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Olga A. Oleinik, Tatiana Shaposhnikova

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Analisi matematica. — *On homogeneization problems for the Laplace operator in partially perforated domains with Neumann's condition on the boundary of cavities.* Nota (*)
di Olga A. Oleinik e Tatiana Shaposhnikova, presentata dal Socio O. A. Oleinik.

Abstract. — In this paper the problem of homogeneization for the Laplace operator in partially perforated domains with small cavities and the Neumann boundary conditions on the boundary of cavities is studied. The corresponding spectral problem is also considered.

Key words: Homogeneization; Perforated domains; Small cavities; Neumann's condition; Spectral problem.

Riassunto. — *Sul problema della omogeneizzazione per l'operatore di Laplace in domini parzialmente perforati con condizioni di Neumann sul contorno delle cavità.* In questa *Nota* viene studiato il problema della omogeneizzazione per l'operatore di Laplace in domini parzialmente perforati con piccole cavità e con condizioni di Neumann nel contorno delle cavità. Viene anche considerato il corrispondente problema spettrale.

Introduction

The problem of homogeneization for the Laplace operator in perforated domains with a small density of cavities and the Dirichlet boundary conditions on the boundary of cavities was considered in many papers (see, for example, [1-5]). In this paper we study the problem of homogeneization for the Laplace operator in partially perforated domains with small cavities and the Neumann boundary conditions on the boundary of cavities. The corresponding spectral problems are also considered.

1. Let Ω be a bounded domain in $\boldsymbol{R}_x^n$ with a smooth boundary $\partial\Omega$, $Q = \{x: 0 < x_j < 1, j = 1, \dots, n\}$, G_0 is a domain in Q, $\overline{G}_0 \subset Q$ and $\overline{G}_0$ is diffeomorphic to a closed ball.

We set $G_\varepsilon = \bigcup_{z \in \boldsymbol{Z}} (a_\varepsilon G_0 + \varepsilon z)$, where ε is a small positive parameter, a_ε is a constant which depends on ε and $a_\varepsilon \varepsilon^{-1} \to 0$ as $\varepsilon \to 0$, $\boldsymbol{Z}$ is the set of vectors z with integer components, $\alpha B = \{x: \alpha^{-1} x \in B\}$, $Y_\varepsilon = \varepsilon Q \setminus \overline{a_\varepsilon G_0}$. We assume that $\Omega \cap \{x: x_1 = 0\} = \gamma \neq \emptyset$.

We denote

$$\Omega^+ = \Omega \cap \{x: x_1 > 0\}, \qquad \Omega^- = \Omega \cap \{x: x_1 < 0\}, \qquad \Omega_\varepsilon^+ = \Omega^+ \setminus \overline{G}_\varepsilon ,$$

$$S_0 = \partial G_0 , \qquad \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \gamma \cup \Omega^- , \qquad S_\varepsilon = \partial\Omega_\varepsilon \cap \Omega , \qquad \Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus S_\varepsilon ,$$

$\langle u \rangle_\omega \equiv |\omega|^{-1} \int_\omega u \, dx$, where $|\omega|$ is the volume of the domain ω.

As usual we denote by $H_1(\Omega, \Gamma)$ the space of functions which is obtained by completion of the set of infinitely differentiable in $\overline{\Omega}$ functions $u(x)$, equal to zero in a

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neighborhood of Γ, by the norm

$$\|u\|_{H_1(\Omega)} = \left(\int_\Omega (u^2 + |\nabla u|^2)\, dx \right)^{1/2}.$$

In partially perforated domain Ω_ε we study the Neumann boundary value problem:

$$(1) \qquad \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } S_\varepsilon, \qquad u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon,$$

where ν is the exterior unit normal vector to S_ε, $f \in C^\alpha(\overline{\Omega})$, $\alpha > 0$.

We consider a weak solution $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ of the problem (1) and study the behaviour of u_ε as $\varepsilon \to 0$.

Let us introduce the function $N_j^\varepsilon(y)$ $(j = 1, \dots, n)$ as a 1-periodic solution in $\varepsilon^{-1} Y_\varepsilon$ of the problem:

$$(2) \qquad \begin{cases} \Delta_y N_j^\varepsilon = 0 \quad \text{in } \varepsilon^{-1} Y_\varepsilon, & \dfrac{\partial N_j^\varepsilon}{\partial \nu} = -\nu_j \quad \text{on } a_\varepsilon \varepsilon^{-1} S_0, \\ \langle N_j^\varepsilon \rangle_{\varepsilon^{-1} Y_\varepsilon} = 0. & \end{cases}$$

In order to estimate N_j^ε, we need some auxiliary results.

LEMMA 1. If $u \in H_1(Y_\varepsilon)$ and $\langle u \rangle_{Y_\varepsilon} = 0$, then

$$(3) \qquad \|u\|_{L_2(Y_\varepsilon)} \leqslant K_1 \varepsilon \|\nabla_x u\|_{L_2(Y_\varepsilon)},$$

where $\nabla_x u \equiv (u_{x_1}, \dots, u_{x_n})$, the constant K_1 does not depend on ε.

LEMMA 2. Let $u \in H_1(Y_\varepsilon)$. Then

$$(4) \qquad \|u\|_{L_2(a_\varepsilon S_0)} \leqslant K_2 \{ a_\varepsilon^{(n-1)/2} \varepsilon^{-n/2} \|u\|_{L_2(Y_\varepsilon)} + \sqrt{a_\varepsilon} \|\nabla_x u\|_{L_2(Y_\varepsilon)} \},$$

if $n \geqslant 3$, and

$$(5) \qquad \|u\|_{L_2(a_\varepsilon S_0)} \leqslant K_3 \left\{ \sqrt{a_\varepsilon}\, \varepsilon^{-1} \|u\|_{L_2(Y_\varepsilon)} + \sqrt{a_\varepsilon \ln \frac{\varepsilon}{2 a_\varepsilon}}\, \|\nabla_x u\|_{L_2(Y_\varepsilon)} \right\},$$

if $n = 2$, where all constants K_j here and in what follows do not depend on ε.

We shall give proofs of these lemmas in the appendix.

Using the integral identity for the problem (2), we obtain the inequality

$$(6) \qquad \|\nabla_y N_j^\varepsilon\|^2_{L_2(\varepsilon^{-1} Y_\varepsilon)} \leqslant K_4 \varepsilon^{-(n-1)} a_\varepsilon^{(n-1)/2} \|N_j^\varepsilon\|_{L_2(a_\varepsilon S_0)}.$$

By virtue of inequalities (3)-(5) we have

$$(7) \qquad \|N_j^\varepsilon\|_{L_2(a_\varepsilon S_0)} \leqslant K_5 (a_\varepsilon^{(n-1)/2} \varepsilon^{-n/2+1} + \sqrt{a_\varepsilon}) \|\nabla_x N_j^\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant$$
$$\leqslant K_6 (a_\varepsilon^{(n-1)/2} + \sqrt{a_\varepsilon}\, \varepsilon^{n/2-1}) \|\nabla_y N_j^\varepsilon\|_{L_2(\varepsilon^{-1} Y_\varepsilon)},$$

if $n \geqslant 3$, and

$$(8) \qquad \|N_j^\varepsilon\|_{L_2(a_\varepsilon S_0)} \leqslant K_7 \left\{ \sqrt{a_\varepsilon} + \sqrt{a_\varepsilon \ln \frac{\varepsilon}{2 a_\varepsilon}} \right\} \|\nabla_y N_j^\varepsilon\|_{L_2(\varepsilon^{-1} Y_\varepsilon)},$$

if $n = 2$.

From (6)-(8) we obtain the following estimates:

$$\|\nabla_y N_j^\varepsilon\|_{L_2(\varepsilon^{-1}Y_\varepsilon)} \leqslant K_8\left(\frac{a_\varepsilon}{\varepsilon}\right)^{n/2}, \quad \text{if } n \geqslant 3, \tag{9}$$

$$\|\nabla_y N_j^\varepsilon\|_{L_2(\varepsilon^{-1}Y_\varepsilon)} \leqslant K_9 \frac{2a_\varepsilon}{\varepsilon}\sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}}, \quad \text{if } n = 2. \tag{10}$$

From Lemma 1 and estimates (9), (10) we obtain

$$\begin{cases} \|N_j^\varepsilon\|_{L_2(Y_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{10} a_\varepsilon^{n/2}, \\ \|N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{11}\left(\frac{a_\varepsilon}{\varepsilon}\right)^{n/2}, \end{cases} \tag{11}$$

for $n \geqslant 3$ and

$$\begin{cases} \|N_j^\varepsilon\|_{L_2(Y_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{12} a_\varepsilon \sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}}, \\ \|N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{13} \frac{2a_\varepsilon}{\varepsilon}\sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}}, \end{cases} \tag{12}$$

for $n = 2$.

Thus, we have

LEMMA 3. Let N_j^ε $(j = 1, \dots, n)$ be a solution of the problem (2). Then the estimates (9)-(12) are valid.

COROLLARY 1. For the functions N_j^ε we have the following estimates:

$$\begin{cases} \max_{y_1=0} |N_j^\varepsilon| \leqslant K_{14}(a_\varepsilon \varepsilon^{-1})^{n/2}, & \max_{y_1=0} |\nabla_y N_j^\varepsilon| \leqslant K_{15}(a_\varepsilon\varepsilon^{-1})^{n/2}, \quad \text{if } n \geqslant 3, \\ \max_{y_1=0} |N_j^\varepsilon| \leqslant K_{16}\frac{2a_\varepsilon}{\varepsilon}\sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}}, & \max_{y_1=0} |\nabla_y N_j^\varepsilon| \leqslant K_{17}\frac{2a_\varepsilon}{\varepsilon}\sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}}, \quad \text{if } n = 2. \end{cases} \tag{13}$$

PROOF. Taking into account that $\dfrac{\partial N_j^\varepsilon}{\partial y_i}$ is a harmonic 1-periodic in y function, we can use the mean value theorem for harmonic functions: if $P \in \gamma$, then

$$\left.\frac{\partial N_j^\varepsilon}{\partial y_i}\right|_P = \left\langle \frac{\partial N_j^\varepsilon}{\partial y_i}\right\rangle_{V_{r_0}^P},$$

where $V_{r_0}^P$ is a ball of radius r_0 and P is the center of $V_{r_0}^P$.

By virtue of the estimates (9), (10) we obtain

$$\max_\gamma \left|\frac{\partial N_j^\varepsilon}{\partial y_i}\right| \leqslant |V_{r_0}^P|^{-1/2}\|\nabla_y N_j^\varepsilon\|_{L_2(V_{r_0}^P)} \leqslant K_{18}(a_\varepsilon \varepsilon^{-1})^{n/2},$$

for $n \geqslant 3$, and

$$\max_\gamma \left|\frac{\partial N_j^\varepsilon}{\partial y_i}\right| \leqslant K_{19}\frac{2a_\varepsilon}{\varepsilon}\sqrt{\ln\frac{\varepsilon}{2a_\varepsilon}},$$

if $n = 2$.

Other estimates (13) are obtained in a similar way.

Let $v_0 \in C^{2+\alpha}(\overline{\Omega})$ be a solution of the problem

$$(14) \qquad \Delta v_0 = f \quad \text{in } \Omega, \qquad v_0 = 0 \quad \text{on } \partial\Omega .$$

Consider the function

$$u_\varepsilon^1 = v_0 + \varepsilon \sum_{j=1}^{n} \widetilde{N}_j^\varepsilon \left(\frac{x}{\varepsilon}\right) \frac{\partial v_0}{\partial x_j}, \qquad x \in \Omega^- \cup \Omega_\varepsilon^+ ,$$

where $\widetilde{N}_j^\varepsilon \equiv N_j^\varepsilon$, if $y_1 > 0$ and $\widetilde{N}_j^\varepsilon \equiv 0$, if $y_1 < 0$, as an approximate solution for u_ε.

In what follows we use the usual convention of repeated indices.

Taking into account the definition of the functions v_0, N_j^ε $(j = 1, \dots, n)$, we obtain that $(u_\varepsilon^1 - u_\varepsilon)$ is a weak solution of the problem:

$$\Delta (u_\varepsilon^1 - u_\varepsilon) = 0, \qquad \text{if } x \in \Omega^- ,$$

$$\Delta (u_\varepsilon^1 - u_\varepsilon) = \varepsilon \frac{\partial}{\partial x_i}\left(N_j^\varepsilon \frac{\partial^2 v_0}{\partial x_i \partial x_j}\right) + \frac{\partial N_j^\varepsilon}{\partial y_s} \frac{\partial^2 v_0}{\partial x_s \partial x_j}, \qquad \text{if } x \in \Omega_\varepsilon^+ ,$$

$$[u_\varepsilon^1 - u_\varepsilon]|_\gamma = \varepsilon N_j^\varepsilon \left.\frac{\partial v_0}{\partial x_j}\right|_{x_1 = +0},$$

$$\left[\frac{\partial}{\partial x_1}(u_\varepsilon^1 - u_\varepsilon)\right]\Bigg|_\gamma = \frac{\partial N_j^\varepsilon}{\partial y_1} \left.\frac{\partial v_0}{\partial x_j}\right|_{x_1 = +0} + \varepsilon N_j^\varepsilon \left.\frac{\partial^2 v_0}{\partial x_1 \partial x_j}\right|_{x_1 = +0},$$

$$\frac{\partial}{\partial \nu}(u_\varepsilon^1 - u_\varepsilon) = \varepsilon N_j^\varepsilon \frac{\partial^2 v_0}{\partial x_j \partial x_i} \nu_i \quad \text{on } S_\varepsilon, \qquad u_\varepsilon^1 - u_\varepsilon = \varepsilon \widetilde{N}_j^\varepsilon \frac{\partial v_0}{\partial x_j} \quad \text{on } \Gamma_\varepsilon ,$$

where $[\varphi]|_{P\in\gamma} \equiv \varphi|_{P+0} - \varphi|_{P-0}$ for any point $P \in \gamma$ and function φ.

We set $u_\varepsilon^1 - u_\varepsilon = v_\varepsilon^1 + v_\varepsilon^2$, where v_ε^1 is a weak solution of the following problem:

$$(15) \qquad \begin{cases} \Delta v_\varepsilon^1 = 0, \quad \text{if } x \in \Omega^-, \qquad \Delta v_\varepsilon^1 = F_\varepsilon^+ + \varepsilon \dfrac{\partial}{\partial x_i} f_{i,\varepsilon}, \quad \text{if } x \in \Omega_\varepsilon^+ , \\ [v_\varepsilon^1]|_\gamma = 0, \qquad \left[\dfrac{\partial v_\varepsilon^1}{\partial x_1}\right]\Bigg|_\gamma = \varepsilon f_{1,\varepsilon}|_{x_1=+0} + l_\varepsilon|_{x_1=+0} , \\ \dfrac{\partial v_\varepsilon^1}{\partial \nu} = \varepsilon f_{i,\varepsilon} \nu_i, \quad \text{on } S_\varepsilon, \qquad v_\varepsilon^1 = 0 \quad \text{on } \Gamma_\varepsilon , \end{cases}$$

where

$$F_\varepsilon^+ \equiv \frac{\partial N_j^\varepsilon}{\partial y_s} \frac{\partial^2 v_0}{\partial x_j \partial x_s}, \qquad f_{i,\varepsilon} \equiv N_j^\varepsilon \frac{\partial^2 v_0}{\partial x_j \partial x_i} \qquad (i = 1, \dots, n),$$

$l_\varepsilon \equiv \dfrac{\partial N_j^\varepsilon}{\partial y_1} \dfrac{\partial v_0}{\partial x_j}$, and v_ε^2 is a weak solution of the problem

$$\text{(16)} \qquad \begin{cases} \Delta v_\varepsilon^2 = 0, \quad \text{if } x \in \Omega^- \cup \Omega_\varepsilon^+, \qquad [v_\varepsilon^2]|_\gamma = \varepsilon N_j^\varepsilon \left. \dfrac{\partial v_0}{\partial x_j} \right|_{x_1 = +0}, \\ \left[\left[\dfrac{\partial v_\varepsilon^2}{\partial x_1} \right]\right]_\gamma = 0, \quad \dfrac{\partial v_\varepsilon^2}{\partial \nu} = 0 \quad \text{on } S_\varepsilon, \qquad v_\varepsilon^2 = \varepsilon \tilde{N}_j^\varepsilon \dfrac{\partial v_0}{\partial x_j} \quad \text{on } \Gamma_\varepsilon. \end{cases}$$

Now we obtain estimates for $v_\varepsilon^1, v_\varepsilon^2$. Using the integral identity for the problem (15) and taking the test-function $\varphi = v_\varepsilon^1$, we deduce

$$\text{(17)} \qquad \int_{\Omega_\varepsilon} |\nabla_x v_\varepsilon^1|^2 dx - \int_\gamma l_\varepsilon v_\varepsilon^1 d\hat{x} = \int_{\Omega_\varepsilon^+} F_\varepsilon^+ v_\varepsilon^1 dx - \varepsilon \int_{\Omega_\varepsilon^+} f_{i,\varepsilon} \frac{\partial v_\varepsilon^1}{\partial x_i} dx.$$

For the function $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ one can prove the Friedrichs type inequality

$$\text{(18)} \qquad \|u\|_{L_2(\Omega_\varepsilon)} \leqslant C_0 \|\nabla_x u\|_{L_2(\Omega_\varepsilon)},$$

where the constant C_0 does not depend on ε. This inequality can be proved in the same way as the Friedrichs type inequality is proved in [4, p.53, Theor. 4.5] for perforated domains.

From the inequality (18), the imbedding theorem and (17) it follows that

$$\text{(19)} \qquad \int_{\Omega_\varepsilon} |\nabla_x v_\varepsilon^1|^2 dx \leqslant K_{20} \|\nabla_x v_\varepsilon^1\|_{L_2(\Omega_\varepsilon)} \cdot \left\{ \max_\gamma |l_\varepsilon| + \|F_\varepsilon^+\|_{L_2(\Omega_\varepsilon^+)} + \varepsilon \sum_{i=1}^n \|f_{i,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \right\}.$$

Using the Friedrichs inequality (18), estimates (11)-(13) and (19), we get

$$\text{(20)} \qquad \begin{cases} \|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leqslant K_{21} (a_\varepsilon \varepsilon^{-1})^{n/2} & \text{for } n \geqslant 3, \\ \|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leqslant K_{22} \dfrac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \dfrac{\varepsilon}{2a_\varepsilon}} & \text{for } n = 2. \end{cases}$$

We set

$$\text{(21)} \qquad \phi_n(\varepsilon) = \begin{cases} (a_\varepsilon \varepsilon^{-1})^{n/2}, & \text{if } n \geqslant 3, \\ \dfrac{2a_\varepsilon}{\varepsilon} \sqrt{\ln \dfrac{\varepsilon}{2a_\varepsilon}}, & \text{if } n = 2. \end{cases}$$

Thus we have

LEMMA 4. Let $v_\varepsilon^1 \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ be a weak solution of the problem (15). Then the following estimate is valid

$$\text{(22)} \qquad \|v_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leqslant K_{23} \phi_n(\varepsilon).$$

Now we derive the estimate for the solution v_ε^2. We define the function $\varphi(x_1)$ as a smooth function for $x_1 \geqslant 0$ and $\varphi \equiv 1$ for $x_1 \in [0, \delta\varepsilon]$, $\varphi \equiv 0$ for $x_1 \geqslant 2\delta\varepsilon$, $|\varphi'| \leqslant K_{24} \varepsilon^{-1}$ and the constant δ is chosen in such a way that $\varphi \equiv 0$ for $x \in S_\varepsilon$. We set $\varphi \equiv 0$ for $x_1 < 0$. We say that v_ε^2 is a weak solution of the problem (16) if the function

$\widetilde{v}_\varepsilon^2 = v_\varepsilon^2 - \varepsilon\varphi \widetilde{N}_j^\varepsilon \dfrac{\partial v_0}{\partial x_j} \in H_1(\Omega_\varepsilon)$ and for any function $\psi \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the following integral identity is valid

$$\int_{\Omega_\varepsilon} (\nabla_x \widetilde{v}_\varepsilon^2, \nabla_x \psi)\, dx = -\varepsilon \int_{\Omega_\varepsilon^+} \left(\nabla_x \left(\varphi N_j^\varepsilon \frac{\partial v_0}{\partial x_j}\right),\ \nabla_x \psi\right) dx .$$

We represent the function $\widetilde{v}_\varepsilon^2$ in the form

$$\widetilde{v}_\varepsilon^2 = w_\varepsilon^1 + w_\varepsilon^2 ,$$

where w_ε^1 is a weak solution of the problem

$$(23)\qquad \begin{cases} \Delta w_\varepsilon^1 = -\varepsilon \Delta\left(\varphi N_j^\varepsilon \dfrac{\partial v_0}{\partial x_j}\right) \ \text{ in } \Omega_\varepsilon^+ ; \qquad \Delta w_\varepsilon^1 = 0 \ \text{ in } \Omega^- , \\ [w_\varepsilon^1]|_\gamma = 0 , \quad \left[\dfrac{\partial w_\varepsilon^1}{\partial x_1}\right]\Bigg|_\gamma = -\varepsilon \dfrac{\partial}{\partial x_1}\left(N_j^\varepsilon \dfrac{\partial v_0}{\partial x_j}\varphi\right)\Bigg|_{x_1 = +0} , \\ \dfrac{\partial w_\varepsilon^1}{\partial \nu} = 0 \ \text{ on } S_\varepsilon , \qquad w_\varepsilon^1 = 0 \ \text{ on } \Gamma_\varepsilon , \end{cases}$$

and w_ε^2 is a weak solution of the problem

$$(24)\qquad \begin{cases} \Delta w_\varepsilon^2 = 0 \ \text{ in } \Omega_\varepsilon , \qquad \dfrac{\partial w_\varepsilon^2}{\partial \nu} = 0 \ \text{ on } S_\varepsilon , \\ w_\varepsilon^2 = \varepsilon(1-\varphi)\widetilde{N}_j^\varepsilon \dfrac{\partial v_0}{\partial x_j} \quad \text{ on } \Gamma_\varepsilon . \end{cases}$$

Taking in the integral identity for the problem (23) the solution w_ε^1 as a test-function and using the Friedrichs inequality (18) for functions from $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and estimates (11), (12) we deduce the following estimate

$$(25)\qquad \|w_\varepsilon^1\|_{H_1(\Omega_\varepsilon)} \leqslant K_{25}\Phi_n(\varepsilon) .$$

In order to estimate w_ε^2 we define the function $\theta \in C^\infty(\overline{\Omega})$ such that $\theta \equiv 1$ when $\varrho(x, \partial\Omega) \leqslant \varepsilon$, $\theta \equiv 0$ when $\varrho(x, \partial\Omega) \geqslant 2\varepsilon$, $0 \leqslant \theta \leqslant 1$.

Let $w_\varepsilon^3 = w_\varepsilon^2 - \varepsilon(1-\varphi)\theta\widetilde{N}_j^\varepsilon \dfrac{\partial v_0}{\partial x_j}$ in Ω_ε. It is easy to see that $w_\varepsilon^3 \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$. Let us take w_ε^3 as a test-function in the integral identity for the problem (24). Then we have

$$(26)\qquad \int_{\Omega_\varepsilon} |\nabla_x w_\varepsilon^2|^2 dx = \varepsilon \int_{\Omega_\varepsilon} \left(\nabla_x\left[(1-\varphi)\theta\widetilde{N}_j^\varepsilon \frac{\partial v_0}{\partial x_j}\right],\ \nabla_x w_\varepsilon^2\right) dx .$$

By virtue of the definition of the function φ and θ and estimates (11), (12) we deduce from (26) that

$$(27)\qquad \|\nabla_x w_\varepsilon^2\|_{L_2(\Omega_\varepsilon)} \leqslant K_{26}\phi_n(\varepsilon) .$$

Using the Friedrichs inequality (18) for w_ε^2, we obtain

$$\|w_\varepsilon^2\|_{L_2(\Omega_\varepsilon)} \leqslant K_{27}\phi_n(\varepsilon). \tag{28}$$

Therefore, from estimates (25), (27), (28) we have

LEMMA 5. Let v_ε^2 be a weak solution of the problem (16). Then the following estimate is valid

$$\|v_\varepsilon^2\|_{H_1(\Omega^-)} + \|v_\varepsilon^2\|_{H_1(\Omega_\varepsilon^+)} \leqslant K_{28}\phi_n(\varepsilon). \tag{29}$$

The next theorem follows from (22) and (29).

THEOREM 1. Let u_ε be a solution of the problem (1), v_0 be a solution of the problem (14). Then

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \leqslant K_{29}\phi_n(\varepsilon),$$

where $\phi_n(\varepsilon)$ is defined by (21).

The case when $a_\varepsilon \varepsilon^{-1} \geqslant C$, $C = \text{const}$, is considered in [6-8].

2. The spectral problem, corresponding to the boundary-value problem (1), can be considered in the same way as in [6, 7], using the theorem from [4] about the spectrum of a sequence of singularly perturbed operators.

On the base of Theorem 1 we have

THEOREM 2. Let $\{\lambda_\varepsilon^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\Delta u_\varepsilon^m + \lambda_\varepsilon^m u_\varepsilon^m = 0 \quad \text{in } \Omega_\varepsilon,$$

$$\frac{\partial u_\varepsilon^m}{\partial \nu} = 0 \quad \text{on } S_\varepsilon, \qquad u_\varepsilon^m = 0 \quad \text{on } \Gamma_\varepsilon,$$

and let $\{\lambda^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\Delta u^m + \lambda^m u^m = 0 \quad \text{in } \Omega,$$

$$u^m = 0 \quad \text{on } \partial\Omega,$$

and every eigenvalue is counted as many times as its multiplicity. Then

$$\left|\frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m}\right| \leqslant C_1\phi_n(\varepsilon),$$

where C_1 is a constant independent of ε.

The homogeneization problem for the Laplace operator in partially perforated domain with the mixed type of boundary conditions is considered in [9].

APPENDIX

PROOF OF LEMMA 1. First of all we extend a function u on εG_0; this means that we construct a new function $\tilde{u} \in H_1(\varepsilon Q)$ such that $\tilde{u} \equiv u$ when $x \in \varepsilon Q \setminus \overline{a_\varepsilon G_0}$ and the following inequality is valid:

$$\|\nabla_x \tilde{u}\|_{L_2(\varepsilon Q)} \leq K_{30} \|\nabla_x u\|_{L_2(Y_\varepsilon)} . \tag{30}$$

In order to get such an extension we introduce a new variable $y' = a_\varepsilon^{-1} x$ and consider the domain $a_\varepsilon^{-1} Y_\varepsilon = (a_\varepsilon^{-1} \varepsilon Q) \setminus \overline{G}_0$. Since $a_\varepsilon \varepsilon^{-1} \to 0$ as $\varepsilon \to 0$ we can take a cube Q_1 with the length of the edge which does not depend on ε. In addition we suppose that the faces of Q_1 and the faces of Q are parallel and $\overline{G}_0 \subset Q_1$.

Then for any function $u \in H_1(Q_1 \setminus \overline{G}_0)$ we can construct such extension $\tilde{v}$ that

$$\|\nabla_{y'} \tilde{v}\|_{L_2(Q_1)} \leq K_{31} \|\nabla_{y'} u\|_{L_2(Q_1 \setminus \overline{G}_0)} . \tag{31}$$

The proof of this inequality can be find in [4].

From the inequality (31) we deduce that

$$\|\nabla_x \tilde{v}\|_{L_2(a_\varepsilon Q_1)} \leq K_{32} \|\nabla_x u\|_{L_2(a_\varepsilon (Q_1 \setminus \overline{G}_0))} .$$

Now we can define the function $\tilde{u}$ setting

$$\tilde{u} = \begin{cases} \tilde{v}, & \text{when } x \in a_\varepsilon G_0 , \\ u, & \text{when } x \in Y_\varepsilon \setminus a_\varepsilon G_0 . \end{cases}$$

For simplicity we assume that u and $\tilde{u}$ are smooth functions. Then, for any points $P, P' \in Y_\varepsilon$ we have

$$u(P') = u(P) + \int_{x_1^P}^{x_1^{P'}} \tilde{u}_{x_1}(x_1, x_2^{P'}, \dots, x_n^{P'})\, dx_1 + \int_{x_2^P}^{x_2^{P'}} \tilde{u}_{x_2}(x_1^P, x_2, x_3^{P'}, \dots, x_n^{P'})\, dx_2 + \dots + \int_{x_n^P}^{x_n^{P'}} \tilde{u}_{x_n}(x_1^P, x_2^P, \dots, x_{n-1}^P, x_n)\, dx_n ,$$

where $P = (x_1^P, \dots, x_n^P)$, $P' = (x_1^{P'}, \dots, x_n^{P'})$.

From this representation we obtain

$$(u(P') - u(P))^2 \leq K_{33}\, \varepsilon \left\{ \int_0^\varepsilon \tilde{u}_{x_1}^2 dx_1 + \dots + \int_0^\varepsilon \tilde{u}_{x_n}^2 dx_n \right\} .$$

Now we integrate the last inequality at first with respect to $P' \in Y_\varepsilon$ and then with respect to $P \in Y_\varepsilon$, and using (30) we deduce

$$2\, |Y_\varepsilon| \int_{Y_\varepsilon} u^2 dx - 2 \left(\int_{Y_\varepsilon} u\, dx \right)^2 \leq K_{34} \varepsilon^2 |Y_\varepsilon| \int_{Y_\varepsilon} |\nabla_x u|^2 dx .$$

Thus, we have

$$\int_{Y_\varepsilon} u^2 dx \leq |Y_\varepsilon|^{-1} \left(\int_{Y_\varepsilon} u\, dx \right)^2 + K_{34} \varepsilon^2 \int_{Y_\varepsilon} |\nabla_x u|^2 dx .$$

Taking into account that $\langle u \rangle_{Y_\varepsilon} = 0$ we obtain the statement of Lemma 1.

PROOF OF LEMMA 2. For simplicity we assume that G_0 is a ball with the radius $\varrho < 1/2$ and its center coincides with the center of Q. Let $P \in a_\varepsilon S_0$, $P' \in \varrho^{-1} r S_0$, $a_\varepsilon \varrho \leq r < \frac{\varepsilon}{2}\varrho$ and P, P' lie at the same vector-radius. We have

$$u^2(P) \leq 2u^2(P') + 2\left(\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} \left|\frac{\partial u}{\partial r}\right| dr\right)^2 . \tag{32}$$

From (32) we deduce

$$u^2(P) \leq 2u^2(P') + 2\left(\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} \left|\frac{\partial u}{\partial r}\right| r^{(n-1)/2} r^{(1-n)/2} dr\right)^2 \leq$$

$$\leq 2u^2(P') + 2\left(\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} r^{1-n} dr\right)\left(\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} \left|\frac{\partial u}{\partial r}\right|^2 r^{n-1} dr\right).$$

We have

$$\begin{cases} u^2(P) \leq 2u^2(P') + \dfrac{2}{n-2}(a_\varepsilon \varrho)^{2-n} \displaystyle\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} \left|\frac{\partial u}{\partial r}\right|^2 r^{n-1} dr, & \text{if } n \geq 3, \\ u^2(P) \leq 2u^2(P') + 2\ln \dfrac{\varepsilon}{2a_\varepsilon} \displaystyle\int_{a_\varepsilon \varrho}^{\varepsilon\varrho/2} \left|\frac{\partial u}{\partial r}\right|^2 r\, dr, & \text{if } n = 2. \end{cases}$$

Multiplying the last inequalities by $(a_\varepsilon \varrho)^{n-1}\ \psi(\varphi_1, \dots, \varphi_{n-1})$, where $J \equiv \equiv r^{n-1}\psi(\varphi_1, \dots, \varphi_{n-1})$ is the Jacobian for the spherical coordinates and integrating with respect to $\varphi_1, \dots, \varphi_{n-1}$, we obtain

$$\int_{a_\varepsilon S_0} u^2 ds \leq K_{35}\left\{a_\varepsilon^{n-1} \int_{S_1} u^2(P')\psi\, d\varphi_1 \dots d\varphi_{n-1} + a_\varepsilon \|\nabla_x u\|^2_{L_2(T_{\varepsilon/2} \setminus T_{a_\varepsilon})}\right\}, \tag{33}$$

if $n \geq 3$, and

$$\int_{a_\varepsilon S_0} u^2 ds \leq K_{36}\left\{a_\varepsilon \int_{S_1} u^2(P')\psi\, d\varphi_1 \dots d\varphi_{n-1} + a_\varepsilon \ln \frac{\varepsilon}{2a_\varepsilon} \|\nabla_x u\|^2_{L_2(T_{\varepsilon/2} \setminus T_{a_\varepsilon})}\right\}, \tag{34}$$

if $n = 2$, where S_1 is a sphere of radius 1, T_σ is a ball of radius $\sigma\varrho$ whose center coincides with the center of G_0.

Then multiplying the inequalities (33), (34) by r^{n-1} and integrating with respect to P' over $r \in \left(a_\varepsilon \varrho, \frac{\varepsilon\varrho}{2}\right)$, we deduce estimates:

$$K_{37}\left(\frac{\varepsilon^n}{2^n} - a_\varepsilon^n\right)\|u\|^2_{L_2(a_\varepsilon S_0)} \leq$$

$$\leq K_{38}\left\{a_\varepsilon^{n-1}\|u\|^2_{L_2(T_{\varepsilon/2} \setminus T_{a_\varepsilon})} + a_\varepsilon\left(\frac{\varepsilon^n}{2^n} - a_\varepsilon^n\right)\|\nabla_x u\|^2_{L_2(T_{\varepsilon/2} \setminus T_{a_\varepsilon})}\right\},$$

if $n \geqslant 3$, and

$$K_{39}\left(\frac{\varepsilon^2}{4}-a_\varepsilon^2\right)\|u\|^2_{L_2(a_\varepsilon S_0)} \leqslant$$

$$\leqslant K_{40}\left\{a_\varepsilon\|u\|^2_{L_2(T_{\varepsilon/2}\setminus T_{a_\varepsilon})}+a_\varepsilon\left(\frac{\varepsilon^2}{4}-a_\varepsilon^2\right)\ln\frac{\varepsilon}{2a_\varepsilon}\|\nabla_x u\|^2_{L_2(T_{\varepsilon/2}\setminus T_{a_\varepsilon})}\right\},$$

if $n = 2$. From these inequalities we can conclude that Lemma 2 is valid.

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Department of Mechanics and Mathematics
Moscow State University
119899 Mosca (Russia)