C^{k+\alpha,(k+\alpha)/2}(Q_T)$ are defined by the formulas

$$|u|_{C^{k+\alpha}(\Omega)} = \sum_{0\leq |j|\leq k} \sup_x |D^j u(x)| + \sum_{|j|=k} \sup_{x,y} \frac{|D^j u(x) - D^j u(y)|}{|x-y|^\alpha},$$

$$|u|_{C^{k+\alpha,(k+\alpha)/2}(Q_T)} = \sum_{0\leq 2i+|j|\leq k} \sup_{x,t,\tau} |D_t^i D_x^j u(x,t)| +$$

$$+ \sum_{2i+|j|=k} \sup_{x,t,\tau} \frac{|D_t^i D_x^j u(x,t) - D_t^i D_x^j u(x,\tau)|}{|t-\tau|^{\alpha/2}} +$$

$$+ \sum_{2i+|j|=k} \sup_{x,y,t} \frac{|D_t^i D_x^j u(x,t) - D_t^i D_x^j u(y,t)|}{|x-y|^\alpha}.$$

We make use of these spaces in the cases $k = 0, 1, 2$.

Main results of the present paper are as follows.

THEOREM 1. 1) *Let* $S\in C^3$, $H_0\in W_p^{2,1}(Q_T)$, $U_0\in W_p^{2,1}(Q_T)$, $p>5/2$. *For arbitrary* $f\in L_p(Q_T)$, $g\in L_p(Q_T)$, $b_0\in W_p^{2-2/p}(\Omega)$, $u_0\in W_p^{2-2/p}(\Omega)$, $a\in W_p^{2-1/p,1-1/2p}(\Sigma_T)$, $a\cdot n=0$, $b\in W_p^{2-1/p,1-1/2p}(\Sigma_T)$, *which satisfy the compatibility conditions* $b_{0\tau}|_S = b(x,0)$, $u_0|_S = a(x,0)$, $\nabla\cdot b_0 = 0$, $\nabla\cdot u_0 = 0$, $\nabla\cdot g = 0$, *in a weak sense*, $b\cdot n = 0$, *problem* (1.4), (1.7)-(A') *has a unique solution* (u,p,b) *such that* $u\in W_p^{2,1}(Q_T)$, $\nabla p\in L_p(Q_T)$, $b\in W_p^{2,1}(Q_T)$. *For this solution the following estimate holds*

$$\|u\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} + \|b\|_{W_p^{2,1}(Q_T)} \leq c_1(T)(\|f\|_{L_p(Q_T)} + \|g\|_{L_p(Q_T)} + \tag{1.9}$$

$$+\|b_0\|_{W_p^{2-2/p}(\Omega)} + \|u_0\|_{W_p^{2-2/p}(\Omega)} + \|a\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)} + \|b\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)}).$$

2) *For arbitrary* $f\in L_p(Q_T)$, $g\in L_p(Q_T)$, $b_0\in W_p^{2-2/p}(\Omega)$, $u_0\in W_p^{2-2/p}(\Omega)$, $b\in W_p^{2-1/p,1-1/2p}(\Sigma_T)$, $c\in W_p^{1-1/p,1/2-1/2p}(\Sigma_T)$, *such that* $\nabla\cdot b_0 = 0$, $\nabla\cdot u_0 = 0$, $\nabla\cdot g = 0$, *in a weak sense*, $b_{0\tau}|_S = b(x,0)$, $u_0\cdot n|_S = 0$, $b\cdot n = 0$, $c\cdot n = 0$,

$$D(u_0)\cdot n - (n\cdot D(u_0)\cdot n)n = c(x,0) \quad \textit{if } p\geq 3 \tag{1.10}$$

problem (1.4), (1.7)-(B') *has a unique solution* (u,p,b) *such that* $u\in W_p^{2,1}(Q_T)$, $\nabla p\in L_p(Q_T)$, $b\in W_p^{2,1}(Q_T)$, *and for this solution the inequality*

$$\|u\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} + \|b\|_{W_p^{2,1}(Q_T)} \leq c_2(T)(\|f\|_{L_p(Q_T)} + \|g\|_{L_p(Q_T)} + \tag{1.11}$$

$$+\|b_0\|_{W_p^{2-2/p}(\Omega)} + \|u_0\|_{W_p^{2-2/p}(\Omega)} + \|a\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)} +$$

$$+\|b\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)} + \|c\|_{W_p^{1-1/p,1/2-1/2p}(\Sigma_T)})$$

holds.

3) *Assume in addition that*

$$H_0\cdot n|_{\Sigma_T} = 0, \qquad U_0\cdot n|_{\Sigma_T} = 0. \tag{1.12}$$

Then for arbitrary f, g, u_0, a *satisfying the hypotheses of n° 1) and arbitrary* $b_0\in W_p^{2-2/p}(\Omega)$,

$d \in W_p^{1-1/p,\,1/2-1/2p}(\Sigma_T)$, *such that* $\nabla\cdot b_0 = 0$, $d\cdot n = 0$, $b_0\cdot n|_S = 0$, $[g - \nabla \times \times d]\cdot n|_S = 0$,

$$B_\tau(b_0)|_S = d(x, 0), \qquad \text{if } p \geq 3 \tag{1.13}$$

problem (1.4), (1.7)-(C′) *has a unique solution* (u, p, b) *such that* $u \in W_p^{2,1}(Q_T)$, $\nabla p \in L_p(Q_T)$, $b \in W_p^{2,1}(Q_T)$, *and for this solution the inequality*

$$\|u\|_{W_p^{2,1}(Q_T)} + \|\nabla p\|_{L_p(Q_T)} + \|b\|_{W_p^{2,1}(Q_T)} \leq c_3(T)(\|f\|_{L_p(Q_T)} + \|g\|_{L_p(Q_T)} + + \|b_0\|_{W_p^{2-2/p}(\Omega)} + \|u_0\|_{W_p^{2-2/p}(\Omega)} + \|a\|_{W_p^{2-1/p,\,1-1/2p}(\Sigma_T)} + \|d\|_{W_p^{1-1/p,\,1/2-1/2p}(\Sigma_T)}) \tag{1.14}$$

holds. The positive constants $C_1(T)$, $C_2(T)$, $C_3(T)$, *are non-decreasing functions of* T.

THEOREM 2. 1) *Let* $S \in C^{3+\alpha}$, $H_0 \in C^{2+\alpha,\,1+\alpha/2}(Q_T)$, $U_0 \in C^{2+\alpha,\,1+\alpha/2}(Q_T)$. $f \in C^{\alpha,(\alpha+\varepsilon)/2}(Q_T)$, $g \in C^{\alpha,\alpha/2}(Q_T)$, b_0, $u_0 \in C^{2+\alpha}(\Omega)$, a, $b \in C^{2+\alpha,\,1+\alpha/2}(\Sigma_T)$, *where* $\alpha \in (0, 1)$ *and* ε *is an arbitrarily small positive number. Assume further* $a\cdot n = 0$ *and that the following compatibility conditions hold:* $\nabla\cdot g = 0$, $\nabla\cdot b_0 = 0$, $\nabla\cdot u_0 = 0$, $u_0|_S = a(x, 0)$, $b_{0\tau} = b(x, 0)$, $b\cdot n = 0$, $a_{\tau t}(x, 0) = u_{(1)\tau}|_S$, $b_t(x, 0) = b_{(1)\tau}|_S$ *where* $u_{(1)}$, $b_{(1)}$ *are found from* (1.4)-(A′) *and initial conditions,* i.e.

$$b_{(1)} = \eta_0 \Delta b_0 + \beta[\nabla \times (H_0 \times (\nabla \times b_0)) + \nabla \times (b_0 \times (\nabla \times H_0))] + \beta_1 \nabla \times \times \{H_0 \times [H_0 \times (\nabla \times b_0)] + H_0 \times [b_0 \times (\nabla \times H_0)] + b_0 \times [H_0 \times (\nabla \times H_0)]\} + + \nabla \times (U_0 \times b_0 + u_0 \times H_0) + g(x, 0). \tag{1.15}$$

$$u_{(1)} = -U_0\cdot\nabla u_0 - u_0\cdot\nabla U_0 + + \mu\rho_0^{-1}[H_0\cdot\nabla b_0 + b_0\cdot\nabla H_0] + \nu\Delta u_0 + f(x, 0) - \rho_0^{-1}\nabla p_0(x), \tag{1.16}$$

and $p_0(x)$ *is a solution of the Neumann problem*

$$\begin{cases} \rho_0^{-1}\Delta p_0 = \nabla\cdot\{\mu\rho_0^{-1}[H_0\cdot\nabla b_0 + b_0\cdot\nabla H_0] - U_0\cdot\nabla u_0 - u_0\cdot\nabla U_0 + f\}, \\ \left.\dfrac{\partial p_0}{\partial n}\right|_S = \\ = \left\{\dfrac{\mu}{\rho_0}[H_0\cdot\nabla b_0 + b_0\cdot\nabla H_0] - U_0\cdot\nabla u_0 - u_0\cdot\nabla U_0 + f - \nu\nabla \times \nabla \times u_0\right\}\cdot n. \end{cases} \tag{1.17}$$

Then problem (1.4), (1.7)-(A′) *has a unique solution* $u \in C^{2+\alpha,\,1+\alpha/2}(Q_T)$, $\nabla p \in \in C^{\alpha,\alpha/2}(Q_T)$, $b \in C^{2+\alpha,\,1+\alpha/2}(Q_T)$ *and*

$$|u|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} + |b|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq \leq c_4(T)(|f|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_T)} + |g|_{C^{\alpha,\alpha/2}(Q_T)} + |u_0|_{C^{2+\alpha}(\Omega)} + + |b_0|_{C^{2+\alpha}(\Omega)} + |a|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + |b|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)}). \tag{1.18}$$

2) *Assume that* S, H_0, U_0 *are as in the case* 1). *For arbitrary* $f \in C^{\alpha,(\alpha+\varepsilon)/2}(Q_T)$, $g \in C^{\alpha,\alpha/2}(Q_T)$, b_0, $u_0 \in C^{2+\alpha}(\Omega)$, $b \in C^{2+\alpha,\,1+\alpha/2}(\Sigma_T)$, $c \in C^{1+\alpha,\,1/2+\alpha/2}(\Sigma_T)$ *such that*

$$\nabla\cdot g = 0, \quad \nabla\cdot b_0 = 0, \quad \nabla\cdot u_0 = 0, \quad u_0\cdot n|_S = 0, \quad b_{0\tau} = b(x, 0), \quad b\cdot n = 0,$$

$$c\cdot n = 0, \quad [D(u_0)\cdot n - (n\cdot D(u_0)\cdot n)n]|_S = c(x, 0), \quad b_t(x, 0) = b_\tau^{(1)}|_S,$$

problem (1.4), (1.7)-(B′) *has a unique solution* $\boldsymbol{u} \in C^{2+\alpha, 1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha, \alpha/2}(Q_T)$, $\boldsymbol{h} \in C^{2+\alpha, 1+\alpha/2}(Q_T)$ *and*

$$(1.19) \qquad |\boldsymbol{u}|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha, \alpha/2}(Q_T)} + |\boldsymbol{h}|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} \leqslant$$
$$\leqslant c_5(T)\big(|\boldsymbol{f}|_{C^{\alpha, (\alpha+\varepsilon)/2}(Q_T)} + |\boldsymbol{g}|_{C^{\alpha, \alpha/2}(Q_T)} + |\boldsymbol{u}_0|_{C^{2+\alpha}(\Omega)} +$$
$$+ |\boldsymbol{h}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{b}|_{C^{2+\alpha, 1+\alpha/2}(\Sigma_T)} + |\boldsymbol{c}|_{C^{1+\alpha, 1/2+\alpha/2}(\Sigma_T)}\big).$$

3) *Assume that* S, $\boldsymbol{H}_0$, $\boldsymbol{U}_0$ *are as in the case* 1). *For arbitrary* $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{a}, \boldsymbol{u}_0$, *satisfying the hypotheses of n°* 1) *and arbitrary* $\boldsymbol{h}_0 \in C^{2+\alpha}(\Omega)$, $\boldsymbol{d} \in C^{1+\alpha, 1/2+\alpha/2}(\Sigma_T)$, *such that*

$$\nabla \cdot \boldsymbol{h}_0 = 0, \qquad \boldsymbol{h}_0 \cdot \boldsymbol{n}|_S = 0, \qquad \boldsymbol{h}^{(1)} \cdot \boldsymbol{n}|_S = 0,$$
$$B\tau(\boldsymbol{h}_0)|_S = \boldsymbol{d}(x, 0), \qquad \boldsymbol{d} \cdot \boldsymbol{n} = 0, \qquad [\boldsymbol{g} - \nabla \times \boldsymbol{d}] \cdot \boldsymbol{n}|_S = 0,$$

problem (1.4), (1.7)-(C′) *has a unique solution* $\boldsymbol{u} \in C^{2+\alpha, 1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha, \alpha/2}(Q_T)$, $\boldsymbol{h} \in C^{2+\alpha, 1+\alpha/2}(Q_T)$ *and for this solution the estimate*

$$(1.20) \qquad |\boldsymbol{u}|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha, \alpha/2}(Q_T)} + |\boldsymbol{h}|_{C^{2+\alpha, 1+\alpha/2}(Q_T)} \leqslant$$
$$\leqslant c_6(T)\big(|\boldsymbol{f}|_{C^{\alpha, (\alpha+\varepsilon)/2}(Q_T)} + |\boldsymbol{g}|_{C^{\alpha, \alpha/2}(Q_T)} + |\boldsymbol{u}_0|_{C^{2+\alpha}(\Omega)} +$$
$$+ |\boldsymbol{h}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{a}|_{C^{2+\alpha, 1+\alpha/2}(\Sigma_T)} + |\boldsymbol{d}|_{C^{1+\alpha, 1/2+\alpha/2}(\Sigma_T)}\big)$$

holds.

The following remarks can be made concerning these two theorems. Under the appropriate hypotheses on $\boldsymbol{U}_0$ and $\boldsymbol{H}_0$, all three assertions of Theorem 1 hold for arbitrary $p > 1$. This can be easily deduced from the results of the papers [17, 18]. In the compatibility conditions (1.10), (1.13), the traces of the functions $\boldsymbol{D}(\boldsymbol{u}_0) \cdot \boldsymbol{n} - (\boldsymbol{n} \cdot \boldsymbol{D}(\boldsymbol{u}_0) \cdot \boldsymbol{n})\boldsymbol{n}$ and $B_\tau(\boldsymbol{u}_0) \in W_p^{1-2/p}(\Omega)$ at the boundary S are defined for $p > 3$. In the limiting case $p = 3$ the condition $f_1(x)|_S = f_2(x, 0)$ with $f_1 \in W_p^{1-2/p}(\Omega)$, $f_2 \in W_p^{1-1/p, 1/2-1/(2p)}(\Sigma_T)$ should be understood in a generalized sense as the conditions of the boundeness of the integral

$$\int_0^T dt \int_S dS_x \int_\Omega |f_1(y) - f_2(x, t)|^3 \cdot [t + |x - y|^2]^{-5/2} \, dy,$$

(see [19]). In particular, when $f_1(y) = 0$ or $f_2(x, 0) = 0$ it is reduced to

$$\int_S \int_0^T \frac{|f_2(x, t)|^3}{t} \, dt \, dS_x < \infty, \quad \text{and} \quad \int_\Omega |f_1(y)|^3 \frac{dy}{\operatorname{dist}(y, S)} < \infty$$

respectively. Condition $[\boldsymbol{g} - \nabla \times \boldsymbol{d}] \cdot \boldsymbol{n}|_S = 0$ should be understood in a weak sense as $\int_\Omega (\boldsymbol{g} - \nabla \times \boldsymbol{d}^*) \cdot \nabla \varphi \, dx = 0$, $\forall t \in (0, T)$, for arbitrary smooth φ (where $\boldsymbol{d}^*$ is an extension of $\boldsymbol{d}$ into Ω). In Theorem 2 the norm $|\boldsymbol{f}|_{C^{\alpha, (\alpha+\varepsilon)/2}(Q_T)}$ is defined by an obvious relation

$$|\boldsymbol{f}|_{C^{\alpha, (\alpha+\varepsilon)/2}(Q_T)} = \sup_{Q_T} |\boldsymbol{f}(x, t)| + \sup_{x, y, t} \frac{|\boldsymbol{f}(x, t) - \boldsymbol{f}(y, t)|}{|x - y|^\alpha} + \sup_{x, t, \tau} \frac{|\boldsymbol{f}(x, t) - \boldsymbol{f}(x, \tau)|}{|t - \tau|^{(\alpha+\varepsilon)/2}}.$$

The number ε can be taken zero if $\boldsymbol{f}(x, t)$ satisfies the conditions $\nabla \cdot \boldsymbol{f} = 0$, $\boldsymbol{n} \cdot \boldsymbol{f}|_{\Sigma_T} = 0$ (see [17]).

2. Auxiliary linear problems

First of all we consider auxiliary linear problems for the magnetic field $\boldsymbol{b}$.

$$
(2.1)\qquad \begin{cases} L\boldsymbol{b} \equiv \boldsymbol{b}_t - \eta_0 \Delta \boldsymbol{b} - \beta\{\nabla \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] + \nabla \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)]\} - \\ \qquad -\beta_1 \nabla \times \{\boldsymbol{H}_0 \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] + \\ \qquad +\boldsymbol{H}_0 \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)] + \boldsymbol{b} \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{H}_0)]\} = \boldsymbol{G}, \\ \nabla \cdot \boldsymbol{b} = 0, \\ \boldsymbol{b}(x, 0) = \boldsymbol{b}_0(x) \quad \text{in } \Omega, \end{cases}
$$

$$
(i)\qquad \boldsymbol{b}_\tau = \boldsymbol{b} \quad \text{on } \Sigma_T,
$$

$$
(ii)\qquad \begin{cases} \boldsymbol{b} \cdot \boldsymbol{n} = 0, \\ B_\tau(\boldsymbol{b}) \equiv \eta_0 [\nabla \times \boldsymbol{b}]_\tau - \beta[\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})]_\tau - \\ \qquad -\beta[\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)]_\tau - \beta_1 \{\boldsymbol{H}_0 \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})]\}_\tau - \\ \qquad -\beta_1 \{\boldsymbol{H}_0 \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)]\}_\tau - \beta_1 \{\boldsymbol{b} \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{H}_0)]\}_\tau = \boldsymbol{d}. \end{cases}
$$

The system (2.1) is an overdetermined parabolic system (see [20]). The following two lemmata show that problems (2.1)-(i), (2.1)-(ii) can be reduced to parabolic initial-boundary value problems

$$
(2.2)\qquad L\boldsymbol{b} = \boldsymbol{G}, \quad \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \quad \nabla \cdot \boldsymbol{b}|_S = 0, \quad \boldsymbol{b}_{1\tau}|_S = \boldsymbol{b},
$$

$$
(2.3)\qquad L\boldsymbol{b} = \boldsymbol{G}, \quad \boldsymbol{b}|_{t=0} = \boldsymbol{b}_0, \quad \boldsymbol{b} \cdot \boldsymbol{n}|_S = 0, \quad B_\tau(\boldsymbol{b})|_S = \boldsymbol{d}.
$$

Lemma 1. *If $\nabla \cdot \boldsymbol{b}_0 = 0$ in Ω, $\nabla \cdot \boldsymbol{G} = 0$ in Q_T, then the solution of problem* (2.2) *satisfies the equation $\nabla \cdot \boldsymbol{b} = 0$ in Q_T.*

Proof. Setting $r = \nabla \cdot \boldsymbol{b}$, we easily obtain from (2.2) the Dirichlet problem

$$
(2.4)\qquad \begin{cases} \partial r / \partial t = \eta_0 \Delta r & \text{in } Q_T, \\ r = 0 & \text{on } \Sigma_T, \\ r(x, 0) = 0 & \text{in } \Omega, \end{cases}
$$

which has the unique solution $\nabla \cdot \boldsymbol{b} = 0$. The lemma is proved.

Lemma 2. *If $\nabla \cdot \boldsymbol{b}_0 = 0$ in Ω, $\nabla \cdot \boldsymbol{G} = 0$ in Q_T, $\boldsymbol{G} \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \nabla \times \boldsymbol{d} = 0$ on Σ_T, then the solution of problem* (2.3) *satisfies the equation $\nabla \cdot \boldsymbol{b} = 0$ in Q_T;* [8].

Proof. We have already seen that $r = \nabla \cdot \boldsymbol{u}$ satisfies the heat equation $\partial r / \partial t - \eta_0 \Delta r = 0$ and a homogeneous initial condition $r(x, 0) = 0$. Let us show that relations (2.3) yield a homogeneous Neumann condition $\partial r / \partial n|_S = 0$. We have

$$
0 = \partial \boldsymbol{b} / \partial t \cdot \boldsymbol{n}|_S = \eta_0 \Delta \boldsymbol{b} \cdot \boldsymbol{n}|_S + \boldsymbol{n} \cdot (\beta\{\nabla \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] + \nabla \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)]\} + \\ +\beta_1 (\nabla \times \{\boldsymbol{H}_0 \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{b})] + \boldsymbol{H}_0 \times [\boldsymbol{b} \times (\nabla \times \boldsymbol{H}_0)] + \\ +\boldsymbol{b} \times [\boldsymbol{H}_0 \times (\nabla \times \boldsymbol{H}_0)]\}))|_S + \boldsymbol{n} \cdot \boldsymbol{G}|_S.
$$

From the identity

$$
(2.5)\qquad \Delta \boldsymbol{u} = -\nabla \times \nabla \times \boldsymbol{u} + \nabla(\nabla \cdot \boldsymbol{u})
$$

and the condition $\nabla \times \boldsymbol{d}\cdot\boldsymbol{n} = \boldsymbol{G}\cdot\boldsymbol{n}$ it follows that $\eta_0(\partial r/\partial n)|_S = -\boldsymbol{G}\cdot\boldsymbol{n} + \nabla \times \times \boldsymbol{d}\cdot\boldsymbol{n} = 0$, hence, r is a solution of a homogeneous Neumann problem for the heat equation which has a unique solution $r = 0$. The lemma is proved.

We observe at the conclusion that relation $(2.4)_1$ and analogous relation in Lemma 2 should be understood in a weak sense, since the derivatives $(\partial\nabla\cdot\boldsymbol{b})/\partial t$ and $(\partial^2\nabla\cdot\boldsymbol{b})/(\partial x_i\,\partial x_j)$ are not defined as elements of $L_p(Q_T)$. Therefore the above arguments have a formal character but it is not difficult to make them rigorous. The idea of elemination of equation $\nabla\cdot\boldsymbol{b} = 0$ was used in the papers [21-23] and in [8] where it had been applied to problem (2.3).

We turn our attention to problems (2.2), (2.3) and show that they are parabolic, *i.e.* that the system $L\boldsymbol{b} = \boldsymbol{G}$ is parabolic and that the *complementing condition* is satisfied. We write the operator L in the form

$$L\boldsymbol{b} = \partial\boldsymbol{b}/\partial t + \mathcal{A}\boldsymbol{b} \tag{2.6}$$

and consider the principal part $\mathcal{A}_0$ of the operator $\mathcal{A}$ with coefficients «frozen» at arbitrary point $(x_0, t_0) \in Q_T$. Clearly,

$$\mathcal{A}_0(x_0, t_0, \partial/\partial x)\boldsymbol{b} = -\eta_0\Delta\boldsymbol{b} - \beta\{\nabla\times[\boldsymbol{H}_0(x_0,t_0)\times(\nabla\times\boldsymbol{b})]\} -$$
$$-\beta_1\nabla\times\{\boldsymbol{H}_0(x_0,t_0)\times[\boldsymbol{H}_0(x_0,t_0)\times(\nabla\times\boldsymbol{b})]\}.$$

Hence, $\mathcal{A}_0(x_0, t_0, i\xi) = \eta_0|\xi|^2 I + \beta R(\xi)R(\boldsymbol{H}_0)R(\xi) + \beta_1 R(\xi)R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(\xi)$, where $\xi \in \boldsymbol{R}^3$, I is 3×3 identity matrix and

$$R(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

For arbitrary $\eta \in \boldsymbol{C}^3$ we consider the quadratic form

$$(\mathcal{A}_0(\xi)\eta, \eta) = \sum_{i,j=1}^{3} \mathcal{A}_{0ij}\eta_j\overline{\eta_i} = \eta_0|\xi|^2|\eta|^2 + \tag{2.7}$$
$$+\beta(R(\xi)R(\boldsymbol{H}_0)R(\xi)\eta, \eta) + \beta_1(R(\xi)R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(\xi)\eta, \eta)$$

where $\bar{a}$ is the complex conjugate of $a \in \boldsymbol{C}$. Since $(R(\xi)\zeta_1, \zeta_2) = -(\zeta_1, R(\xi)\zeta_2)$, we have $\operatorname{Re}(R(\xi)\zeta, \zeta) = 0$, $\forall\zeta \in \boldsymbol{C}^3$ and $\operatorname{Re}(\mathcal{A}_0(\xi)\eta, \eta) = \eta_0|\xi|^2|\eta|^2 + \beta_1|R(\boldsymbol{H}_0)R(\xi)\eta|^2 \geqslant \eta_0|\xi|^2|\eta|^2$. This shows that the operator $\mathcal{A}_0$ is strongly elliptic [24], hence, the operator L is strongly parabolic which implies the parabolicity in the sense of Petrovskii.

To verify the complementing conditions for problem (2.2), (2.3), we have to prove the solvability of model problems for a homogeneous system of ordinary differential equations arising after «freezing» the coefficients of the operator $\partial/\partial t + \mathcal{A}_0$ at an arbitrary point $x_0 \in S$, $t_0 \in (0, T)$ and making (formally) the Laplace transform with respect to t and the Fourier transform with respect to the tangential space variables at the point

x_0. Consider first problem (2.2). Since the translation and the rotation of coordinate axes leaves the system $\partial \boldsymbol{b}/\partial t + \mathcal{A}_0(x_0, t_0, \partial/\partial x)\boldsymbol{b} = 0$ invariant, the model problem mentioned above has a form

$$(2.8)\quad \begin{cases} p\tilde{\boldsymbol{b}} + \eta_0(\xi_1^2+\xi_2^2 - d^2/dz^2)\tilde{\boldsymbol{b}} - \beta R(i\xi_1, i\xi_2, d/dz)R(\boldsymbol{H}_0)R(i\xi_1, i\xi_2, d/dz)\tilde{\boldsymbol{b}} - \\ \quad -\beta_1 R(i\xi_1, i\xi_2, d/dz)R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(i\xi_1, i\xi_2, d/dz)\tilde{\boldsymbol{b}} = 0 \quad \text{for } z > 0, \\ \tilde{b}_\alpha|_{z=0} = b_\alpha, \quad \alpha = 1, 2, \\ [d\tilde{b}_3/dz + i\xi_1\tilde{b}_1 + i\xi_2\tilde{b}_2]|_{z=0} = e, \quad \tilde{\boldsymbol{b}} \underset{z\to\infty}{\longrightarrow} 0. \end{cases}$$

The complementing condition is equivalent to the assertion that problem (2.8) has a unique solution for arbitrary complex-valued b_α, e, arbitrary $\xi \in \boldsymbol{R}^2$ and $p \in \boldsymbol{C}$ with $\mathrm{Re}\, p \geqslant 0$ (or more generally, $\mathrm{Re}\, p \geqslant -\delta\xi^2$, $\delta \in (0, \eta_0)$), $|p| + \xi^2 \neq 0$. In fact, it suffices to prove the uniqueness of the solution, *i.e.* to show that the homogeneous problem has only a trivial solution $\tilde{\boldsymbol{b}} = 0$. This can be deduced from «energy estimate» for our model problem (see [23] in this connection). Let $\tilde{\boldsymbol{b}}$ be a solution of a homogeneous problem (2.8) (*i.e.* with $b_\alpha = 0$, $e = 0$). Multiplying scalarly $(2.8)_1$ by $\tilde{\boldsymbol{b}}$ and integrating from 0 to ∞, we obtain

$$(2.9)\quad \mathrm{Re}\,[p + \eta_0(\xi_1^2 + \xi_2^2)]\int_0^\infty |\tilde{\boldsymbol{b}}|^2\, dz + \eta_0\int_0^\infty |d\tilde{\boldsymbol{b}}/dz|^2\, dz -$$
$$-\beta\, \mathrm{Re}\int_0^\infty (R(i\xi_1, i\xi_2, d/dz)R(\boldsymbol{H}_0)R(i\xi_1, i\xi_2, d/dz)\tilde{\boldsymbol{b}}, \tilde{\boldsymbol{b}})\, dz -$$
$$-\beta_1\, \mathrm{Re}\int_0^\infty (R(i\xi_1, i\xi_2, d/dz)R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(i\xi_1, i\xi_2, d/dz)\tilde{\boldsymbol{b}}, \tilde{\boldsymbol{b}})\, dz = 0.$$

Now we observe that the operator R possesses the properties $R_{lj} = -R_{jl}$, $\forall l, j = 1, 2, 3$,

$$\int_0^\infty (R\boldsymbol{u}, \boldsymbol{v})\, dz = \int_0^\infty (\boldsymbol{u}, R\boldsymbol{v})\, dz, \quad \text{when } u_\alpha|_{z=0} = 0, \quad \text{and } u_\alpha \to 0 \text{ for } z \to \infty.$$

Hence, after integration by parts in (2.9) we obtain

$$\mathrm{Re}\,[p + \eta_0(\xi_1^2 + \xi_2^2)]\int_0^\infty |\tilde{\boldsymbol{b}}|^2\, dz + \eta_0\int_0^\infty |d\tilde{\boldsymbol{b}}/dz|^2 dz +$$
$$+\beta_1\, \mathrm{Re}\int_0^\infty |R(\boldsymbol{H}_0)R(i\xi_1, i\xi_2, d/dz)\tilde{\boldsymbol{b}}|^2\, dz = 0.$$

This implies that $\tilde{\boldsymbol{b}} = 0$ for $z > 0$ and hence the complementing condition is verified.

By similar arguments it can be shown that the complementing condition holds also for problem (2.3). In this case it is necessary to show that the problem

$$(2.10)\quad \begin{cases} p\tilde{\boldsymbol{b}}+\eta_0(\xi_1^2+\xi_2^2-d^2/dz^2)\tilde{\boldsymbol{b}}-\beta R(i\xi_1,i\xi_2,d/dz)R(\boldsymbol{H}_0)R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}- \\ \quad -\beta_1 R(i\xi_1,i\xi_2,d/dz)R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}=0 \quad \text{for } z>0, \\ \tilde{b}_3|_{z=0}=0, \\ [\eta_0 R\tilde{\boldsymbol{b}}-\beta R(\boldsymbol{H}_0)R\tilde{\boldsymbol{b}}-\beta_1 R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R\tilde{\boldsymbol{b}}]_\alpha|_{z=0}=0, \quad \alpha=1,2, \\ \tilde{\boldsymbol{b}}\underset{z\to\infty}{\longrightarrow}0, \end{cases}$$

has only a trivial solution. $(2.10)_1$ can be written in the form

$$p\tilde{\boldsymbol{b}}-\eta_0(i\xi_1,i\xi_2,d/dz)(i\xi_1\tilde{b}_1+i\xi_2\tilde{b}_2+d\tilde{b}_3/dz)+$$
$$+R(i\xi_1,i\xi_2,d/dz)[\eta_0 R(i\xi_1,i\xi_2,d/z)-\beta R(\boldsymbol{H}_0)R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}-$$
$$-\beta_1 R(\boldsymbol{H}_0)R(\boldsymbol{H}_0)R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}]=0.$$

Multiplying this equation by $\tilde{\boldsymbol{b}}$ and integrating from 0 to ∞, we obtain after integration by parts

$$\int_0^\infty [p|\tilde{\boldsymbol{b}}|^2+\eta_0|(i\xi_1\tilde{b}_1+i\xi_2\tilde{b}_2+d\tilde{b}_3/dz)|^2+\eta_0|R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}|^2-$$
$$-\beta(R(\boldsymbol{H}_0)R\tilde{\boldsymbol{b}},R\tilde{\boldsymbol{b}})+\beta_1|R(\boldsymbol{H}_0)R(i\xi_1,i\xi_2,d/dz)\tilde{\boldsymbol{b}}|^2]\,dz=0.$$

Taking the real part of the last expression we easily obtain as before $\tilde{\boldsymbol{b}}=0$ for $z>0$ which is equivalent to the complementing condition for problem (2.3).

Now we are able to apply to (2.2), (2.3) the theory of general parabolic initial-boundary value problems. The following propositions are particular cases of Theorems 1.2 in [18] and 4.9 in [25].

THEOREM 3. 1) *Let* $S\in C^3$, $\boldsymbol{H}_0\in W_p^{2,1}(Q_T)$, $p>5/2$. *For arbitrary* $\boldsymbol{G}\in L_p(Q_T)$, $\boldsymbol{b}_0\in W_p^{2-2/p}(\Omega)$, $\boldsymbol{b}\in W_p^{2-1/p,1-1/2p}(\Sigma_T)$, *satisfying the conditions* $\boldsymbol{b}_{0\tau}|_S=\boldsymbol{b}(x,0)$, $\boldsymbol{b}\cdot\boldsymbol{n}=0$, $\nabla\cdot\boldsymbol{b}_0|_S=0$, *problem* (2.2) *has a unique solution* $\boldsymbol{b}\in W_p^{2,1}(Q_T)$ *satisfying the inequality*

$$(2.11)\quad \|\boldsymbol{b}\|_{W_p^{2,1}(Q_T)}\le C_7(T)(\|\boldsymbol{G}\|_{L_p(Q_T)}+\|\boldsymbol{b}_0\|_{W_p^{2-2/p}(\Omega)}+\|\boldsymbol{b}\|_{W_p^{2-1/p,1-1/2p}(\Sigma_T)}).$$

2) *For arbitrary* $\boldsymbol{G}\in L_p(Q_T)$, $\boldsymbol{b}_0\in W_p^{2-2/p}(\Omega)$, $\boldsymbol{d}\in W_p^{1-1/p,1/2-1/2p}(\Sigma_T)$, *satisfying conditions* $\boldsymbol{b}_0\cdot\boldsymbol{n}|_S=0$, $\boldsymbol{d}\cdot\boldsymbol{n}=0$, *and* $B_\tau(\boldsymbol{b}_0)|_S=\boldsymbol{d}(x,0)$, *if* $p\ge 3$, *problem* (2.3) *has a unique solution* $\boldsymbol{b}\in W_p^{2,1}(Q_T)$ *satisfying the inequality*

$$(2.12)\quad \|\boldsymbol{b}\|_{W_p^{2,1}(Q_T)}\le C_8(T)(\|\boldsymbol{G}\|_{L_p(Q_T)}+\|\boldsymbol{b}_0\|_{W_p^{2-2/p}(\Omega)}+\|\boldsymbol{d}\|_{W_p^{1-1/p,1/2-1/2p}(\Sigma_T)}).$$

THEOREM 4. 1) *Let* $S\in C^{3+\alpha}$, $\boldsymbol{H}_0\in C^{2+\alpha,1+\alpha/2}(Q_T)$, *with* $\alpha\in(0,1)$. *For arbitrary* $\boldsymbol{G}\in C^{\alpha,\alpha/2}(Q_T)$, $\boldsymbol{b}_0\in C^{2+\alpha}(\Omega)$, $\boldsymbol{b}\in C^{2+\alpha,1+\alpha/2}(\Sigma_T)$, *satisfying the conditions* $\boldsymbol{b}_{0\tau}|_S=\boldsymbol{b}(x,0)$, $\boldsymbol{b}\cdot\boldsymbol{n}=0$, $\nabla\cdot\boldsymbol{b}_0|_S=0$, $\boldsymbol{b}_{(1)\tau}|_S=\boldsymbol{b}_t(x,0)$ *where* $\boldsymbol{b}_{(1)}(x)=$ $=\mathcal{Q}(x,0,\partial/\partial x)\boldsymbol{b}_0$ *problem* (2.2) *has a unique solution* $\boldsymbol{b}\in C^{2+\alpha,1+\alpha/2}(Q_T)$ *and for this* *solution the inequality*

$$(2.13)\qquad |\boldsymbol{b}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq C_9(T)\big(|\boldsymbol{G}|_{C^{\alpha,\alpha/2}(Q_T)} + |\boldsymbol{b}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{b}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)}\big)$$

holds.

2) *For arbitrary* $\boldsymbol{G} \in C^{\alpha,\alpha/2}(Q_T)$, $\boldsymbol{b}_0 \in C^{2+\alpha}(\Omega)$, $\boldsymbol{d} \in C^{1+\alpha,1/2+\alpha/2}(\Sigma_T)$, *satisfying the conditions* $\boldsymbol{b}_0 \cdot \boldsymbol{n}|_S = b(x, 0)$, $\boldsymbol{d}\cdot\boldsymbol{n} = 0$, $B_\tau(\boldsymbol{b}_0)|_S = \boldsymbol{d}(x, 0)$, $\boldsymbol{b}_{(1)}\cdot\boldsymbol{n}|_S = 0$, *problem* (2.3) *has a unique solution* $\boldsymbol{b} \in C^{2+\alpha,1+\alpha/2}(Q_T)$ *and*

$$(2.14)\qquad |\boldsymbol{b}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq C_{10}(T)\big(|\boldsymbol{G}|_{C^{\alpha,\alpha/2}(Q_T)} + |\boldsymbol{b}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{d}|_{C^{1+\alpha,1/2+\alpha/2}(\Sigma_T)}\big).$$

The constants C_7, C_8, C_9, C_{10} *are non-decreasing functions of T.*

Taking account of Lemmata 1 and 2 we arrive at the following existence theorems for problems (2.1)-(*i*) and (2.1)-(*ii*).

THEOREM 5. 1) *Let S,* $\boldsymbol{H}_0$, $\boldsymbol{G}$, $\boldsymbol{b}_0$, $\boldsymbol{b}$, *p, satisfy the hypotheses of Theorem* 3, *n°* 1) *and of Lemma* 1. *Then problem* (2.1)-(*i*) *has a unique solution* $\boldsymbol{b} \in W_p^{2,1}(Q_T)$ *and this solution satisfies inequality* (2.11).

2) *If* $\boldsymbol{G}$, $\boldsymbol{b}_0$, $\boldsymbol{d}$, *satisfy the hypotheses of theorem,* 3, *n°* 2) *and of Lemma* 2, *then problem* (2.1)-(*ii*) *has a unique solution* $\boldsymbol{b} \in W_p^{2,1}(Q_T)$ *satisfying inequality* (2.12).

THEOREM 6. 1) *Let S,* $\boldsymbol{H}_0$, $\boldsymbol{G}$, $\boldsymbol{b}_0$, $\boldsymbol{b}$, *p, satisfy the hypotheses of Theorem* 4, *n°* 1) *and of Lemma* 1. *Then problem* (2.1)-(*i*) *has a unique solution* $\boldsymbol{b} \in C^{2+\alpha,1+\alpha/2}(Q_T)$ *and for this solution inequality* (2.13) *holds.*

2) *If* $\boldsymbol{G}$, $\boldsymbol{b}_0$, $\boldsymbol{d}$, *satisfy the hypotheses of Theorem* 4, *n°* 2) *and of Lemma* 2, *then problem* (2.1)-(*ii*) *has a unique solution* $\boldsymbol{b} \in C^{2+\alpha,1+\alpha/2}(Q_T)$ *satisfying inequality* (2.14).

3. PROOF OF THEOREMS 1 AND 2

In this section we prove n° 1) of Theorem 1 and n° 3) of Theorem 2. We start with Theorem 1. To establish the solvability of problem (1.4), (1.7)-(A′), we use the method of successive approximations. We set $\boldsymbol{u}^{(0)} = 0$, $p^{(0)} = 0$, $\boldsymbol{b}^{(0)} = 0$ and define $\boldsymbol{u}^{(m+1)}$, $p^{(m+1)}$, $\boldsymbol{b}^{(m+1)}$, $m \geq 0$, as the solutions of linear problems

$$(3.1)\qquad \begin{cases} L_{01}(\boldsymbol{u}^{(m+1)}, p^{(m+1)}) = \boldsymbol{f} - l_1(\boldsymbol{u}^{(m)}, \boldsymbol{b}^{(m)}), & \nabla\cdot\boldsymbol{u}^{(m+1)} = 0, \\ \boldsymbol{u}^{(m+1)}|_{t=0} = \boldsymbol{u}_0(x), & \boldsymbol{u}^{(m+1)}|_{\Sigma_T} = \boldsymbol{a}, \end{cases}$$

$$(3.2)\qquad \begin{cases} L_{02}(\boldsymbol{b}^{(m+1)}) = \boldsymbol{g} - l_2(\boldsymbol{u}^{(m)}, \boldsymbol{b}^{(m)}), & \nabla\cdot\boldsymbol{u}^{(m+1)} = 0, \\ \boldsymbol{b}^{(m+1)}|_{t=0} = \boldsymbol{b}_0(x), & \boldsymbol{b}_\tau^{(m+1)}|_{\Sigma_T} = \boldsymbol{b} \end{cases}$$

where

$$L_{01}(\boldsymbol{u},p)=\boldsymbol{u}_t-\nu\Delta\boldsymbol{u}+\rho_0^{-1}\nabla p\,,$$

$$l_1(\boldsymbol{u},\boldsymbol{b})=\boldsymbol{U}_0\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{U}_0+\mu\rho_0^{-1}[\boldsymbol{H}_0\cdot\nabla\boldsymbol{b}+\boldsymbol{b}\cdot\nabla\boldsymbol{H}_0]\,,$$

$$L_{02}(\boldsymbol{b})=\boldsymbol{b}_t-\eta_0\Delta\boldsymbol{b}-\beta[\nabla\times(\boldsymbol{H}_0\times(\nabla\times\boldsymbol{b}))+\nabla\times(\boldsymbol{b}\times(\nabla\times\boldsymbol{H}_0))]-$$
$$-\beta_1\nabla\times\{\boldsymbol{H}_0\times[\boldsymbol{H}_0\times(\nabla\times\boldsymbol{b})]+\boldsymbol{H}_0\times[\boldsymbol{b}\times(\nabla\times\boldsymbol{H}_0)]+\boldsymbol{b}\times[\boldsymbol{H}_0\times(\nabla\times\boldsymbol{H}_0)]\}\,,$$

$$l_2(\boldsymbol{u},\boldsymbol{b})=-\nabla\times(\boldsymbol{U}_0\times\boldsymbol{b}+\boldsymbol{u}\times\boldsymbol{H}_0)\,.$$

Making use of the assumptions $\boldsymbol{U}_0\in W_p^{2,1}(Q_T)$, $\boldsymbol{H}_0\in W_p^{2,1}(Q_T)$, it is not hard to prove that

$$(3.3)\qquad \|l_1(\boldsymbol{u},\boldsymbol{b})\|_{L_p(Q_t)}+\|l_2(\boldsymbol{u},\boldsymbol{b})\|_{L_p(Q_t)}\leqslant$$
$$\leqslant\varepsilon(\|\boldsymbol{u}\|_{W_p^{2,1}(Q_t)}+\|\boldsymbol{b}\|_{W_p^{2,1}(Q_t)})+c(\varepsilon)(\|\boldsymbol{u}\|_{L_p(Q_t)}+\|\boldsymbol{b}\|_{L_p(Q_t)}$$

where ε is an arbitrarily small positive number and $t\in(0,T]$. For instance,

$$\left(\int_0^t\int_\Omega(|\boldsymbol{U}_0\cdot\nabla\boldsymbol{u}|^p+|\boldsymbol{u}\cdot\nabla\boldsymbol{U}_0|^p)\,dx\,d\tau\right)^{1/p}\leqslant$$

$$\leqslant C\sup_{\tau\leqslant t}\|\boldsymbol{U}_0\|_{L_\infty(\Omega)}\left(\int_0^t\|\nabla\boldsymbol{u}\|^p_{L_p(\Omega)}\,d\tau\right)^{1/p}+C\sup_{\tau\leqslant t}\|\nabla\boldsymbol{U}_0\|_{L_p(\Omega)}\left(\int_0^t\|\boldsymbol{u}\|^p_{L_\infty(\Omega)}\,d\tau\right)^{1/p}.$$

From inequality (1.8) and from interpolation inequalities

$$\|\nabla\boldsymbol{u}\|_{L_p(\Omega)}\leqslant\varepsilon_1\|\boldsymbol{u}\|_{W_p^2(\Omega)}+c(\varepsilon_1)\|\boldsymbol{u}\|_{L_p(\Omega)}\,,\qquad\forall\varepsilon_1\in(0,1)\,,$$
$$\|\boldsymbol{u}\|_{L_\infty(\Omega)}\leqslant\varepsilon_2\|\boldsymbol{u}\|_{W_p^2(\Omega)}+c(\varepsilon_2)\|\boldsymbol{u}\|_{L_p(\Omega)}\,,\qquad\forall\varepsilon_2\in(0,1)\,,$$

we easily deduce the estimate

$$\left(\int_0^t\|\boldsymbol{U}_0\cdot\nabla\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{U}_0\|^p\,d\tau\right)^{1/p}\leqslant\varepsilon\left(\int_0^t\|\boldsymbol{u}\|^p_{W_p^2(\Omega)}\,d\tau\right)^{1/p}+c(\varepsilon)\left(\int_0^t\|\boldsymbol{u}\|^p_{L_p(\Omega)}\,d\tau\right)^{1/p}.$$

Other terms in l_1 and l_2 can be evaluated in a similar way. Estimates (3.3) show that l_1, $l_2\in L_p(Q_T)$ and consequently problems (3.1), (3.2) are uniquely solvable. Let us show that the sequences $\{\boldsymbol{u}^{(m)}\}$, $\{p^{(m)}\}$, $\{\boldsymbol{b}^{(m)}\}$ are convergent. The differences

$$\boldsymbol{w}^{(m+1)}=\boldsymbol{u}^{(m+1)}-\boldsymbol{u}^{(m)}\,,\qquad s^{(m+1)}=p^{(m+1)}-p^{(m)}\,,\qquad \boldsymbol{z}^{(m+1)}=\boldsymbol{b}^{(m+1)}-\boldsymbol{b}^{(m)}\,,$$

are solutions of the problems

$$L_{01}(\boldsymbol{w}^{(m+1)},s^{(m+1)})=-l_1(\boldsymbol{w}^{(m)},\boldsymbol{z}^{(m)})\,,$$
$$\nabla\cdot\boldsymbol{w}^{(m+1)}=0\,,\qquad\boldsymbol{w}^{(m+1)}|_{t=0}=0\,,\qquad\boldsymbol{w}^{(m+1)}|_S=0\,,$$
$$L_{02}(\boldsymbol{z}^{(m+1)})=-l_2(\boldsymbol{w}^{(m)},\boldsymbol{z}^{(m)})\,,$$
$$\nabla\cdot\boldsymbol{z}^{(m+1)}=0\,,\qquad\boldsymbol{z}^{(m+1)}|_{t=0}=0\,,\qquad\boldsymbol{z}_\tau^{(m+1)}|_{t=0}=0\,.$$

Taking into account the results of [17], Theorem 5 n° 1) and (3.3) we obtain for arbitrary $t \leqslant T$

$$(3.4)\qquad \|w^{(m+1)}\|_{W_p^{2,1}(Q_t)} + \|\nabla s^{(m+1)}\|_{L_p(Q_t)} + \|z^{(m+1)}\|_{W_p^{2,1}(Q_t)} \leqslant$$
$$\leqslant C(T)[\|l_1(w^{(m)}, z^{(m)})\|_{L_p(Q_t)} + \|l_2(w^{(m)}, z^{(m)})\|_{L_p(Q_t)}] \leqslant$$
$$\leqslant \varepsilon(\|w^{(m)}\|_{W_p^{2,1}(Q_t)} + \|z^{(m)}\|_{W_p^{2,1}(Q_t)}) + c(\varepsilon)(\|w^{(m)}\|_{L_p(Q_t)} + \|z^{(m)}\|_{L_p(Q_t)}) .$$

For the norms in $L_p(Q_t)$ we have

$$(3.5)\qquad \|z^{(m)}\|_{L_p(Q_t)} \leqslant \max_{[0,t]} \|z^{(m)}(\cdot, \tau)\|_{L_p(\Omega)}^{1/p'} \left[\int_0^t \|z^{(m)}(\tau)\|_{L_p(\Omega)}\, d\tau\right]^{1/p} \leqslant$$
$$\leqslant t^{1/(pp')} \|z^{(m)}\|_{W_p^{2,1}(Q_t)}^{1/p'} \left[\int_0^t \|z^{(m)}(\tau)\|_{L_p(\Omega)}\, d\tau\right]^{1/p} \leqslant$$
$$\leqslant t^{1/p} \|z^{(m)}\|_{W_p^{2,1}(Q_t)}^{1/p'} \left[\int_0^t \|z^{(m)}(\tau)\|_{W_p^{2,1}(Q_\tau)}\, d\tau\right]^{1/p},$$

where p' is the conjugate of p. By applying the Young inequality, we easily obtain from (3.5)

$$N_{m+1}(t) \leqslant \varepsilon_3 N_m(t) + c(\varepsilon_3)\int_0^t N_m(\tau)\, d\tau, \qquad m \in \mathbf{N}, \qquad m \geqslant 1, \qquad \varepsilon_3 \geqslant 0,$$

where $N_m(t) = \|w^{(m)}\|_{W_p^{2,1}(Q_t)} + \|\nabla s^{(m)}\|_{L_p(Q_t)} + \|z^{(m)}\|_{W_p^{2,1}(Q_t)}$. Setting $\Sigma_N(t) = \sum_{m=1}^{N} N_m(t)$, and making the summation with respect to $m \in [1, N]$ we obtain

$$\Sigma_N(t) \leqslant \Sigma_{N+1}(t) \leqslant \varepsilon_3 \Sigma_N(t) + c(\varepsilon_3)\int_0^t \Sigma_N(\tau)\, d\tau + N_1(t).$$

Assuming $\varepsilon_3 < 1$ we have $\Sigma_N(t) \leqslant c_1(\varepsilon_3)\int_0^t \Sigma_N(\tau)\, d\tau + N_1(t)/(1-\varepsilon_3)$. By the Gronwall's lemma there exists a positive constant $C(t)$ such that

$$(3.6)\qquad \Sigma_N(t) \leqslant C(t) N_1(t) \leqslant C(t)(\|u^{(1)}\|_{W_p^{2,1}(Q_t)} + \|\nabla p^{(1)}\|_{L_p(Q_t)} + \|b^{(1)}\|_{W_p^{2,1}(Q_t)}) \leqslant$$
$$\leqslant C_1(T)(\|f\|_{L_p(Q_T)} + \|g\|_{L_p(Q_T)} + \|b_0\|_{W_p^{2-2/p}(\Omega)} +$$
$$+ \|u_0\|_{W_p^{2-2/p}(\Omega)} + \|a\|_{W_p^{2-1/p,\,1-1/2p}(\Sigma_T)} + \|b\|_{W_p^{2-1/p,\,1-1/2p}(\Sigma_T)}) .$$

The last inequality implies the convergence of the series $\sum_{m=1}^{\infty} N_{m+1}(t)$ and the existence of the solution of the problem (1.4), (1.7)-(A') $u = \lim_{m\to\infty} u^{(m+1)}$, $s = \lim_{m\to\infty} s^{(m+1)}$, $b = \lim_{m\to\infty} b^{(m+1)}$. Estimate (2.11) follows from (3.6). Uniqueness of the solution can also be established by the same kind of arguments. Let (w, p, b) be a solution of homogeneous problem (1.4), (1.7)-(A') Then

$$\|w\|_{W_p^{2,1}(Q_t)} + \|\nabla p\|_{L_p(Q_t)} + \|b\|_{W_p^{2,1}(Q_t)} \leqslant C(\|l_1\|_{L_p(Q_t)} + \|l_2\|_{L_p(Q_t)})$$

and, as a consequence, $N(t) = \|w\|_{W_p^{2,1}(Q_t)} + \|\nabla p\|_{L_p(Q_t)} + \|b\|_{W_p^{2,1}(Q_t)}$ satisfy the inequality $N(t) \leqslant \varepsilon_3 N(t) + c(\varepsilon_3)\int_0^t N(\tau)\,d\tau$, which implies $w = 0$, $\nabla p = 0$, $b = 0$. The first part of theorem is proved.

Other two statements of Theorem 1 are proved in the same way. All the necessary estimates for $u^{(m+1)}$, $p^{(m+1)}$, $b^{(m+1)}$ follow from Theorem 5 and from the results of [26, 27].

Let us turn our attention to Theorem 2, n° 3. We define again iterations $u^{(m)}, p^{(m)}$, $b^{(m)}$ by $u^{(0)} = 0$, $p^{(0)} = 0$, $b^{(0)} = 0$,

$$
(3.7)\qquad \begin{cases} L_{01}(u^{(m,+1)}, p^{(m+1)}) = f - l_1(u^{(m)}, b^{(m)}), & \nabla\cdot u^{(m+1)} = 0\,, \\ u^{(m+1)}|_{t=0} = u_0(x)\,, \quad u^{(m+1)}|_S = a\,, \end{cases}
$$

$$
(3.8)\qquad \begin{cases} L_{02}(b^{(m+1)}) = g - l_2(u^{(m)}, b^{(m)}), \quad \nabla\cdot b^{(m+1)} = 0\,, \\ b^{(m+1)}|_{t=0} = b_0(x)\,, \quad b^{(m+1)}\cdot n|_S = 0\,, \quad B_\tau(b^{(m+1)}) = d\,. \end{cases}
$$

For $l_1(u, b)$ and $l_2(u, b)$ we have the estimate

$$
(3.9)\qquad |l_1(u,b)|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_t)} + |l_2(u,b)|_{C^{\alpha,\alpha/2}(Q_t)} \leqslant
$$

$$
\leqslant C_2(|u|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_t)} + |b|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_t)} + |Du|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_t)} + |Db|_{C^{\alpha,(\alpha+\varepsilon)/2}(Q_t)}) \leqslant
$$

$$
\leqslant \varepsilon_4(|u|_{C^{2+\alpha,1+\alpha/2}(Q_t)} + |b|_{C^{2+\alpha,1+\alpha/2}(Q_t)} + c(\varepsilon_4)\Big(\sup_{Q_T}|u| + \sup_{Q_T}|b|\Big), \qquad \varepsilon_4 > 0
$$

and we observe also that in virtue of boundary conditions $(3.7)_2$, $(3.8)_2$ and (1.12), $(U_0 \times b + u \times H_0)_\tau|_S = 0$ and, as a consequence, $l_2(u\cdot b)\cdot n|_S = 0$. This makes it possible to apply Theorem 6, n° 2) to problem (3.8). From this theorem and from results of papers [17, 25] it follows that $(u^{(m)}, p^{(m)}, b^{(m)})$ are well defined. The differences

$$
w^{(m+1)} = u^{(m+1)} - u^{(m)}\,, \qquad s^{(m+1)} = p^{(m+1)} - p^{(m)}\,, \qquad z^{(m+1)} = b^{(m+1)} - b^{(m)}\,,
$$

are solutions to the problems

$$
\begin{cases} L_{01}(w^{(m+1)}, s^{(m+1)}) = -l_1(w^{(m)}, z^{(m)}), & \nabla\cdot w^{(m+1)} = 0\,, \\ w^{(m+1)}|_{t=0} = 0\,, \quad w^{(m+1)}|_S = 0\,, \end{cases}
$$

$$
\begin{cases} L_{02}(z^{(m+1)}) = -l_2(w^{(m)}, z^{(m)}), \quad \nabla\cdot z^{(m+1)} = 0\,, \\ z^{(m+1)}|_{t=0} = 0\,, \quad z^{(m+1)}\cdot n|_{t=0} = 0\,, \quad B_\tau(z^{(m+1)})|_S = 0\,. \end{cases}
$$

Making use of Theorems of [17, 25], of theorem 6, n° 2) and of the inequality (3.9) and taking into account that

$$\max_{Q_t} |z^{(m)}(x, \tau)| \leq \int_0^t |z^{(m)}(x, \tau)|_{C^{2+\alpha, 1+\alpha/2}(Q_\tau)} d\tau$$

we easily arrive at the estimate $Q_{m+1}(t) \leq \varepsilon_5 Q_m(t) + c(\varepsilon_5) \int_0^t Q_m(\tau) d\tau, \quad \varepsilon_5 > 0$ where

$$Q_{m+1}(t) = |\boldsymbol{w}^{(m+1)}|_{C^{2+\alpha, 1+\alpha/2}(Q_t)} + |\nabla s^{(m+1)}|_{C^{\alpha, \alpha/2}(Q_t)} + |\boldsymbol{z}^{(m+1)}|_{C^{2+\alpha, 1+\alpha/2}(Q_t)}.$$

Further arguments are absolutely the same as in the proof of Theorem 1 and they can be omitted.

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G. Mulone: Dipartimento di Matematica e Applicazioni «Renato Caccioppoli»
Università degli Studi di Napoli «Federico II»
Complesso Monte S. Angelo
Edificio T, Via Cinthia - 80126 Napoli

V. A. Solonnikov: Steklov Institute of Mathematics St. Petersburg
Department Fontanka 27 - 191011 St. Petersburg (Russia)