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Rigidity of holomorphic maps and distortion of biholomorphic maps in operator Siegel domains

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Geometria. — *Rigidity of holomorphic maps and distortion of biholomorphic maps in operator Siegel domains.* Nota di KAZIMIERZ WŁODARCZYK, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Results concerning the rigidity of holomorphic maps and the distortion of biholomorphic maps in infinite dimensional Siegel domains of J^*-algebras are established. The homogeneity of the open unit balls in these algebras plays a key role in the arguments.

KEY WORDS: Rigidity of holomorphic maps; Distortion of biholomorphic maps; Operator Siegel domains; J^*-algebras.

RIASSUNTO. — *Rigidità di applicazioni olomorfe e distorsione di applicazioni biolomorfe in dominî di Siegel di operatori.* Si stabiliscono alcuni risultati sulla rigidità di applicazioni olomorfe e sulla distorsione di biolomorfismi di dominî di Siegel di algebre J^*. L'omogeneità dei dischi unità in queste algebre ha un ruolo essenziale nelle dimostrazioni.

1. INTRODUCTION

Let H and K be Hilbert spaces over $\mathbf{C}$, let $\mathcal{L}(H, K)$ denote the Banach space of all bounded linear operators X from H to K with the operator norm, and let $\mathfrak{U} \subset \mathcal{L}(H, K)$ be a J^*-algebra, *i.e.* a closed complex linear subspace of $\mathcal{L}(H, K)$ such that $XX^*X \in \mathfrak{U}$ whenever $X \in \mathfrak{U}$. J^*-algebras are natural generalizations of C^*-algebras, B^*-algebras, JC^*-algebras, ternary algebras, complex Hilbert spaces and others.

For partial isometries $U \in \mathfrak{U}$, let $\mathfrak{U}_U$ be operator Siegel domains defined by the formulae

$$\mathfrak{U}_U = \{X \in \mathfrak{U}\colon 2 \operatorname{Re} U^*X - X^*(I_K - UU^*)X + I_H - U^*U > 0\} \tag{1.1}$$

when $U^*U \neq I_H$ and

$$\mathfrak{U}_U = \{X \in \mathfrak{U}\colon \operatorname{Re} U^*X - X^*(I_K - UU^*)X > 0\} \tag{1.2}$$

when $U^*U = I_H$. Obviously, if $U = 0$ in (1.1), then

$$\mathfrak{U}_0 = \{X \in \mathfrak{U}\colon \|X\| < 1\}. \tag{1.3}$$

In particular, for J^*-algebras $\mathfrak{U} = \mathcal{L}(\mathbf{C}, H) = H$, if $u \in H$ and $\|u\| = 1$, by (1.2), we have $H_u = \{x \in H\colon \operatorname{Re}\langle x, u\rangle - \|x - \langle x, u\rangle u\|^2 > 0\}$ and, by (1.3), $H_0 = \{x \in H\colon \|x\| < 1\}$. As is well known, the unit balls $\mathfrak{U}_0$ are bounded symmetric homogeneous domains and, moreover, $\mathfrak{U}_0$ and unbounded convex domains $\mathfrak{U}_U$, $U \neq 0$, are holomorphically equivalent (see L. A. Harris [16, 18, 19] in infinite dimensions and E. Cartan [5] and I. I. Pjateckij-Shapiro [24] in finite dimensions).

We study the problems concerning the rigidity of holomorphic maps in $\mathfrak{U}_U$ and the distortion of biholomorphic maps in H_u.

There are many variations, generalizations and applications of Schwarz's lemma, concerning the rigidity of holomorphic maps, and Koebe or Bloch theorems concerning

(*) Nella seduta del 15 giugno 1995.

the distortion of biholomorphic or holomorphic maps. There are still many open problems. For details, see *e.g.* [14, 15, 7, 10, 20-23, 3, 29].

Pick's differential version of the Schwarz lemma reads as follows [33]:

$$|DF(x)| \leq (1 - |F(x)|^2)/(1 - |x|^2) \tag{1.4}$$

everywhere in $\Delta = \{x \in \mathbf{C}: |x| < 1\}$ for all holomorphic maps $F: \Delta \to \Delta$. Assuming that $F: \Delta \to \mathbf{C}$ is biholomorphic in Δ and $z \in \Delta$, Koebe proved, in particular, the following result [25, p. 22]:

$$(1 - |x|^2)\,|DF(x)|/4 \leq \operatorname{dist}[F(x), \partial F(\Delta)] \leq (1 - |x|^2)\,|DF(x)|\,. \tag{1.5}$$

In this paper we present infinite dimensional forms of Pick's differential version of the Schwarz lemma for maps holomorphic in $\mathfrak{U}_U$ (see Theorems 2.1, 2.3 and 2.4). Moreover, we prove the upper bound in (1.5) for biholomorphic maps in H_u (see Theorem 2.2).

Recall that, in the spaces of complex dimensions greater than one, the lower bounds in inequalities (1.5) (in particular, the Koebe 1/4-theorem when $x = 0$, $F(0) = 0$ and $DF(0) = 1$) do not hold and the Bieberbach conjecture and the Bloch theorem are false, either (some additional assumptions, such as *e.g.* the starlikeness or the convexity [28] of the maps considered, are necessary). Speculations about the extensions of the Koebe 1/4-theorem, the Bieberbach conjecture or the Bloch theorem are probably as old as H. Cartan's article [4]. We refer the reader to [1, 2, 6, 8, 9, 11-13, 17, 21, 30] for details, counter-examples, additional assumptions and results in these directions.

2. Statement of results

Before stating our results, we must introduce some definitions and notations.

Let H and K be complex Hilbert spaces. Let H_u, where $u \in H$ and $\|u\| = 1$ or $u = 0$, and K_V, where $v \in K$ and $\|v\| = 1$ or $v = 0$, be domains defined by the formulae

$$H_u = \{x \in H: \operatorname{Re}\langle x, u\rangle - \|x - \langle x, u\rangle u\|^2 > 0\} \quad \text{when } \|u\| = 1\,; \tag{2.1}$$

$$H_0 = \{x \in H: \|x\| < 1\} \text{ when } u = 0\,; \tag{2.2}$$

$$K_V = \{y \in K: \operatorname{Re}\langle y, v\rangle - \|y - \langle y, v\rangle v\|^2 > 0\} \quad \text{when } \|v\| = 1\,; \tag{2.3}$$

$$K_0 = \{y \in K: \|y\| < 1\} \text{ when } v = 0\,. \tag{2.4}$$

Throughout the paper, for $x \in H$,

$$Q_x = E_x + (1 - \|x\|^2)^{1/2}[I_H - E_x] \text{ if } x \in H\setminus\{0\}, \qquad E_0 = 0,\ Q_0 = I_H\,, \tag{2.5}$$

and, for $y \in K$,

$$Q_y = E_y + (1 - \|y\|^2)^{1/2}[I_K - E_y] \text{ if } \quad y \in K\setminus\{0\}, \qquad E_0 = 0,\ Q_0 = I_K\,, \tag{2.6}$$

where E_x and E_y denote linear projections of H and K onto the one-dimensional subspaces spanned by x and y, respectively.

Let H, K, M and N be complex Hilbert spaces and let $\mathfrak{U} \subset \mathcal{L}(H, K)$ and $\mathfrak{B} \subset \mathcal{L}(M, N)$ be J^*-algebras. For partial isometries $U \in \mathfrak{U}$ and $V \in \mathfrak{B}$, let $\mathfrak{U}_U$ and $\mathfrak{B}_V$ be domains defined by the formulae

(2.7) $\mathfrak{U}_U=\{X\in\mathfrak{U}: 2\ \mathrm{Re}\ U^*X-X^*(I_K-UU^*)X+I_H-U^*U>0\}$ when $U^*U\neq I_H$;

(2.8) $\mathfrak{U}_U=\{X\in\mathfrak{U}: \mathrm{Re}\ U^*X-X^*(I_K-UU^*)X>0\}$ when $U^*U=I_H$;

(2.9) $\mathfrak{B}_V=\{Y\in\mathfrak{B}: 2\ \mathrm{Re}\ V^*Y-Y^*(I_N-VV^*)Y+I_M-V^*V>0\}$ when $V^*V\neq I_M$;

(2.10) $\mathfrak{B}_V=\{Y\in\mathfrak{B}: \mathrm{Re}\ V^*Y-Y^*(I_K-VV^*)Y>0\}$ when $V^*V=I_M$.

In particular, from (2.7) when $U=0$ and from (2.9) when $V=0$ we have

(2.11) $$\mathfrak{U}_0=\{X\in\mathfrak{U}: \|X\|<1\}$$

and

(2.12) $$\mathfrak{B}_0=\{Y\in\mathfrak{B}: \|Y\|<1\},$$

respectively.

For $X\in\mathfrak{U}_U$ when $U^*U\neq I_H$, let

(2.13) $$A_X(\mathfrak{U}_U)=A_U^{1/2}(I_H+X^*U)^{-1}(\mathrm{RE}\,U^*X)(I_H+U^*X)^{-1}A_U^{1/2},$$

(2.14) $$B_X(\mathfrak{U}_U)=B_U^{1/2}(I_K+XU^*)^{-1}(\mathrm{RE}\,XU^*)(I_K+UX^*)^{-1}B_U^{1/2},$$

where

(2.15) $$A_U=I_H+U^*U,\quad B_U=I_K+UU^*,$$

(2.16) $$\mathrm{RE}\,U^*X=2\ \mathrm{Re}\ U^*X-X^*(I_K-UU^*)X+I_H-U^*U,$$

(2.17) $$\mathrm{RE}\,XU^*=2\ \mathrm{Re}\ XU^*-X(I_H-U^*U)X^*+I_K-UU^*.$$

In particular, when $U=0$, we have

(2.18) $$A_X(\mathfrak{U}_0)=I_H-X^*X,\quad B_X(\mathfrak{U}_0)=I_K-XX^*.$$

For $X\in\mathfrak{U}_U$ when $U^*U=I_H$, let

(2.19) $$A_X(\mathfrak{U}_U)=(I_H+X^*U)^{-1}(\mathrm{RE}\,U^*X)(I_H+U^*X)^{-1},$$

(2.20) $$B_X(\mathfrak{U}_U)=(I_K+XU^*)^{-1}(\mathrm{RE}\,XU^*)(I_K+UX^*)^{-1},$$

where

(2.21) $$\mathrm{RE}\,U^*X=\mathrm{Re}\ U^*X-X^*(I_K-UU^*)X,$$

(2.22) $$\mathrm{RE}\,XU^*=\mathrm{Re}\ XU^*-X(I_H-U^*U)X^*.$$

For $Y\in\mathfrak{B}_V$ when $V^*V\neq I_M$, let

(2.23) $$A_Y(\mathfrak{B}_V)=A_V^{1/2}(I_M+Y^*V)^{-1}(\mathrm{RE}\,V^*Y)(I_M+V^*Y)^{-1}A_V^{1/2},$$

(2.24) $$B_Y(\mathfrak{B}_V)=B_V^{1/2}(I_N+YV^*)^{-1}(\mathrm{RE}\,YV^*)(I_N+VY^*)^{-1}B_V^{1/2},$$

where

(2.25) $$A_V=I_M+V^*V,\qquad B_V=I_N+VV^*,$$

(2.26) $$\mathrm{RE}\,V^*Y=2\ \mathrm{Re}\ V^*Y-Y^*(I_N-VV^*)Y+I_M-V^*V,$$

(2.27) $$\mathrm{RE}\,YV^*=2\ \mathrm{Re}\ YV^*-Y(I_M-V^*V)Y^*+I_N-VV^*.$$

In particular, when $V=0$, we have

(2.28) $$A_Y(\mathfrak{B}_0)=I_M-Y^*Y,\qquad B_Y(\mathfrak{B}_0)=I_N-YY^*.$$

For $Y \in \mathfrak{B}_V$ when $V^* V = I_M$, let

$$A_Y(\mathfrak{B}_V) = (I_M + Y^* V)^{-1} (\mathrm{RE}\, V^* Y)(I_M + V^* Y)^{-1}, \tag{2.29}$$

$$B_Y(\mathfrak{B}_V) = (I_N + YV^*)^{-1} (\mathrm{RE}\, YV^*)(I_N + VY^*)^{-1}, \tag{2.30}$$

where

$$\mathrm{RE}\, V^* Y = \mathrm{Re}\, V^* Y - Y^* (I_N - VV^*) Y, \tag{2.31}$$

$$\mathrm{RE}\, YV^* = \mathrm{Re}\, YV^* - Y(I_M - V^* V) Y^*. \tag{2.32}$$

Domains defined by (2.1), (2.3) and (2.7)-(2.10) are called Siegel domains (see [18, 16]).

Our main results are summarized in the following theorems.

THEOREM 2.1. *Let H and K be complex Hilbert spaces, let H_u, where $u \in H$ and $\|u\| = 1$ or $u = 0$, and K_V, where $v \in K$ and $\|v\| = 1$ or $v = 0$, be domains defined by formulae* (2.1)-(2.4), *respectively, and let $F: H_u \to K$ be a map holomorphic in H_u such that $F(H_u) \subset K_v$. If $x \in H_u$ and $p \in H$, then*

$$\|(1 + \langle F(x), v\rangle) Q_{F(x) - v}[DF(x)(p)] - [F(x) - v]\langle DF(x)(p), v\rangle\| \leqslant \tag{2.33}$$

$$\leqslant \frac{|1 + \langle F(x), v\rangle|^2 - \|F(x) - v\|^2}{|1 + \langle x, u\rangle|^2 - \|x - u\|^2} \cdot \|[1 + \langle x, u\rangle] Q_{x-u}(p) - (x - u)\langle p, u\rangle\|.$$

Here Q_x, $x \in H$, and Q_y, $y \in K$, are defined by (2.5) *and* (2.6), *respectively.*

THEOREM 2.2. *Let H and K be complex Hilbert spaces and let H_u, where $u \in H$ and $\|u\| = 1$ or $u = 0$, be domains defined by* (2.1) *or* (2.2), *respectively. If $F: H_u \to K$ is a biholomorphic map and $x \in H_u$, then*

$$\operatorname{dist}[F(x), \partial F(H_u)] \leqslant \frac{|1 - \langle U, u\rangle| + \|u\| \|U + u\|}{|1 - \langle U, u\rangle|^2} (1 - \|U\|^2)^{1/2} \cdot \tag{2.34}$$

$$\cdot \|I_H - [1 - (1 - \|U\|^2)^{1/2}] E_{x-u}\| \|DF(x)\|$$

where

$$U = (x - u)/(1 + \langle x, u\rangle). \tag{2.35}$$

REMARK 2.1. From (2.33) it follows that if $u = 0$ and $v = 0$, *i.e.* if $F: H_0 \to K_0$ is a holomorphic map, then

$$\|Q_{F(x)}[DF(x)(p)]\| \leqslant (1 - \|F(x)\|^2)/(1 - \|x\|^2) \|Q_x(p)\|, \quad x \in H_0, \quad p \in H, \tag{2.36}$$

and, in particular, $\|Q_{F(x)}[DF(x)(x)]\| \leqslant \|x\| (1 - \|F(x)\|^2)/(1 - \|x\|^2)$, $x \in H_0$, $\|Q_{F(0)}[DF(0)(p)]\| \leqslant \|p\| [1 - \|F(0)\|^2]$, $p \in H$. If $u = 0$ then (2.34) (by (2.35)) is of the form

$$\operatorname{dist}[F(x), \partial F(H_0)] \leqslant (1 - \|x\|^2)^{1/2} \|I_H - [1 - (1 - \|x\|^2)^{1/2}] E_x\| \|DF(x)\|. \tag{2.37}$$

Thus, in the case when $\dim_C(H) = \dim_C(K) = 1$, inequality (2.36) is equivalent to (1.4), and inequality (2.37) and the upper bound in (1.5) are identical since $E_x = Q_x = I_H$ for $x \in H$ and $E_y = Q_y = I_K$ for $y \in K$. If $\dim_C(H) > 1$, then inequality (2.34) is identical with $\operatorname{dist}[F(x), \partial F(H_u)] \leqslant (|1 - \langle U, u\rangle| + \|u\| \|U + u\|)/|1 - \langle U, u\rangle|^2 \cdot$

$\cdot(1-\|U\|^2)^{1/2}\,\|DF(x)\|$, $x\in H_u$, and, in particular, $\operatorname{dist}[F(x),\partial F(H_0)]\leq(1-\|x\|^2)^{1/2}\cdot\|DF(x)\|$, $x\in H_0$.

THEOREM 2.3. *Let* $\mathfrak{U}\subset\mathcal{L}(H,K)$ *and* $\mathfrak{B}\subset\mathcal{L}(M,N)$ *be* J^**-algebras containing partial isometries* U *and* V*, respectively. Let* $F\colon\mathfrak{U}_U\to\mathfrak{B}$ *be a map holomorphic in* $\mathfrak{U}_U$ *such that* $F(\mathfrak{U}_U)\subset\mathfrak{B}_V$*. Let* $X\in\mathfrak{U}_U$ *and* $P\in\mathfrak{U}$.

(*a*) *If* $U^*U\neq I_H$ *and* $V^*V\neq I_M$*, then*

$$(2.38)\qquad \|[B_{F(X)}(\mathfrak{B}_V)]^{-1/2}B_V^{1/2}[I_N+F(X)V^*]^{-1}DF(X)\cdot P[I_M+V^*F(X)]^{-1}A_V^{1/2}[A_{F(X)}(\mathfrak{B}_V)]^{-1/2}\|\leq$$
$$\leq\|[B_X(\mathfrak{U}_U)]^{-1/2}B_U^{1/2}(I_K+XU^*)^{-1}P(I_H+U^*X)^{-1}A_U^{1/2}[A_X(\mathfrak{U}_U)]^{-1/2}\|.$$

Here $\mathfrak{U}_U$, $\mathfrak{B}_V$, $A_X(\mathfrak{U}_U)$, $B_X(\mathfrak{U}_U)$, $A_{F(X)}(\mathfrak{B}_V)$ *and* $B_{F(X)}(\mathfrak{B}_V)$ *are defined by* (2.7), (2.11), (2.9), (2.12), (2.13)-(2.18) *and* (2.23)-(2.28), *respectively.*

(*b*) *If* $U^*U\neq I_H$ *and* $V^*V=I_M$*, then*

$$(2.39)\qquad \|[B_{F(X)}(\mathfrak{B}_V)]^{-1/2}[I_N+F(X)V^*]^{-1}DF(X)\cdot P[I_M+V^*F(X)]^{-1}[A_{F(X)}(\mathfrak{B}_V)]^{-1/2}\|\leq$$
$$\leq\|[B_X(\mathfrak{U}_U)]^{-1/2}B_U^{1/2}(I_K+XU^*)^{-1}P(I_H+U^*X)^{-1}A_U^{1/2}[A_X(\mathfrak{U}_U)]^{-1/2}\|.$$

Here $\mathfrak{U}_U$, $\mathfrak{B}_V$, $A_X(\mathfrak{U}_U)$, $B_X(\mathfrak{U}_U)$, $A_{F(X)}(\mathfrak{B}_V)$ *and* $B_{F(X)}(\mathfrak{B}_V)$ *are defined by* (2.7), (2.11), (2.10), (2.13)-(2.18) *and* (2.29)-(2.32), *respectively.*

(*c*) *If* $U^*U=I_H$ *and* $V^*V\neq I_M$*, then*

$$(2.40)\qquad \|[B_{F(X)}(\mathfrak{B}_V)]^{-1/2}B_V^{1/2}[I_N+F(X)V^*]^{-1}DF(X)\cdot P[I_M+V^*F(X)]^{-1}A_V^{1/2}[A_{F(X)}(\mathfrak{B}_V)]^{-1/2}\|\leq$$
$$\leq\|[B_X(\mathfrak{U}_U)]^{-1/2}(I_K+XU^*)^{-1}P(I_H+U^*X)^{-1}[A_X(\mathfrak{U}_U)]^{-1/2}\|.$$

Here $\mathfrak{U}_U$, $\mathfrak{B}_V$, $A_X(\mathfrak{U}_U)$, $B_X(\mathfrak{U}_U)$, $A_{F(X)}(\mathfrak{B}_V)$ *and* $B_{F(X)}(\mathfrak{B}_V)$ *are defined by* (2.8), (2.9), (2.12) *and* (2.19)-(2.28), *respectively.*

(*d*) *If* $U^*U=I_H$ *and* $V^*V=I_K$*, then*

$$(2.41)\qquad \|[B_{F(X)}(\mathfrak{B}_V)]^{-1/2}[I_N+F(X)V^*]^{-1}DF(X)P[I_M+V^*F(X)]^{-1}\cdot[A_{F(X)}(\mathfrak{B}_V)]^{-1/2}\|\leq\|[B_X(\mathfrak{U}_U)]^{-1/2}(I_K+XU^*)^{-1}P(I_H+U^*X)^{-1}[A_X(\mathfrak{U}_U)]^{-1/2}\|.$$

Here $\mathfrak{U}_U$, $\mathfrak{B}_V$, $A_X(\mathfrak{U}_U)$, $B_X(\mathfrak{U}_U)$, $A_{F(X)}(\mathfrak{B}_V)$ *and* $B_{F(X)}(\mathfrak{B}_V)$ *are defined by* (2.8), (2.10), (2.19)-(2.22) *and* (2.29)-(2.32), *respectively.*

REMARK 2.2. If $U=0$ and $V=0$, then, by (2.18) and (2.28), inequality (2.38) may be rewritten in the form

$$(2.42)\qquad \|[I_N-F(X)F(X)^*]^{-1/2}DF(X)(P)[I_M-F(X)^*F(X)]^{-1/2}\|\leq$$
$$\leq\|(I_K-XX^*)^{-1/2}P(I_H-X^*X)^{-1/2}\|,\qquad X\in\mathfrak{U}_0,\quad P\in\mathfrak{U}.$$

Thus, if $\mathfrak{U}=\mathfrak{B}=\mathcal{L}(C,C)=C$, then from (2.42) we immediately obtain (1.4).

THEOREM 2.4. *Let* $\mathfrak{U} \subset \mathcal{L}(H, K)$ *be a* J^**-algebra containing a partial isometry* U.

(*a*) *Let* $F: \mathfrak{U}_U \to \mathfrak{U}$ *be a map holomorphic in* $\mathfrak{U}_U$. *Suppose the following holds:* (*i*) F *is expansive,* i.e. $\mathfrak{U}_U \subset F(\mathfrak{U}_U)$; (*ii*) F *is injective on* $\mathfrak{U}_U$; (*iii*) F^{-1} *is holomorphic in* $\mathfrak{U}_U$; *and* (*iv*) $F(Z) = Z$ *for some* $Z \in \mathfrak{U}_U$. *If* $U^*U \neq I_H$ *and* $P \in \mathfrak{U}$, *then*

$$\|P\| \leq \|[B_Z(\mathfrak{U}_U)]^{-1/2} B_U^{1/2} (I_K + ZU^*)^{-1} DF(Z)(I_K + ZU^*) B_U^{-1/2} [B_Z(\mathfrak{U}_U)]^{1/2} P\| \tag{2.43}$$

where $B_Z(\mathfrak{U}_U)$ *are defined by* (2.14) *and* (2.18). *If* $U^*U = I_H$ *and* $P \in \mathfrak{U}$, *then*

$$\|P\| \leq \|[B_Z(\mathfrak{U}_U)]^{-1/2} (I_K + ZU^*)^{-1} DF(Z)(I_K + ZU^*)[B_Z(\mathfrak{U}_U)]^{1/2} P\| \tag{2.44}$$

where $B_Z(\mathfrak{U}_U)$ *are defined by* (2.20).

(*b*) *Let* $F: \mathfrak{U}_0 \to \mathfrak{U}$ *be a map holomorphic in* $\mathfrak{U}_0$. *Suppose the following holds:* (*i*) F *is expansive,* i.e. $\mathfrak{U}_0 \subset F(\mathfrak{U}_0)$; (*ii*) F *is injective on* $\mathfrak{U}_0$; (*iii*) F^{-1} *is holomorphic in* $\mathfrak{U}_0$; *and* (*iv*) $F(Z) = Z$ *for some* $Z \in \mathfrak{U}_0$. *Then*

$$\|Z\| \leq \|(I_K - ZZ^*)^{-1/2} DF(Z)(Z)(I_H - Z^*Z)^{1/2}\|. \tag{2.45}$$

In (2.45) *the equality holds for biholomorphic maps* $F_W: \mathfrak{U}_0 \to \mathfrak{U}_0$, $W \in \mathfrak{U}_0$, *of the forms*

$$F_W(X) = B_W^{-1/2}(W - X)(I_H - W^*X)^{-1} A_W^{1/2} \tag{2.46}$$

and for their fixed points

$$Z = W(I_H + A_W^{1/2})^{-1}, \tag{2.47}$$

where $A_W = I_H - W^*W$, $B_W = I_K - WW^*$.

(*c*) *Let* $F: H_0 \to H$ *be a map holomorphic in* H_0. *Suppose the following holds:* (*i*) F *is expansive,* i.e. $H_0 \subset F(H_0)$; (*ii*) F *is injective on* H_0; (*iii*) F^{-1} *is holomorphic in* H_0 *and* (*iv*) $F(z) = z$ *for some* $z \in H_0$. *Then*

$$\|z\| \leq \|(I_H - \|z\|^2 E_z)^{-1/2} DF(z)(z)\| (1 - \|z\|^2)^{1/2}. \tag{2.48}$$

In (2.48) *the equality holds for biholomorphic maps* $F_W: H_0 \to H_0$, $w \in H_0$, *of the forms* $F(x) = (1 - \langle x, w\rangle)^{-1} Q_w(w - x)$ *and for their fixed points* $z = w[1 + (1 - \|w\|^2)^{1/2}]^{-1}$.

REMARK 2.3. A consequence of the above result is that inequality (2.1) in [32, Theorem 2.3] may be replaced by (2.45).

3. PROOF OF THEOREM 2.1

In the sequel, fixing $x \in H_u$, let $F: H_u \to K$ be a map holomorphic in H_u such that $F(H_u) \subset K_V$ where H_u and K_v are arbitrary and fixed sets defined by (2.1)-(2.4), respectively; and let

$$W = F(x). \tag{3.1}$$

We denote

$$\begin{cases} U = f_u^{-1}(x) = (x - u)/(1 + \langle x, u\rangle), \\ V = f_v^{-1}(W) = (W - v)/(1 + \langle W, v\rangle), \end{cases} \tag{3.2}$$

where f_u and f_v are biholomorphic maps of H_0 onto H_u and K_0 onto K_v (cf. [16, Theorem 12] and [30, Theorem 5]) defined by the formulae

$$(3.3)\qquad \begin{cases} f_u(x) = (x+u)/(1-\langle x, u\rangle), & x \in H_0, \\ f_v(y) = (y+v)/(1-\langle y, v\rangle), & y \in K_0, \end{cases}$$

respectively.

Let now $T_{\boldsymbol{U}}: H_0 \to H_0$ and $T_{\boldsymbol{V}}: K_0 \to K_0$ be Möbius biholomorphic maps of the forms [16, Theorem 2]

$$\begin{cases} T_{\boldsymbol{U}}(s) = Q_{\boldsymbol{U}}(\boldsymbol{U}-s)/(1-\langle s, \boldsymbol{U}\rangle), & s \in H_0, \\ T_{\boldsymbol{V}}(s) = Q_{\boldsymbol{V}}(\boldsymbol{V}-s)/(1-\langle s, \boldsymbol{V}\rangle), & s \in K_0, \end{cases}$$

respectively.

Consider a map $g: H_0 \to K$ holomorphic in H_0, such that $g(H_0) \subset K_0$, of the form

$$g(s) = (T_{\boldsymbol{V}} \circ f_V^{-1} \circ F \circ f_u \circ T_{\boldsymbol{U}}^{-1})(s), \qquad s \in H_0.$$

Since $T_{\boldsymbol{U}}^{-1}(0) = \boldsymbol{U}$ and $T_{\boldsymbol{V}}(\boldsymbol{V}) = 0$, we get $g(0) = 0$. Thus (see L. A. Harris [14])

$$(3.4)\qquad \|Dg(0)(p)\| \leq \|p\|, \qquad p \in H.$$

For $p \in H$, by the inverse map theorem (see [26]) and by the chain rule, we have

$$Dg(0)(p) = DT_{\boldsymbol{V}}(\boldsymbol{V})\{Df_v^{-1}(\boldsymbol{W})\{DF(x)\{[Df_u^{-1}(x)]^{-1}\{[DT_{\boldsymbol{U}}(\boldsymbol{U})]^{-1}(p)\}\}\}\}.$$

Thus, by (3.4), if we replace p with $DT_{\boldsymbol{U}}(\boldsymbol{U})\{Df_u^{-1}(x)(p)\}$, we obtain

$$(3.5)\qquad \|DT_{\boldsymbol{V}}(\boldsymbol{V})\{Df_v^{-1}(\boldsymbol{W})\{DF(x)(p)\}\}\| \leq \|DT_{\boldsymbol{U}}(\boldsymbol{U})\{Df_u^{-1}(x)(p)\}\|, \qquad p \in H.$$

But

$$DT_{\boldsymbol{U}}(s)(p) = -(Q_{\boldsymbol{U}}[p(1-\langle s, \boldsymbol{U}\rangle) + (s-\boldsymbol{U})\langle p, \boldsymbol{U}\rangle])/(1-\langle s, \boldsymbol{U}\rangle)^2, \quad s \in H_0, \ p \in H,$$

$$DT_{\boldsymbol{V}}(s)(p) = -(Q_{\boldsymbol{V}}[p(1-\langle s, \boldsymbol{V}\rangle) + (s-\boldsymbol{V})\langle p, \boldsymbol{V}\rangle])/(1-\langle s, \boldsymbol{V}\rangle)^2, \quad s \in K_0, \ p \in K,$$

and, in particular,

$$(3.6)\qquad \begin{cases} DT_{\boldsymbol{U}}(\boldsymbol{U})(p) = -Q_{\boldsymbol{U}}(p)/(1-\|\boldsymbol{U}\|^2), & p \in H, \\ DT_{\boldsymbol{V}}(\boldsymbol{V})(p) = -Q_{\boldsymbol{V}}(p)/(1-\|\boldsymbol{V}\|^2), & p \in K. \end{cases}$$

Moreover, we obtain,

$$(3.7)\qquad Df_u^{-1}(x)(p) = (p(1+\langle x, u\rangle) - (x-u)\langle p, u\rangle)/(1+\langle x, u\rangle)^2, \qquad p \in H,$$

$$(3.8)\qquad Df_v^{-1}(\boldsymbol{W})(p) = (p(1+\langle \boldsymbol{W}, v\rangle) - (\boldsymbol{W}-v)\langle p, v\rangle)/(1+\langle \boldsymbol{W}, v\rangle)^2, \qquad p \in K,$$

Consequently, using (3.6)-(3.8), from (3.5) we get, for $p \in H$,

$$(3.9)\qquad \|[1+\langle \boldsymbol{W}, v\rangle]Q_{\boldsymbol{V}}[DF(x)(p)] - \langle DF(x)(p), v\rangle Q_{\boldsymbol{V}}(\boldsymbol{W}-v)\| \leq$$

$$\leq \frac{|1+\langle \boldsymbol{W}, v\rangle|^2 - \|\boldsymbol{W}-v\|^2}{|1+\langle x, u\rangle|^2 - \|x-u\|^2} \cdot \|[1+\langle x, u\rangle]Q_{\boldsymbol{U}}(p) - \langle p, u\rangle Q_{\boldsymbol{U}}(x-u)\|.$$

Finally, let us observe that

(3.10) $Q_{\boldsymbol{U}}(p) = Q_{x-u}(p)$ for $p \in H$ and $Q_{\boldsymbol{V}}(p) = Q_{W-v}(p)$ for $p \in K$.

By (3.1), (3.2) and (3.10), inequality (2.33) is a simple consequence of (3.9).

4. Proof of Theorem 2.2

Fixing $x \in H_u$, let $F: H_u \to K$ be a map biholomorphic in H_u where H_u is an arbitrary and fixed set defined by (2.1) or (2.2) and let $\boldsymbol{U}$ be defined by (3.2).

Let $g_{\boldsymbol{U}}: H_0 \to K$ be a biholomorphic map defined by the formula

$$g_{\boldsymbol{U}}(s) = (F \circ f_u \circ T_{\boldsymbol{U}})(s) - (F \circ f_u)(\boldsymbol{U}), \quad s \in H_0, \tag{4.1}$$

where $T_{\boldsymbol{U}}$ is a Möbius biholomorphic map of H_0 onto H_0 defined by the formula

$$T_{\boldsymbol{U}}(s) = Q_{\boldsymbol{U}}(\boldsymbol{U} + s)/(1 + \langle s, \boldsymbol{U}\rangle), \quad s \in H_0,$$

and f_u is defined by (3.3). Since

$$\begin{aligned} Dg_{\boldsymbol{U}}(0)(p) &= DF[(f_u \circ T_{\boldsymbol{U}})(0)]\{Df_u[T_{\boldsymbol{U}}(0)]\{DT_{\boldsymbol{U}}(0)(p)\}\} = \\ &= DF(x)\{Df_u(\boldsymbol{U})\{DT_{\boldsymbol{U}}(0)(p)\}\}, \quad p \in H, \end{aligned}$$

therefore, by using the inverse map theorem, the map $h_{\boldsymbol{U}}: H_0 \to H$ defined by the formula

$$h_{\boldsymbol{U}}(s) = [DT_{\boldsymbol{U}}(0)]^{-1}\{[Df_u(\boldsymbol{U})]^{-1}\{[DF(x)]^{-1}[g_{\boldsymbol{U}}(s)]\}\}, \quad s \in H_0, \tag{4.2}$$

is biholomorphic and satisfies the conditions

$$h_{\boldsymbol{U}}(0) = 0 \quad \text{and} \quad Dh_{\boldsymbol{U}}(0) = I_H. \tag{4.3}$$

Now, let us observe that

$$Df_u(\boldsymbol{U})(p) = \frac{(1 - \langle \boldsymbol{U}, u\rangle)p + (\boldsymbol{U} + u)\langle p, u\rangle}{(1 - \langle \boldsymbol{U}, u\rangle)^2}, \quad p \in H,$$

and

$$DT_{\boldsymbol{U}}(s)(p) = -\frac{Q_{\boldsymbol{U}}[p(1 + \langle s, \boldsymbol{U}\rangle) - (s + \boldsymbol{U})\langle p, \boldsymbol{U}\rangle]}{(1 + \langle s, \boldsymbol{U}\rangle)^2}, \quad s \in H_0, \ p \in H.$$

Thus $\|Df_u(\boldsymbol{U})(p)\| \leqslant A_{x,u} \cdot \|p\|$ where

$$A_{x,u} = \frac{|1 - \langle \boldsymbol{U}, u\rangle| + \|u\| \|\boldsymbol{U} + u\|}{|1 - \langle \boldsymbol{U}, u\rangle|^2}, \quad p \in H. \tag{4.4}$$

Further, for $p \in H$, $DT_{\boldsymbol{U}}(0)(p) = -Q_{\boldsymbol{U}}[p - \boldsymbol{U}\langle p, \boldsymbol{U}\rangle] = -B_{x,u}(p)$ where

$$B_{x,u} = (1 - \|\boldsymbol{U}\|^2)^{1/2}\{I_H - [1 - (1 - \|\boldsymbol{U}\|^2)^{1/2}]E_{x-u}\}. \tag{4.5}$$

By (4.2),

$$\begin{aligned} \|g_{\boldsymbol{U}}(s)\| &\leqslant \|DF(x)\| \|Df_u(\boldsymbol{U})\{DT_{\boldsymbol{U}}(0)[h_{\boldsymbol{U}}(s)]\}\| \leqslant \\ &\leqslant \|DF(x)\| A_{x,u} \cdot \|B_{x,u}\| \|h_{\boldsymbol{U}}(s)\|, \quad s \in H_0, \end{aligned}$$

and, from (4.1) and (4.3) we get

$$\operatorname{dist}[(F \circ f_u)(s), \partial(F \circ f_u)(H_0)] \leqslant \|Df(x)\| A_{x,u} \cdot \|B_{x,u}\| \cdot \operatorname{dist}[0, \partial h_{\boldsymbol{U}}(H_0)], \quad s \in H_0.$$

Thus, for $s = \boldsymbol{U}$, we obtain

$$\text{(4.6)} \qquad \operatorname{dist}[F(x), \partial F(H_u)] \leqslant \|DF(x)\| A_{x,u} \cdot \|B_{x,u}\| \cdot \lambda$$

where $\lambda = \operatorname{dist}[0, \partial h_{\boldsymbol{U}}(H_0)]$.

Now, let us observe that

$$\text{(4.7)} \qquad \lambda \leqslant 1 .$$

Indeed, by (4.3) and [17, 21], we have $\lambda > 0$. Moreover, $\lambda < \infty$; otherwise, by the Liouville theorem for holomorphic maps in complex Banach spaces, the map h_U^{-1} is constant, a contradiction. Let $\omega(s) = h_{\boldsymbol{U}}^{-1}(\lambda s)$. Since $\omega: H_0 \to H_0$ is holomorphic, $\omega(0) = 0$ and $Dh_{\boldsymbol{U}}^{-1}(0) = I_H$, therefore, by the Schwarz lemma [14] and the chain rule, $\|\lambda s\| = \|D\omega(0)(s)\| \leqslant \|s\|$, $s \in H$, which implies (4.7).

Inequality (2.34) is a simple consequence of (4.4)-(4.7).

5. Proof of Theorem 2.3

In the sequel, fixing $X \in \mathfrak{U}_U$, let F be a holomorphic map of $\mathfrak{U}_U$ into $\mathfrak{B}$ such that $F(\mathfrak{U}_U) \subset \mathfrak{B}_V$ where $\mathfrak{U}_U$ and $\mathfrak{B}_V$ are arbitrary and fixed sets defined by (2.7)-(2.12) and let

$$\text{(5.1)} \qquad \boldsymbol{W} = F(X) .$$

Let us observe that, when $U^* U \neq I_H$,

$$\text{(5.2)} \quad UA_U^{-1/2} = B_U^{-1/2} U = 2^{-1/2} U , \quad A_U^{-1} = I_H - 2^{-1} U^* U, \quad B_U^{-1} = I_K - 2^{-1} UU^* ,$$

and, when $V^* V \neq I_M$,

$$\text{(5.3)} \quad VA_V^{-1/2} = B_V^{-1/2} V = 2^{-1/2} V , \quad A_V^{-1} = I_M - 2^{-1} V^* V, \quad B_V^{-1} = I_N - 2^{-1} VV^* .$$

If we denote

$$\text{(5.4)} \quad \boldsymbol{U} = f_U^{-1}(X) = B_U^{-1/2}(X - U)(I_H + U^* X)^{-1} A_U^{1/2} \quad \text{when } U^* U \neq I_H ,$$

$$\text{(5.5)} \quad \boldsymbol{U} = f_U^{-1}(X) = (X - U)(I_H + U^* X)^{-1} \quad \text{when } U^* U = I_H ,$$

and

$$\boldsymbol{V} = f_V^{-1}(\boldsymbol{W}) = B_V^{-1/2}(\boldsymbol{W} - V)(I_M + V^* \boldsymbol{W})^{-1} A_V^{1/2} \quad \text{when} \quad V^* V \neq I_M ,$$

$$\boldsymbol{V} = f_V^{-1}(\boldsymbol{W}) = (\boldsymbol{W} - V)(I_M + V^* \boldsymbol{W})^{-1} \quad \text{when} \quad V^* V = I_M ,$$

where f_U and f_V are biholomorphic maps of $\mathfrak{U}_0$ onto $\mathfrak{U}_U$ and $\mathfrak{B}_0$ onto $\mathfrak{B}_V$ (cf. [18, Proposition 2] and [30, Theorem 5]) defined by the formulae

$$f_U(X) = B_U^{-1/2}(X + U)(I_H - U^* X)^{-1} A_U^{1/2}, X \in \mathfrak{U}_0 , \quad \text{when} \quad U^* U \neq I_H ,$$

$$f_U(X) = (X + U)(I_H - U^* X)^{-1}, X \in \mathfrak{U}_0 , \quad \text{when} \quad U^* U = I_H ,$$

and

$$f_V(Y) = B_V^{-1/2}(Y + V)(I_M - V^* Y)^{-1} A_V^{1/2}, \quad Y \in \mathfrak{B}_0 , \text{ when } V^* V \neq I_M ,$$

$$f_V(Y) = (Y + V)(I_M - V^* Y)^{-1}, \quad Y \in \mathfrak{B}_0 , \text{ when } V^* V = I_M ,$$

respectively, then, by (5.2) and (5.3), we obtain

$$\text{(5.6)} \quad A_{\boldsymbol{U}} = I_H - \boldsymbol{U}^* \boldsymbol{U} = A_X(\mathfrak{U}_U) \quad \text{and} \quad A_{\boldsymbol{V}} = I_M - \boldsymbol{V}^* \boldsymbol{V} = A_{\boldsymbol{W}}(\mathfrak{B}_V) .$$

Moreover, since

$$(X-U)(I_H+U^*X)^{-1}=B_U(I_K+XU^*)^{-1}(X-U)A_U^{-1} \quad \text{when } U^*U\neq I_H\,,$$
$$(X-U)(I_H+U^*X)^{-1}=(I_K+XU^*)^{-1}(X-U) \quad \text{when } U^*U=I_H\,,$$

and

$$(\boldsymbol{W}-V)(I_M+V^*\boldsymbol{W})^{-1}=B_V(I_N+\boldsymbol{W}V^*)^{-1}(\boldsymbol{W}-V)A_V^{-1} \quad \text{when } V^*V\neq I_M\,,$$
$$(\boldsymbol{W}-V)(I_M+V^*\boldsymbol{W})^{-1}=(I_N+\boldsymbol{W}V^*)^{-1}(\boldsymbol{W}-V) \quad \text{when } V^*V=I_M\,,$$

we also get

(5.7) $$B_{\boldsymbol{U}}=I_K-\boldsymbol{U}\boldsymbol{U}^*=B_X(\mathfrak{U}_U) \quad \text{and} \quad B_{\boldsymbol{V}}=I_N-\boldsymbol{V}\boldsymbol{V}^*=B_{\boldsymbol{W}}(\mathfrak{B}_V)\,.$$

Hence, in particular, after taking account of $B_{\boldsymbol{U}}>0$ (since $\boldsymbol{U}\in\mathfrak{U}_0$) and $B_{\boldsymbol{V}}>0$ (since $\boldsymbol{V}\in\mathfrak{B}_0$), we obtain, by (5.6), (5.7) and (2.13)-(2.32),

$$\mathrm{RE}\,XU^*>0 \quad \text{and} \quad \mathrm{RE}\,\boldsymbol{W}V^*>0\,.$$

Let now $T_{\boldsymbol{U}}:\mathfrak{U}_0\to\mathfrak{U}_0$ and $T_{\boldsymbol{V}}:\mathfrak{B}_0\to\mathfrak{B}_0$ be Möbius biholomorphic maps of the forms [16, Theorem 2]

$$T_{\boldsymbol{U}}(S)=B_{\boldsymbol{U}}^{-1/2}(\boldsymbol{U}-S)(I_H-\boldsymbol{U}^*S)^{-1}A_{\boldsymbol{U}}^{1/2}\,, \qquad S\in\mathfrak{U}_0\,,$$

where

$$A_{\boldsymbol{U}}=I_H-\boldsymbol{U}^*\boldsymbol{U} \quad \text{and} \quad B_{\boldsymbol{U}}=I_K-\boldsymbol{U}\boldsymbol{U}^*\,,$$

and

$$T_{\boldsymbol{V}}(S)=B_{\boldsymbol{V}}^{-1/2}(\boldsymbol{V}-S)(I_M-\boldsymbol{V}^*S)^{-1}A_{\boldsymbol{V}}^{1/2}\,, \qquad S\in\mathfrak{B}_0\,,$$

where

$$A_{\boldsymbol{V}}=I_M-\boldsymbol{V}^*\boldsymbol{V} \quad \text{and} \quad B_{\boldsymbol{V}}=I_N-\boldsymbol{V}\boldsymbol{V}^*\,,$$

respectively.

Consider the map $g\in\mathfrak{U}_0\to\mathfrak{B}$ holomorphic in $\mathfrak{U}_0$, such that $g(\mathfrak{U}_0)\subset\mathfrak{B}_0$, of the form

$$g(S)=(T_{\boldsymbol{V}}\circ f_V^{-1}\circ F\circ f_U\circ T_{\boldsymbol{U}}^{-1})(S)\,, \qquad S\in\mathfrak{U}_0\,.$$

Since $T_{\boldsymbol{U}}^{-1}(0)=\boldsymbol{U}$ by (5.2) and $T_{\boldsymbol{V}}(\boldsymbol{V})=0$, we get $g(0)=0$. Thus (see [14])

(5.8) $$\|Dg(0)(P)\|\leqslant\|P\|\,, \qquad P\in\mathfrak{U}\,.$$

For $S\in\mathfrak{U}_0$ and $P\in\mathfrak{U}$, by the chain rule, we get

$$Dg(S)(P)=DT_{\boldsymbol{V}}[(f_V^{-1}\circ F\circ f_U\circ T_{\boldsymbol{U}}^{-1})(S)]\big\{Df_V^{-1}[(F\circ f_U\circ T_{\boldsymbol{U}}^{-1})(S)]\big\{DF[(f_U\circ T_{\boldsymbol{U}}^{-1})(S)]\cdot$$
$$\cdot\big\{Df_U[T_{\boldsymbol{U}}^{-1}(S)]\{DT_{\boldsymbol{U}}^{-1}(S)(P)\}\big\}\big\}\big\}$$

and, in particular,

(5.9) $$Dg(0)(P)=DT_{\boldsymbol{V}}(\boldsymbol{V})\big\{Df_V^{-1}(\boldsymbol{W})\big\{DF(X)\big\{Df_U(\boldsymbol{U})\{DT_{\boldsymbol{U}}^{-1}(0)(P)\}\big\}\big\}\big\}=$$
$$=DT_{\boldsymbol{V}}(\boldsymbol{V})\big\{Df_V^{-1}(\boldsymbol{W})\big\{DF(X)\big\{[Df_U^{-1}(X)]^{-1}\{[DT_{\boldsymbol{U}}(\boldsymbol{U})]^{-1}(P)\}\big\}\big\}\big\}\,.$$

Let us observe that if $P\in\mathfrak{U}$, then also $R\in\mathfrak{U}$ where

(5.10) $$R=DT_{\boldsymbol{U}}(\boldsymbol{U})\{Df_U^{-1}(X)(P)\}\,.$$

From (5.8), where we replace P by R, using (5.9), we get

$$(5.11)\qquad \|DT_{\boldsymbol{V}}(V)\{Df_V^{-1}(W)\{DF(X)(P)\}\}\| \leqslant \|R\|$$

where R is of form (5.10) and $P \in \mathfrak{U}$.

We have

$$(5.12)\qquad \begin{cases} DT_{\boldsymbol{U}}(S)(P) = -B_{\boldsymbol{U}}^{1/2}(I_K - SU^*)^{-1}P(I_H - U^*S)^{-1}A_{\boldsymbol{U}}^{1/2}, & S \in \mathfrak{U}_0,\ P \in \mathfrak{U}, \\ DT_{\boldsymbol{V}}(S)(P) = -B_{\boldsymbol{V}}^{1/2}(I_N - SV^*)^{-1}P(I_M - V^*S)^{-1}A_{\boldsymbol{V}}^{1/2}, & S \in \mathfrak{B}_0,\ P \in \mathfrak{B}, \end{cases}$$

and, in particular,

$$(5.13)\qquad DT_{\boldsymbol{U}}(U)(P) = -B_{\boldsymbol{U}}^{-1/2}PA_{\boldsymbol{U}}^{-1/2}, \qquad P \in \mathfrak{U},$$

$$(5.14)\qquad DT_{\boldsymbol{V}}(V)(P) = -B_{\boldsymbol{V}}^{-1/2}PA_{\boldsymbol{V}}^{-1/2}, \qquad P \in \mathfrak{B}.$$

Moreover, for $S \in \mathfrak{U}_0$, $P \in \mathfrak{U}$,

$$(5.15)\qquad Df_U^{-1}(S)(P) = B_U^{1/2}(I_K + SU^*)^{-1}P(I_H + U^*S)^{-1}A_U^{1/2} \quad \text{when } U^*U \neq I_H,$$

$$(5.16)\qquad Df_U^{-1}(S)(P) = (I_K + SU^*)^{-1}P(I_H + U^*S)^{-1} \quad \text{when } U^*U = I_H,$$

and, for $S \in \mathfrak{B}_V$, $P \in \mathfrak{B}$,

$$(5.17)\qquad Df_V^{-1}(S)(P) = B_V^{1/2}(I_N + SV^*)^{-1}P(I_M + V^*S)^{-1}A_V^{1/2} \quad \text{when } V^*V \neq I_M,$$

$$(5.18)\qquad Df_V^{-1}(S)(P) = (I_N + SV^*)^{-1}P(I_M + V^*S)^{-1} \quad \text{when } V^*V = I_M.$$

Thus, by (5.11), using (5.10) and (5.12)-(5.18), for $P \in \mathfrak{U}$, we get

$$(5.19)\qquad \|B_{\boldsymbol{V}}^{-1/2}B_V^{1/2}(I_N + WV^*)^{-1}DF(X)(P)(I_M + V^*W)^{-1}A_V^{1/2}A_{\boldsymbol{V}}^{-1/2}\| \leqslant$$
$$\leqslant \|B_{\boldsymbol{U}}^{-1/2}B_U^{1/2}(I_K + XU^*)^{-1}P(I_H + U^*X)^{-1}A_U^{1/2}A_{\boldsymbol{U}}^{-1/2}\|$$

when $U^*U \neq I_H$ and $V^*V \neq I_M$;

$$(5.20)\qquad \|B_{\boldsymbol{V}}^{-1/2}(I_N + WV^*)^{-1}DF(X)(P)(I_M + V^*W)^{-1}A_{\boldsymbol{V}}^{-1/2}\| \leqslant$$
$$\leqslant \|B_{\boldsymbol{U}}^{-1/2}B_U^{1/2}(I_K + XU^*)^{-1}P(I_H + U^*X)^{-1}A_U^{1/2}A_{\boldsymbol{U}}^{-1/2}\|$$

when $U^*U \neq I_H$ and $V^*V = I_M$;

$$(5.21)\qquad \|B_{\boldsymbol{V}}^{-1/2}B_V^{1/2}(I_N + WV^*)^{-1}DF(X)(P)(I_M + V^*W)^{-1}A_V^{1/2}A_{\boldsymbol{V}}^{-1/2}\| \leqslant$$
$$\leqslant \|B_{\boldsymbol{U}}^{-1/2}(I_K + XU^*)^{-1}P(I_H + U^*X)^{-1}A_{\boldsymbol{U}}^{-1/2}\|$$

when $U^*U = I_H$ and $V^*V \neq I_M$;

$$(5.22)\qquad \|B_{\boldsymbol{V}}^{-1/2}(I_N + WV^*)^{-1}DF(X)(P)(I_M + V^*W)^{-1}A_{\boldsymbol{V}}^{-1/2}\| \leqslant$$
$$\leqslant \|B_{\boldsymbol{U}}^{-1/2}(I_K + XU^*)^{-1}P(I_H + U^*X)^{-1}A_{\boldsymbol{U}}^{-1/2}\|$$

when $U^*U = I_H$ and $V^*V = I_M$. Consequently, for $P \in \mathfrak{U}$, in virtue of (5.6) and (5.7), by using notations (5.1) and (2.7)-(2.32), inequalities (5.19)-(5.22) imply assertions (2.38)-(2.41), respectively.

6. Proof of Theorem 2.4

(a) Using (2.38) for the map F^{-1} and the point $X = Z$ and, next, replacing P by $DF(Z)(I_K + ZU^*)B_U^{-1/2}[B_Z(\mathfrak{U}_U)]^{1/2}P[A_Z(\mathfrak{U}_U)]^{1/2}A_U^{-1/2}(I_H + U^*Z)$, we immediately obtain (2.43). Inequality (2.44) is a consequence of (2.41) by an analogous argument.

(b) Let $A_Z = I_H - Z^*Z$ and $B_Z = I_K - ZZ^*$. Using inequality (2.43) for $U = 0$ and $P = Z$, we get $\|Z\| \leq \|B_Z^{-1/2} DF(Z) B_Z^{1/2} Z\|$ which is identical with (2.45) since $B_Z^{1/2} Z = Z A_Z^{1/2}$.

Estimate (2.45) is precise for F_W defined by (2.46) and their fixed points (2.47) determined in [31, Theorem 2.1 (c)]. Indeed, we have $DF_W(Z)(Z) = -B_W^{1/2}(I_K - ZW^*)^{-1} Z (I_H - W^*Z)^{-1} A_W^{1/2}$, and (2.45) is then of the form

$$(6.1) \qquad \|Z\| \leq \|B_Z^{-1/2} B_W^{1/2} (I_K - ZW^*)^{-1} Z (I_H - W^*Z)^{-1} A_W^{1/2} A_Z^{1/2}\|.$$

Now, let us observe that, by (2.47),

$$(6.2) \qquad (I_H - W^*Z)^{-1} = (I_H + A_W^{1/2})(A_W + A_W^{1/2})^{-1} = A_W^{-1/2}.$$

Moreover, since $A_W^{1/2} W^* = W^* B_W^{1/2}$, we get $W(I_H + A_W^{1/2})^{-1} W^* = WW^*(I_K + B_W^{1/2})^{-1}$, and thus,

$$(6.3) \qquad (I_K - ZW^*)^{-1} = (I_K + B_W^{1/2})(B_W + B_W^{1/2})^{-1} = B_W^{-1/2}.$$

Consequently, inequality (6.1) (by (6.2) and (6.3)) is, in fact, the equality

$$\|Z\| \leq \|B_Z^{-1/2} Z A_Z^{1/2}\| = \|Z A_Z^{-1/2} A_Z^{1/2}\| = \|Z\|.$$

(c) Identifying H with the J^*-algebra $\mathcal{L}(C, H)$, from (b) we get (c).

References

[1] R. W. Barnard - C. H. FitzGerald - S. Gong, *The growth and 1/4-theorems for starlike functions in C^n.* Pacific J. Math., 150, 1991, 13-22.

[2] R. W. Barnard - C. H. FitzGerald - S. Gong, *A distortion theorem for biholomorphic mappings in C^2.* Trans. Amer. Math. Soc., 344, 1994, 907-924.

[3] D. M. Burns - S. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary.* J. Amer. Math. Soc., 7, 1994, 661-676.

[4] H. Cartan, *Sur la possibilité d'étendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalentes.* Note added to P. Montel, *Leçons sur les Fonctions Univalentes ou Multivalentes,* Gauthier-Villars, Paris 1933, 129-155.

[5] E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes.* Abh. Math. Sem. Univ. Hamburg, 11, 1935, 116-162.

[6] J. M. Cohen - F. Colonna, *Bounded holomorphic functions on bounded symmetric domains.* Trans. Amer. Math. Soc., 343, 1994, 135-156.

[7] S. Dineen, *The Schwarz Lemma.* Oxford Mathematical Monographs, Clarendon Press, Oxford 1989.

[8] P. Duren - W. Rudin, *Distortion in several variables.* Complex Variables Theory Appl., 5, 1986, 323-326.

[9] C. H. FitzGerald - C. R. Thomas, *Some bounds on convex mappings in several complex variables.* Pacific J. Math., 165, 1994, 295-320.

[10] T. Franzoni - E. Vesentini, *Holomorphic Maps and Invariant Distances.* North-Holland Mathematics Studies, 40, Amsterdam-New York-Oxford 1980.

[11] S. Gong - S. K. Wang - Q. H. Yu, *Biholomorphic convex mappings of ball in C^n.* Pacific J. Math., 161, 1993, 287-306.

[12] S. Gong - S. K. Wang - Q. H. Yu, *The growth theorem for biholomorphic mappings in several complex variables.* Chinese Ann. Math., Ser. B 14, 1993, 93-104.

[13] I. GRAHAM, *Distortion theorems for holomorphic maps between convex domains in C^n.* Complex Variables Theory Appl., 15, 1990, 37-42.

[14] L. A. HARRIS, *Schwarz's lemma in normed linear spaces.* Proc. Nat. Acad. Sci. U.S.A., 62, 1969, 1014-1017.

[15] L. A. HARRIS, *A continuous form of Schwarz's lemma in normed linear spaces.* Pacific J. Math., 38, 1971, 635-639.

[16] L. A. HARRIS, *Bounded Symmetric Homogeneous Domains in Infinite Dimensional Spaces.* Lecture Notes in Mathematics, 364, Springer-Verlag, Berlin-Heidelberg-New York 1974, 13-40.

[17] L. A. HARRIS, *On the size of balls covered by analytic transformations.* Monatsh. Math., 83, 1977, 9-23.

[18] L. A. HARRIS, *Operator Siegel domains.* Proc. Roy. Soc. Edinburgh, 79 A, 1977, 137-156.

[19] L. A. HARRIS, *Linear fractional transformations of circular domains in operator spaces.* Indiana Univ. Math. J., 41, 1992, 125-147.

[20] M. HERVÉ, *Lindelöf's Principle in Infinite Dimensions.* Lecture Notes in Mathematics, 364, Springer-Verlag, Berlin-Heidelberg-New York 1974, 41-57.

[21] M. HERVÉ, *Some Properties of the Images of Analytic Maps.* North-Holland Mathematics Studies 12, Amsterdam-New York-Oxford 1977, 217-229.

[22] M. HERVÉ, *Analyticity in Infinite Dimensional Spaces.* de Gruyter Studies in Mathematics, 10, Berlin-New York 1989.

[23] Y. KUBOTA, *Some results on rigidity of holomorphic mappings.* Kodai Math. J., 17, 1994, 246-255.

[24] I. I. PJATECKIJ-SHAPIRO, *Automorphic Functions and the Geometry of Classical Domains.* Gordon-Breach, New York 1969.

[25] CH. POMMERENKE, *Univalent Functions.* Vandenhoeck and Ruprecht, Göttingen 1975.

[26] W. RUDIN, *Functional Analysis.* McGraw-Hill, New York 1973.

[27] W. RUDIN, *Function Theory in the Unit Ball of C^n.* Springer-Verlag, New York-Heidelberg-Berlin 1980.

[28] T. J. SUFFRIDGE, *Starlikeness, Convexity and other Geometric Properties of Holomorphic Maps in Higher Dimensions.* Lecture Notes in Mathematics, 599, Springer-Verlag, New York 1976, 146-159.

[29] E. VESENTINI, *Rigidity of holomorphic isometries.* Rend. Mat. Acc. Lincei, s. 9, vol. 5, 1994, 55-62.

[30] K. WŁODARCZYK, *On holomorphic maps in Banach spaces and J^*-algebras.* Quart. J. Math. Oxford, (2), 36, 1985, 495-511.

[31] K. WŁODARCZYK, *Hyperbolic geometry in bounded symmetric homogeneous domains of J^*-algebras.* Atti Sem. Mat. Fis. Univ. Modena, 39, 1991, 201-211.

[32] K. WŁODARCZYK, *Fixed points and invariant domains of expansive holomorphic maps in complex Banach spaces.* Adv. in Math., 110, 1995, 247-254.

[33] S. YAMASHITA, *The Pick version of the Schwarz lemma and comparison of the Poincaré densities.* Ann. Acad. Sci. Fenn., 19, 1994, 291-322.

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