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Calcolo delle variazioni. — *On a variational theory of light rays on Lorentzian manifolds.* Nota di Fabio Giannoni e Antonio Masiello, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — In this *Note*, by using a generalization of the classical Fermat principle, we prove the existence and multiplicity of lightlike geodesics joining a point with a timelike curve on a class of Lorentzian manifolds, satisfying a suitable compactness assumption, which is weaker than the globally hyperbolicity.

Key words: Lorentzian manifolds; Lightlike geodesics; Fermat principle.

Riassunto. — *Su una teoria variazionale dei raggi di luce su una varietà Lorentziana*. In questa *Nota*, usando una generalizzazione del principio di Fermat, si studia l'esistenza e la molteplicità di geodetiche di tipo luce congiungenti un punto con una curva di tipo tempo su una classe di varietà Lorentziane, soddisfacente una condizione di compattezza più debole della globale iperbolicità.

A Lorentzian manifold is a couple $(\mathcal{M}, g)$, where $\mathcal{M}$ is a smooth connected finite dimensional manifold, and g is a Lorentzian metric on $\mathcal{M}$, that is a metric tensor of index 1. In General Relativity, space-times are represented by 4-dimensional Lorentzian manifolds (for details see [1, 13]).

We study existence and multiplicity of lightlike geodesics joining a given point of the manifold with a timelike curve, and the relations between such geodesics and the topology of $\mathcal{M}$, under nondegeneration assumptions. Let $\gamma: \boldsymbol{R} \to \mathcal{M}$ be a C^1 timelike curve, (*i.e.* $g(\gamma(s))[\dot{\gamma}(s), \dot{\gamma}(s)] < 0$ for any $s \in \boldsymbol{R}$), and a point $p \in \mathcal{M}$. Suppose $\mathcal{M}$ to be time oriented (cf. [13]). We consider the set $\mathcal{L}_{p,\gamma}$ of the future pointing lightlike unparametrized curves joining p with γ, that is

$$(1) \qquad \mathcal{L}_{p,\gamma} = \{z: [0, 1] \to \mathcal{M} \mid z \text{ is smooth, } g(z(s))[\dot{z}(s), \dot{z}(s)] = 0, \text{ for any } s \in [0, 1],$$
$$z(0) = p,\ z(1) \text{ is in the future of } p, \text{ there exists } t(z) \in \boldsymbol{R}, \text{ such that } z(1) = \gamma(t(z))\}.$$

The number $t(z)$ above is called *arrival time* of the curve z at γ. We want to study the set $\mathcal{G}_{p,\gamma}$ of the geodesics (*i.e.* smooth curves z such that $D_s \dot{z} = 0$, where D_s denotes the covariant derivative induced by the Levi-Civita connection) contained in $\mathcal{L}_{p,\gamma}$.

The first results for these problems were obtained by [16], for a class of globally hyperbolic Lorentzian manifolds, and by [6] for a class of conformally stationary Lorentzian manifolds with boundary, with applications to classical space-times of General Relativity, as Schwarzschild, Reissner-Nordström and Kerr space-times (see [10]). In the results above, the problem was reduced (by suitable variational principles)

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to the search of critical points of a functional defined on an infinite dimensional manifold.

In the paper [15], a very general variational principle is proved. Indeed, the author showed that in an *arbitrary* Lorentzian manifold, the lightlike geodesics joining p with γ are the critical points of the *arrival time* functional, defined above.

However, such a functional is not easy to handle (in particular it presents the same problems that the length functional for a Riemannian manifold). Moreover, the set $\mathcal{L}_{p,\gamma}$ could be empty, as proved by the following example.

EXAMPLE 1. *Consider the manifold $\mathbf{R} \times \mathbf{R}$, equipped with the Lorentzian metric $ds^2 = (1 + t^2)^2 dx^2 - dt^2$. Then, simple calculations show the non existence of a lightlike curve joining the point $p = (0, 0)$ with the timelike (vertical) line $\gamma(s) = (\pi/2, s)$.*

We present now some results on lightlike geodesics, assuming a weaker condition than globally hyperbolicity, and based on a sort of compactness only of lightlike curves. We require that the sublevels of the arrival time functional are precompact for the compact-open topology of the set of the curves of the manifold.

DEFINITION 2. *A Lorentzian manifold $(\mathcal{M}, g)$ is said* globally lightlike complete, *if for any $p \in \mathcal{M}$, for any timelike curve $\gamma : \mathbf{R} \to \mathcal{M}$, and for any $c \in \mathbf{R}$, the set $T^c_{p,\gamma} = \{z \in \mathcal{L}_{p,\gamma} \mid t(z) \leq c\}$ is precompact for the compact-open topology.*

We point out that the notion of global lightlike completeness is independent on isometries.

REMARK 3. *The notion of global lightlike completeness is weaker than global hyperbolicity. Indeed the Anti-de Sitter space-time $]-\pi/2, \pi/2[\times \mathbf{R}$ with metric $ds^2 = (dx^2 - dt^2)/\cos^2 x$ is globally lightlike complete, but not globally hyperbolic* (see [14]). *On the other hand, it is not difficult to see that every globally hyperbolic Lorentzian manifold is globally lightlike complete.*

DEFINITION 4. *A Lorentzian manifold $(\mathcal{M}, g)$ is said* orthogonal splitting *if $\mathcal{M} = \mathcal{M}_0 \times \mathbf{R}$, and the metric g has the following form. For any $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbf{R}$,*

$$g(z)[\zeta, \zeta] = \langle \alpha(z)\xi, \xi \rangle - \beta(z)\tau^2, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ is a Riemannian metric on $\mathcal{M}_0$, $\alpha(z)$ is a positive linear operator on $T_x\mathcal{M}_0$, smoothly depending on z, and $\beta(z)$ is a smooth positive scalar field on $\mathcal{M}$.

Let X be a topological space, and denote by *cat* X the Ljusternik-Schnirelmann category of X, that is the minimal number of closed, contractible subsets in X, and covering X itself. The first result is the following

THEOREM 5. *Let $(\mathcal{M}, g)$ be a Lorentzian manifold, p a point of $\mathcal{M}$ and γ a timelike curve. Moreover, assume that:*

L_1) $(\mathcal{M}, g)$ *is globally lightlike complete;*

L_2) $(\mathcal{M}, g)$ *is isometric to a splitting manifold* $(\mathcal{M}_0 \times \boldsymbol{R}, g_0)$.

Then, there exists at least cat $\mathcal{L}_{p,\gamma}$ *of future pointing lightlike geodesics joining p with* γ. *Moreover, if cat* $\mathcal{L}_{p,\gamma} = +\infty$, *the arrival time is unbounded on* $\mathcal{G}_{p,\gamma}$.

REMARK 6. *It can be proved that, using a result of Fadell-Husseini (see* [5]), *cat* $\mathcal{L}_{p,\gamma} = +\infty$, *if* $\mathcal{M}$ *is noncontractible and the following growth condition holds.*

L_3) *For any curve* $x(s) \in C^1([0, 1], \mathcal{M}_0)$, *such that* $x(0) = x_0$, $x(1) = x_1$, *there exists* $t(s) \in C^1([0, 1], \boldsymbol{R})$, *satisfying the Cauchy problem*

$$\begin{cases} \dot{t}(s) = \sqrt{\langle \alpha(x, t)\dot{x}, \dot{x}\rangle}, \\ t(0) = 0. \end{cases} \tag{3}$$

REMARK 7. *We recall that a deep result of Geroch (cf.* [7]) *shows that every globally hyperbolic Lorentzian manifold is isometric to an orthogonal splitting. We do not know if this is true also for globally lightlike complete manifolds.*

REMARK 8. *Condition* L_3) *is comparable with the «metric growth condition» of* [16]. *It is satisfied, for instance, if the metric is stationary.*

Under nondegenerate assumptions, we relate the set $\mathcal{G}_{p,\gamma}$ to the uniform topology of $\mathcal{L}_{p,\gamma}$. Such relation is obtained by proving a Morse Theory for the lightlike geodesics of $\mathcal{G}_{p,\gamma}$, in the spirit of the classical Morse Theory for Riemannian geodesics (see [4, 11, 12]). We refer also to [2] for a Morse Theory for geodesics on static Lorentzian manifolds, [8] for a Morse Theory for geodesics on stationary Lorentzian manifold with boundary, [16] for a Morse Theory for timelike and null geodesics on a globally hyperbolic Lorentzian manifold and [1] for a general Index Theorem for light rays.

We first recall some definitions. Let $z: [0, 1] \to \mathcal{M}$ be a geodesic on a Lorentzian manifold. A smooth vector field ζ along z is said Jacobi field, if it is a solution of the system of differential equations

$$D_s^2 \zeta + R(\zeta, \dot{z})\dot{z} = 0, \tag{4}$$

where R denotes the curvature tensor of the metric. A point $z(s_0)$ is said *conjugate* to $z(0)$ along z, if there exists a nonnull Jacobi field ζ along $z|_{[0,s_0]}$, such that

$$\zeta(0) = \zeta(s_0) = 0. \tag{5}$$

The multiplicity of $z(s_0)$ is the maximal number of linearly independent Jacobi fields satisfying (5). It is clearly a finite number. Finally, the *geometric index* $\mu(z)$ is the number of conjugate points to $z(0)$ along z, counted with their multiplicity.

In general $\mu(z)$ can be equal to $+\infty$ (see [9] for an example). The classical Morse Index Theorem of Riemannian Geometry proves that it is finite for Riemannian geodesics. Moreover, $\mu(z)$ is finite for timelike and lightlike geodesics of any

Lorentzian manifold, see [1] (in particular $\mu(z)$ is finite for any geodesic in $\mathcal{G}_{p,\gamma}$). In [3] it is proved that it is finite for any geodesic of a splitting orthogonal Lorentzian manifold (in particular for static and globally hyperbolic metrics), while in [8] it is proved that it is finite for any geodesic of a stationary metric.

In the next theorem, we shall obtain the Morse Relations for the lightlike geodesics joining p and γ, under a nondegeneration assumption on p and γ.

DEFINITION 9. *The point p and the timelike curve γ are said* nonconjugate *if for every geodesic $z \in \mathcal{G}_{p,\gamma}$, $z(1)$ is nonconjugate to p along z.*

THEOREM 10. *Assume that L_1) and L_2) hold, and assume that p and γ are nonconjugate. Then, for any field $\mathcal{K}$, there exists a formal series $Q(\lambda)$ with positive cardinal integer coefficients, such that*

$$\sum_{z \in \mathcal{G}_{p,\gamma}} \lambda^{\mu(z)} = \mathcal{P}_\lambda(\mathcal{L}_{p,\gamma}; \mathcal{K}) + (1+\lambda)\,Q(\lambda), \tag{6}$$

where

$$\mathcal{P}_\lambda(\mathcal{L}_{p,\gamma}; \mathcal{K}) = \sum_{k=0}^{\infty} \dim H_k(\mathcal{L}_{p,\gamma}; \mathcal{K})\,\lambda^k$$

is the Poincaré polynomial of $\mathcal{L}_{p,\gamma}$ with coefficients in $\mathcal{K}$.

REMARK 11. *Whenever L_3) holds, we can replace $\mathcal{L}_{p,\gamma}$ with the based loop space Ω on $\mathcal{M}$. In this case the lightlike geodesics joining p with γ carry on all the topological information of the path space of $\mathcal{M}$.*

The Proofs of Theorems 5 and 10 rely on a variational principle, which is a Lorentzian version of the classical Fermat principle in optics and is also related to the Maupertuis principle for mechanical systems. Consider the functional

$$F(z) = \int_0^1 \langle \alpha(x,t)\dot{x}, \dot{x} \rangle\, ds\,, \tag{7}$$

defined on the set $\mathcal{L}_{p,\gamma}$ of the lightlike curves joining p and γ. It can be proved that under assumptions L_1)-L_2) the set $\mathcal{L}_{p,\gamma}$ is an infinite dimensional Lipschitz manifold, when it is equipped with the Sobolev $W^{1,2}$ topology. For this reason, it is not completely clear the meaning of critical point of F (which is a nonsmooth functional). Anyway, we can approximate respectively $\mathcal{L}_{p,\gamma}$ with a family $\mathcal{T}_\varepsilon$ of smooth manifolds consisting of timelike curves with constant energy, and F by a family of functionals F_ε defined on $\mathcal{T}_\varepsilon$. We get, by a priori estimates on the critical points of F_ε, that the limit (as $\varepsilon \to 0$) of suitable families of critical points of F_ε are lightlike geodesic joining p and γ.

Then, using the a priori estimates quoted above and the Ljusternik-Schnirelmann category, the proof of Theorem 7 follows.

Finally the Morse Relations are proved by studying the topology of the sublevels of the approximating functionals. A limit procedure to get the topology of the sublevels of F, and the study of the Hessian (defined in a subtle way) of a critical point of F, allow to conclude the proof of Theorem 10. Indeed, it is proved that the linear map associated to the Hessian form is a compact perturbation of a positive definite linear map. Hence the «Morse index» of a critical point z of F is finite. Finally it is proved that such Morse index is equal to the geometric index $\mu(z)$.

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