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Seshadri positive curves in a smooth projective 3-fold

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Geometria algebrica. — *Seshadri positive curves in a smooth projective 3-fold.*
Nota (*) di Roberto Paoletti, presentata dal Corrisp. M. Cornalba.

Abstract. — A notion of positivity, called *Seshadri ampleness*, is introduced for a smooth curve C in a polarized smooth projective 3-fold (X, A), whose motivation stems from some recent results concerning the gonality of space curves and the behaviour of stable bundles on $\mathbb{P}^3$ under restriction to C. This condition is stronger than the normality of the normal bundle and more general than C being defined by a regular section of an ample rank-2 vector bundle. We then explore some of the properties of Seshadri-ample curves.

Key words: Seshadri constant; Ampleness; Bigness; Normal bundle; Cohomogical dimension.

Riassunto. — *Curve Seshadri-positive in una 3-varietà proiettiva liscia.* Si introduce una nozione di positività, denominata *Seshadri ampiezza*, per una curva non-singolare C in una varietà proiettiva liscia 3-dimensionale polarizzata (X, A), motivata da alcuni recenti risultati concernenti la gonalità di una curva nello spazio e il comportamento di fibrati vettoriali stabili su $\mathbb{P}^3$ sotto restrizione a una curva data. Questa condizione è più forte della normalità del fibrato vettoriale, e più generale dell'essere C definita da una sezione regolare di un fibrato ampio di rango due. Si esplorano quindi alcune proprietà delle curve Seshadri-ampie.

1. – Introduction

Let X be a projective manifold and $Y \subset X$ a non-singular ample divisor in it. It is a well-established and ubiquitous heuristic principle in algebraic geometry that the intrinsic geometry of Y should be reflected in the global properties of X. Apart from the best known classical results, arising for example from the work of Lefschetz, some striking more recent statements are due among others to Sommese and Fujita [19, 4], and of course the classification theory for varieties with suitable hyperplane sections is another vein of this area of research. It would be desirable to have a notion of positivity in higher codimensions, leading to a similar principle; this has at least partly motivated the theory of ample vector bundles (see [11] for a quick overview). In this paper we consider the simplest case of curves in 3-folds, and suggest that an ingredient of such a notion of positivity should be the Seshadri constant of the curve, and a related invariant of the embedding. Recall the following definition from [15]:

Definition 1.1. Let (X, A) be a smooth polarized projective threefold, $C \subset X$ a smooth connected curve, $N = N_{C/X}$ the normal bundle.

A) Let $f: X_C = \mathrm{Bl}_C(X) \to X$ be the blow-up of X along C, and $E \subset X_C$ the exceptional divisor. Then set $\varepsilon(C, A) = \sup\{x \in \mathbb{Q} \mid f^* H - xE \text{ is ample }\}$. The invariant $\varepsilon(C, A)$ is the *Seshadri constant* of C with respect to A.

B) For $\gamma \in \mathbb{Q}$ set $\delta_\gamma(C, A) = \gamma \cdot \deg(N) - A \cdot C$. Also set $\delta(C, A) = \delta_{\varepsilon(C,A)}(C, A)$. We shall often drop the polarization A.

(*) Pervenuta all'Accademia il 27 luglio 1995.

Let $A = \mathcal{O}_{\mathbb{P}^3}(1)$ in the above expressions; we then have [15]:

THEOREM A. *Let $C \subset \mathbb{P}^3$ be a smooth connected curve of degree d. Set* $\alpha = \min\{1, \sqrt{d}(1 - \varepsilon(C)\sqrt{d})\}$. *Then* $\mathrm{gon}(C) \geqslant \min\{\delta(C)/4\varepsilon(C), \alpha \cdot (d - \alpha/\varepsilon(C))\}$.

THEOREM B. *Let $\mathcal{E}$ be a stable rank-2 vector bundle on $\mathbb{P}^3$ with $c_1(\mathcal{E}) = 0$, and $C \subset \mathbb{P}^3$ smooth connected curve of degree d. Set* $\gamma =: \sup\{\eta \in [0, \varepsilon(C)] \,|\, f^*\mathcal{E}$ *is* $(H, H - \eta E)$-*stable*$\}$, $\alpha =: \min\{1, \sqrt{d}(\sqrt{3/4} - \gamma\sqrt{d})\}$. *If* $\mathcal{E}|_C$ *is not stable, then* $c_2(\mathcal{E}) \geqslant \min\{\delta_\gamma(C)/4, \alpha\gamma(d - \alpha/\gamma)\}$.

Both theorems actually generalize to arbitrary projective threefolds with $NS(X) \cong \mathbb{Z}$.

Comparison with corresponding results for the codimension one case (*e.g.*, from [17, 14, 2]), as well with some results of Lazarsfeld on the gonality of complete intersection curves in projective space, shows that the assumption $\delta(C) > 0$ plays the role of the positivity hypothesis on the divisor in the former case, and generalizes the one that C be a complete intersection in the latter. In this paper, we look at the class of curves to which these results may be applied, and add some substance to the idea that they form a generalization of complete intersection curves. To this end, we introduce the following:

DEFINITION 1.2. Let (X, A), $C \subset X$ and $E \subset X_C$ be as in Definition 1.1. Then C is Seshadri-big (respectively, Seshadri-ample) with respect to A if for some m, $n > 0$ such that $mA - nE$ is very ample, $\mathcal{O}_S(E)$ is big and nef (respectively, ample), for a general $S \in |mA - nE|$. For brevity, we shall also say that C is A-big or A-ample, respectively. C is Seshadri-ample (respectively, Seshadri-big) if it is Seshadri-ample (resp., Seshadri-big) with respect to some polarization A.

REMARK 1.1. Let (A,A) be a smooth polarized threefold, $C \subset X$ a smooth curve in it. If C is A-big, then it is connected. If it is connected and $\delta(C, A) > 0$ then it is A-big.

REMARK 1.2. A statement of Hard-Lefschetz type holds for the Poincaré dual classes of Seshadri ample curves. In other words, we have:

THEOREM. *Suppose that X is a smooth projective 3-fold and let $C \subset X$ be a smooth Seshadri-ample curve. If η_C is the Poincaré dual class to C, then* $\eta_C \wedge : H^1(X, \mathbb{C}) \to H^5(X, \mathbb{C})$ *is an isomorphism.*

A proof of this will be given elsewhere.

2. LEMMA 2.1. *Let (X,A) be a smooth polarized projective threefold, $C \subset X$ a smooth curve. If C is A-ample, then $H^1(X, \mathbb{Z}) \to H^1(C, \mathbb{Z})$ is injective.*

PROOF. Let $X_C = \mathrm{Bl}_C(X)$, $E \subset X_C$ the exceptional divisor. By assumption, there exist integers $m, n > 0$, such that $mA - nE$ is very ample, and for $S \in |mA - nE|$ general the line bundle $\mathcal{O}_S(E)$ is ample. We may assume that the curve $C' = S \cap E$ is smooth and irreducible. The inclusions $C' \subset S \subset X_C$ are thus embeddings of smooth ample divisors, and therefore all maps in the composition $f: H^1(X, \mathbb{Z}) \to H^1(X_C, \mathbb{Z}) \to H^1(S, \mathbb{Z}) \to \to H^1(C', \mathbb{Z})$ are injective. On the other hand, f can also be decomposed as $H^1(X, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to H^1(C', \mathbb{Z})$. ∎

LEMMA 2.2. *If C is A-ample, then it meets every surface in X.*

PROOF. Let $W \subset X$ be a surface, and let $\widetilde{W} \subset X_C$ be the inverse image of W in X_C. We want to show that E meets $\widetilde{W}$. Suppose that $mA - nE$ is very ample and that for $S \in |mA - nE|$ general $\mathcal{O}_S(E)$ is ample. For $S \in |mA - nE|$ general, the intersection $W_S = \widetilde{W} \cap S$ is a curve in S, and therefore $W_S \cdot E > 0$. ∎

COROLLARY 2.1. *Let $f: X \to Y$ be a dominant morphism, and suppose* $\dim(Y) > 0$. *If C is A-ample, then $f_*([C]) \neq 0$.*

Using the following result, it is easy to generalize Corollary 2.1 to A-big curves. However, this generalization follows more transparently once one shows that the normal bundle to an A-big curve is ample.

PROPOSITION 2.1. *Let (X,A) be a nonsingular polarized threefold, $C \subset X$ and A-ample curve. Then there exist quasi-projective varieties $\mathcal{C}$, $\mathcal{Y}$ and morphisms $\phi: \mathcal{C} \to X$, $p: \mathcal{C} \to \mathcal{Y}$, such that (i) ϕ is dominant and p is proper; (ii) $\forall y \in \mathcal{Y}$, the inverse image $C_y := p^{-1}(y)$ is a connected projective curve; (iii) if $y \in \mathcal{Y}$ is general, the curve C_y is smooth and irreducible, and $\phi|_{C_y}: C_y \to X$ is a closed embedding; (iv) for some $y_0 \in \mathcal{Y}$, the equality $\phi_*([C_{y0}]) = = s[C]$ holds at the cycle level; (v) the curves $\phi_*([C_y])$, $y \in \mathcal{Y}$, are all rationally equivalent.*

PROOF. Let us start by assuming that C is A-ample. Then there exist $m, n > 0$, such that $\mathcal{O}_{X_C}(mA - nE)$ is very ample, and that for $S \in |mA - nE|$ general the line bundle $\mathcal{O}_S(E)$ is ample. Hence, for such an S, there exists $r > 0$ such that $\mathcal{O}_S(rE)$ is very ample. Thus we can find $U \subset \mathbb{P}H^0(X_C, \mathcal{O}_{X_C}(mA - nE))$, such that for $u \in U$ the surface $S_u = = \operatorname{div}(u)$ is smooth and irreducible, the line bundle $\mathcal{O}_{S_u}(rE)$ is very ample, and $N = = h^0(S_u, \mathcal{O}_{S_u}(rE)) - 1$ is independent of $u \in U$. Let next $\mathcal{S} \subset X_C \times U$ be the incidence correspondence, and $\pi_1: \mathcal{S} \to X_C$, $\pi_2: \mathcal{S} \to U$ the projections. If we let $\mathcal{O}_{\mathcal{S}}(rE) := := \pi_1^*(\mathcal{O}_{X_C}(rE))$, then $\mathcal{E} := \pi_{2*}(\mathcal{O}_{\mathcal{S}}(rE))$ is a vector bundle on U, of rank $N+1$. Let us set $\mathcal{Y} = \mathbb{P}\mathcal{E}$, and $q: \mathcal{Y} \to U$ the projection.

On $\mathcal{Y}$, we have the tautological exact sequence $0 \to \mathcal{O}_{\mathcal{Y}}(-1) \to q^*\mathcal{E} \to \mathcal{Q} \to 0$, where $\mathcal{O}_{\mathcal{Y}}(-1) \subset q^*\mathcal{E}$ is the universal subbundle. Hence, on $\mathcal{S} \times_U \mathcal{Y}$ (with projections $p_1: \mathcal{S} \times_U \mathcal{Y} \to \mathcal{S}$ and $p_2: \mathcal{S} \times_U \mathcal{Y} \to \mathcal{Y}$) we have the exact sequence $0 \to p_2^*\mathcal{O}_{\mathcal{Y}}(-1) \to \to p_2^* q^* \mathcal{E} \to p_2^* \mathcal{Q} \to 0$. On the other hand, there is a morphism $p_2^* q^* \mathcal{E} = = p_1^* \pi_2^* \pi_{2*}(\mathcal{O}_{\mathcal{S}}(rE)) \to p_1^* \mathcal{O}_{\mathcal{S}}(rE)$. We thus have a composition $\psi: p_2^* \mathcal{O}_{\mathcal{Y}}(-1) \to \to p_1^* \mathcal{O}_{\mathcal{S}}(rE)$, and we can set $\mathcal{C} := \operatorname{div}(\psi) \subset \mathcal{S} \times_U \mathcal{Y}$.

Given $u \in U$, the inverse image of u in $\mathcal{C}$ can be identified with the incidence correspondence $\mathcal{C}_u \subset S_u \times Y_u$ of the linear series $|\mathcal{O}_{S_u}(rE)|$, and therefore the projection $\mathcal{C}_u \to S_u$ is onto. It follows that the sequence of morphisms $\phi: \mathcal{C} \to \mathcal{S} \to X_C \to X$ is dominant. Next, let $p\colon \mathcal{C} \to \mathcal{Y}$ the projection, and pick $y \in Y_u \cong |\mathcal{O}_{S_u}(rE)|$. Then $C_y = p^{-1}(y) \subset S_u$ can be identified with $\operatorname{div}(y)$. Hence, it is a connected projective curve, and for general $y \in Y_u$ it is smooth. Since for general $u \in U$ the map $S_u \to X$ is finite and birational onto its image, it follows from [3, § II.3] that for general $y \in Y_u$ the restriction $\phi_u\colon C_y \to X$ is a closed embedding.

Next the curves C_y, for $y \in Y_u$ and for a fixed $u \in U$, form the linear series $|\mathcal{O}_{S_u}(rE)|$, and therefore they are all linearly equivalent in $S_u \subset X_C$ to $r(S_u \cap E)$. The curves $r(S_u \cap E)$, on the other hand, move in the linear series $|\mathcal{O}_E(r(mA - nE))|$ when u moves in U, and thus they are all rationally equivalent in X_C. The proper push-forward of the C_y's in X, therefore, are also all linearly equivalent. Finally it is clear that $\phi_*(r[S_u \cap E]) = s[C]$, for some $s > 0$.

Suppose now that C is only A-big. Then for $S \in |mA - nE|$ general, and for $r \gg 0$, we know at least that the linear series $|\mathcal{O}_{S_u}(rE)|$ is base-point-free, and that the corresponding projective morphism $\phi_{\mathcal{O}_S(rE)}\colon S \to \mathbb{P}^N$ is birational onto its image, and an isomorphism in the neighbourhood of $S \cap E$. The argument then proceed as in the case where C is A-ample. ■

Remark 2.1. As shown in §1 of [6], similar conclusions do not hold in general dimensions under the simple assumption of ample normal bundle.

Proposition 2.2. *Let $X' \to X$ be a finite morphism of smooth projective threefolds, A a polarization on X, and $A' = f^*A$. Let $C \subset X$ be an irreducible smooth curve, and suppose that $C' := f^{-1}(C)$ is also smooth and irreducible. Then C is A-big (resp., A-ample) if and only if C' is A'-big (resp., A'-ample).*

Proof. Let us set $p\colon X_C = \mathrm{Bl}_C(X) \to X$ and $p'\colon X'_{C'} = \mathrm{Bl}_{C'}(X') \to X'$, and let $E \subset X_C$, $E' \subset X'_{C'}$ be the exceptional divisors. As $(f \circ p')^{-1}(C) = E'$, the universal property of blow-up implies the existence of a morphism $\tilde{f}\colon X'_{C'} \to X_C$, such that $f \circ p' = p \circ \tilde{f}$. It is easy to see that $\tilde{f}^*(E) = E'$, and that $\tilde{f}$ is a finite morphism. Hence $\mathcal{O}'_{X_{C'}}(mA' - nE') = \tilde{f}^*\mathcal{O}_{X_C}(mA - nE)$ is ample if and only if $\mathcal{O}_{X_C}(mA - nE)$ is. This shows that

$$\varepsilon(C, A) = \varepsilon(C', A'). \tag{1}$$

By the projection formula, we also have $C' \cdot A' = \deg(f) \cdot (C \cdot A)$ and clearly $\deg(N_{C'/A'}) = \deg(f) \cdot \deg(N_{C/A})$. Thus,

$$\delta(C', A') = \deg(f) \cdot \delta(C, A). \tag{2}$$

Hence, it follows that C is A-big if and only if C' is A'-big. Next, suppose that $mA - nE$ is very ample on X_C, and pick $S \in |mA - nE|$ general. Then, $S' = f^*S$ is smooth and irreducible, and the induced morphism $S' \to S$ is finite. Thus $\mathcal{O}_{S'}(E')$ is ample if and only if $\mathcal{O}_S(E)$ is, and this proves that C is A-ample if and only if C' is A'-ample. ■

Example 2.1. Let $C \subset \mathbb{P}^3$ be a complete intersection of type (a, b), with $a \geqslant b$. Then $\varepsilon(C) = 1/a$, and $\delta(C) = b^2$ [15].

More generally, we have:

PROPOSITION 2.3. *Let X be a smooth projective threefold, $\mathcal{E}$ and $\mathcal{F}$ vector bundles on X of ranks r and $r+1$, respectively, such that $\mathcal{E}^* \otimes \mathcal{F}$ is ample. Let $\phi: \mathcal{E} \to \mathcal{F}$ be a morphism, and suppose that the degeneracy locus $C = X_{r-1}(\phi)$ is smooth. If $\mathcal{E}$ (resp., $\mathcal{F}$) has a filtration whose successive quotients are line bundles algebraically equivalent to the trivial line bundle, then C is $\det(\mathcal{F})$-big (resp., $\det(\mathcal{E}^*)$-big).*

PROOF. Let us consider first the case where $\mathcal{E} = \mathcal{O}_X$, and $\mathcal{F}$ is an ample rank-2 vector bundle on X, $s \in H^0(X, \mathcal{F})$ a regular section of $\mathcal{F}$, and $C = Z(s)$. Suppose that C is smooth; it is then irreducible. We want to show that then C is $\det(\mathcal{F})$-big. The normal bundle to C in X is $N = \mathcal{F}|_C$, and furthermore $[C] = c_2(\mathcal{F})$. Hence, $\delta(C, A) = (\varepsilon(C, A) - 1)\det(\mathcal{F})\cdot c_2(\mathcal{F})$. The second factor is clearly positive, so it remains to check that $\varepsilon(C, \det(\mathcal{F})) > 1$. In other words, if X_C is the blow-up of X along C, and if $E \subset X$ is the exceptional divisor, then we need to check that $\det(\mathcal{F})(-E)$ is ample on X_C. By the hypothesis, we have an exact sequence

$$0 \to \det(\mathcal{F})^* \to \mathcal{F} \to \mathfrak{J} \to 0,$$

where $\mathfrak{J} \subset \mathcal{O}_X$ is the ideal sheaf of C. This yields a surjection of graded $\mathcal{O}_X$-algebras $\bigoplus S^n \mathcal{F}^* \to \bigoplus \mathfrak{J}^n$ and dualizing we obtain a closed embedding $X_C \hookrightarrow \mathbb{P}\mathcal{F}$, under which $\mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$ restricts to $\mathcal{O}_{X_C}(-E)$. Hence $\det(\mathcal{F})(-E)$ is the restriction of $\det(\mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}\mathcal{F}}(1)$, which is ample by assumption.

Let us consider the general case. We have $[C] = c_2(\mathcal{F} - \mathcal{E})$. Furthermore, $K = \operatorname{Ker}(\phi|_C)$ is a line bundle on C, and $\deg(K) = -c_3(\mathcal{F} - \mathcal{E})$ [1, § II.4]. Hence, if $N = N_{C/X}$ is the normal bundle, then

$$\deg(N) = (c_1(\mathcal{F}) - c_1(\mathcal{E}))\cdot c_2(\mathcal{F} - \mathcal{E}) + c_3(\mathcal{F} - \mathcal{E}). \tag{3}$$

Let us consider first the case where $\mathcal{E}$ has a filtration whose successive quotients are line bundles algebraically equivalent to the trivial line bundle. Then $\mathcal{F}$ is ample itself. We have $C = Z(\wedge^r \phi)$ and therefore a surjective morphism of sheaves $\{\wedge^r \mathcal{F} \otimes \det(\mathcal{E}^*)\}^* \to \mathfrak{J}$. As above, this gives rise to an embedding $X_C \hookrightarrow \mathbb{P}(\wedge^r \mathcal{F} \otimes \det(\mathcal{E})^{-1})$. Writing $\wedge^r \mathcal{F} \cong \mathcal{F}^* \otimes \det(\mathcal{F})$, and arguing as before, we thus see that $\mathcal{O}_{\mathbb{P}\mathcal{F}^*}(1)|_{X_C} = \det(\mathcal{F}) \otimes \det(\mathcal{E})^{-1}(-E)$ is ample, and therefore so is $\det(\mathcal{F})(-E)$. Thus, $\varepsilon(C, \det(\mathcal{F})) > 1$. Hence, using equation (3), we obtain $\delta(C, \det(\mathcal{F})) > c_3(\mathcal{F}) > 0$ [8].

Finally, let us consider the case where $\mathcal{F}$ has a filtration whose successive quotients are line bundles algebraically equivalent to the trivial line bundle. Then, clearly, $\mathcal{E}^*$ is ample.

CLAIM 2.1. *$\mathcal{F}$ is nef.*

PROOF. We want to show that $\mathcal{O}_{\mathbb{P}\mathcal{F}^*}(1)$ is nef. Suppose, by contradiction, that there exists a reduced irreducible curve $D \subset \mathbb{P}\mathcal{F}^*$, such that $\mathcal{O}_{\mathbb{P}\mathcal{F}^*}(1)\cdot D < 0$. Let $g: \widetilde{D} \to D \subset \mathbb{P}\mathcal{F}^*$ be the normalization, and $f: \widetilde{D} \to X$ the induced morphism. Then, g corresponds to a line bundle $L = g^* \mathcal{O}_{\mathbb{P}\mathcal{F}^*}(-1) \subset f^* \mathcal{F}^*$, with $\deg(L) = \mathcal{O}_{\mathbb{P}\mathcal{F}^*}(-1)\cdot D > 0$.

Suppose first that $\operatorname{rank}(\mathcal{F}^*) = 2$. By assumption, there exists an exact sequence $0 \to N^* \to \mathcal{F}^* \to M^* \to 0$, with M and N algebraically equivalent to the trivial line bundle. Pulling this sequence back to $\widetilde{D}$, we easily obtain a contradiction. The general case is then reduced to this one by induction on the rank. ■

Arguing as above, we then obtain that $\det(\mathcal{E}^*)(-E)$ is nef on X_C, and therefore that $\varepsilon(C, \det(\mathcal{E}^*)) \geq 1$. Hence, $\delta(C, \det(\mathcal{E}^*)) \geq 2c_1(\mathcal{E})c_2(\mathcal{E}) - c_1(\mathcal{E})^3 - c_3(\mathcal{E}) > 0$ [8]. ■

Corollary 2.2. *Let C be a smooth curve of genus 3, $u: C \hookrightarrow J$ its Abel-Jacobi map. If $\Theta \subset J$ is the theta divisor, then C is Θ-big.*

Proof. This follows from the proposition, together with the well-known determinental description on the varieties of special divisors [1, chapters IV and VII; 7]. However, under the additional hypothesis that C is Petri-general, we shall show explicitly that $\delta(C, \Theta) \geq 1$. Namely, we shall prove «by hand» that $\varepsilon(C, \Theta) \geq 1$. Since $\Theta \cdot C = 3$ and $\deg(N_{C/J}) = 4$, this will then imply $\delta(C, \Theta) \geq 4 - 3 = 1$. Let then $\widetilde{J} = \mathrm{Bl}_C(J)$, $E_C \subset \widetilde{J}$ the exceptional divisor. We want to show that $\Theta - E_C$ is nef. By fixing $p_0 \in C$, we identify $J \cong \mathrm{Pic}^d(C)$ $(L \leftrightarrow L(dp_0))$ for all d. Similarly, $u_1: C \to \mathrm{Pic}^1(C)$, given by $p \mapsto [\mathcal{O}_C(p)]$, gets identified with $u_d: C \to \mathrm{Pic}^d(C)$, defined by $p \mapsto [\mathcal{O}_C(p + (d - 1)p_0)]$. The image of u_1 is W_1, and so $u_d(C) = W_1 + (d-1)p_0$. Since $\varepsilon(C, \Theta)$ only depends on the numerical class of Θ, by Riemann's theorem we can identify $\Theta = W_2 \subset \mathrm{Pic}^2(C)$.

Consider then an integral curve $D \subset J$, $D \neq C$, and let $\widetilde{D} \subset \widetilde{J}$ be its proper transform.

Lemma 2.3. *There exists $q_0 \in C$ such that $D \not\subset W_2 + u(p_0 - q_0)$.*

Proof of Lemma 2.3. Suppose that the statement is false. Then $D \subset W_2 \cap (W_2 + u(p_0) - u(q))$, for every $q \in C$. We have [1, p. 266]:

$$W_2 \cap (W_2 + u(p_0) - u(q)) = (W_1 + u(p_0)) \cup (W_3^1 - u(q)) .$$

The first term on the right is $u(C)$. Hence, it follows that $D \subset W_3^1 - u(q)$, $\forall q \in C$. By a well-known theorem of Fulton-Lazarsfeld [7], W_3^1 is connected. Furthermore, since C is Petri-general, W_d^r has the expected dimension, given by the Brill-Noether number $\varrho = g - (r+1)(g - d + r)$, and is smooth off W_d^{r+1}. Hence $\dim(W_3^1) = 1$ and W_3^1 is smooth, and therefore irreducible. Thus $D = W_3^1 - u(q)$, for all $q \in C$, and $W_2 \cap (W_2 + u(p_0) - u(q)) = C \cup D$ does not depend on q; but this is impossible (see [1, p. 268]). ■

Choose then $q_0 \in C$ such that $D \not\subset W_2 + u(p_0) - u(q_0)$. Letting $t: D_n \to J$ be the normalization of D, it follows that $\widetilde{D} \cdot E_C = \deg\{t^{-1}C\} \leq \deg\{t^{-1}(W_2 + u(p_0) - u(q_0))\} = \Theta \cdot D$.

Next, we look at the restriction $(\Theta - E_C)|_{E_C}$. By Proposition 3.2 of [15], this is nef if and only if $1 \leq \Theta \cdot C / s(N)$, where $N = N_{C/J}$ is the normal bundle. Now $N \cong M_K^*$, where M_K sits in the exact sequence $0 \to M_K \to H^0(C, K) \otimes \mathcal{O}_C \to K \to 0$, and therefore it is semistable [16], Hence $s(N) = 1/2 \deg(N) = 2$. Thus, $\Theta \cdot C/s(N) = 3/2 > 1$. ■

EXAMPLE 2.2. Let $C \subset \mathbb{P}^3$ be residual to a line in a complete intersection of type (a,b), with $a, b \geqslant 2$. Then $\varepsilon(C) = 1/(a+b-2)$, and $\delta(C) = (2ab-a-b)/(a+b-2) > 0$ [15].

LEMMA 2.4. *Let $C_t \subset \mathbb{P}^3$ be a flat family of curves, and denote by $\varepsilon(C_t)$ the Seshadri constant of C_t. For t generic, $\varepsilon(C_t) \geqslant \varepsilon(C_0)$.*

PROOF. The blow-ups $P_t = \mathrm{Bl}_{C_t}(\mathbb{P}^3)$ also form a flat family. For each t, let H_t and E_t denote, respectively, the hyperplane class and the exceptional divisor on P_t. Pick $\eta \in (0, \varepsilon(C_0)) \cap \mathbb{Q}$, and write $\eta = n/m$, with $m, n > 0$. Then, by definition, $mH_0 - nE_0$ is ample on P_0; since ampleness is an open property, it follows that $mH_t - nE_t$ is also ample, for general t. ■

EXAMPLE 2.3. Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve, embedded by a non-special line bundle, *i.e.* $H^1(C, \mathcal{O}_C(1)) = 0$. Let $\Delta \subset \mathbb{P}^3$ be a smooth complete intersection curve of type (a,b), with $a \geqslant b$, and meeting C non-tangentially at a point P, and nowhere else. Let d_C, g_C and d_Δ, g_Δ denote, respectively, the degree and the genus of C and Δ. Let $C' = C \cup \Delta$. Then $d_{C'} = d_C + d_\Delta$ and $p_{C'} = g_C + g_\Delta$, where p denotes the arithmetic genus. We have an exact sequence $0 \to \mathcal{O}_{C'} \to \mathcal{O}_C \oplus \mathcal{O}_\Delta \to \mathbb{C}_P \to 0$; twisting by $T_{\mathbb{P}^3}$ we get the other

$$0 \to T_{\mathbb{P}^3}|_{C'} \to T_{\mathbb{P}^3}|_C \oplus T_{\mathbb{P}^3}|_\Delta \to \mathbb{C}_P^3 \to 0 .$$

Since $T_{\mathbb{P}^3}$ is globally generated, the above sequence is exact on global sections. Hence, we obtain an isomorphism $H^1(T_{\mathbb{P}^3}|_{C'}) \cong H^1(T_{\mathbb{P}^3}|_C) \oplus H^1(T_{\mathbb{P}^3}|_\Delta) = 0$ where the latter vanishing follows from the Euler sequence and the fact that C and Δ are nonspecial. Looking at the exact sequence

$$0 \to T_{C'} \to T_{\mathbb{P}^3}|_{C'} \to N \to T_P^1 \to 0 ,$$

we then conclude that $H^1(N) = 0$, and $H^0(N) \to T_P^1$ is surjective. Hence, C' can be smoothed.

Next, since C can be cut out by surfaces of degree d, and Δ by surfaces of degree a, C' can be cut out by surfaces of degree $a+d$. Hence, $\varepsilon(C') \geqslant 1/(a+d)$. Now let C_t be a flat family smoothing C'. Then, by Lemma 2.4, we have that $\delta(C_t) \geqslant (1/(a+d))(4(d+ab) + 2g_C + 2g_\Delta - 2) - d - ab$, and one easily sees that this behaves like $b^2 - bd - d$; if $b \gg 0$, then, C_t is big.

3. Let us now consider the relation between A-ampleness and A-bigness: by analogy with the divisor case, one would expect a big curve to be ample if and only if it meets every surface in X; as we shall see, this is indeed the case.

If C is A-ample, then clearly it is A-big. On the other hand, as the following example shows, the converse is false in general.

EXAMPLE 3.1. Let $C \subset \mathbb{P}^3$ be an irreducible smooth curve, with $\delta(C) > 0$. Let $\eta \in$

$\in (0, \varepsilon(C))$ be a rational number, such that $\eta \deg(N) > d$, where $N = N_{C/\mathbb{P}^3}$ is the normal bundle. Pick a point $p \in \mathbb{P}^3 \setminus C$, and let $X = \mathrm{Bl}_p(\mathbb{P}^3)$, with exceptional divisor $F \subset X$. We can find a rational number a, with $0 < a \ll 1$, such that $i)$ $L_a := H - aF$ is ample on X, and $ii)$ if $x = (1 - a)\eta$, then $x \deg(N) > d$. Now let $X_C = \mathrm{Bl}_C(X)$, where we identify C with its proper transform in X_C, and denote by $E \subset X_C$ the exceptional divisor over C.

CLAIM 3.1. *$L_a - xE$ is ample in X_C.*

PROOF. By Seshadri's criterion, we need to show that there exists $\alpha > 0$, such that for all integral curves $D \subset X_C$ one has

$$(L_a - xE) \cdot D \geqslant \alpha m(D), \tag{4}$$

where $m(D) := \sup\{m_Q(D) \mid Q \in D\}$. To see this, we distinguish three cases:

$i)$ $D \subset F$. Clearly, $F \cong \mathbb{P}^2$, and $\mathcal{O}_F(F) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. Hence, $-F \cdot D \geqslant m(D)$, and so $(L_a - xE) \cdot D = -aF \cdot D \geqslant am(D)$, for all such curves.

$ii)$ D is disjoint from F. Then D can be identified with its image in $P_C = \mathrm{Bl}_C(\mathbb{P}^3)$, and $(L_a - xE) \cdot D = (H - xE) \cdot D$. Since $x < \eta < \varepsilon(C)$, $H - xE$ is ample on P_C. Hence, there exists $\alpha_1 > 0$, such that $(L_a - xE) \cdot D = (H - xE) \cdot D \geqslant \alpha_1 m(D)$.

$iii)$ $D \cdot F = m > 0$, *i.e.*, $D = \widetilde{Z}$, the proper transform of a curve $Z \subset P_C$, having multiplicity m at p. Clearly, $m \leqslant H \cdot \widetilde{Z}$, and $m(Z) \geqslant m(\widetilde{Z})$. Since $H - \eta E$ is ample on P_C, there exists $\alpha_2 > 0$, such that $(H - \eta E) \cdot Z \geqslant \alpha_2 m(Z)$ for all integral curves $Z \subset P_C$. Thus, $(L_a - xE) \cdot \widetilde{Z} = a(H - F)\widetilde{Z} + (1 - a)(H - \eta E) Z \geqslant (1 - a)\alpha_2 m(Z) \geqslant (1 - a)\alpha_2 m(\widetilde{Z})$.

Hence, by letting $\alpha = \min\{a, \alpha_1, (1 - a)\alpha_2\}$, we see that (4) is satisfied. ■

Let then γ be the Seshadri constant of $C \subset X$ with respect to L_a; by the claim, $\gamma \geqslant x$. Hence, $\delta(C, L_a) \geqslant x \deg(N) - d > 0$, and therefore C is L_a-big. On the other hand, every ample surface $S \subset X_C$ intersetcs F along a curve disjoint from E, and thus C is not L_a-ample.

However, the following theorem gives a simple geometric reason for the failure of an A-big curve to be A-ample.

THEOREM 3.1. *Let (X, A) be a polarized threefold and $C \subset X$ a smooth curve. If C is A-big, but not A-ample, then there exists a surface $D \subset X$ disjoint from C.*

PROOF. As usual, we shall let $X = \mathrm{Bl}_C(X)$ and denote by E the exceptional divisor. By assumption, $\delta(C, A) > 0$. Hence, there exists a rational number η, with $0 < \eta < \varepsilon(C, A)$, such that $\eta \deg(N_{C/X}) > A \cdot C$. We may write $\eta = n/m$, where m, $n > 0$ are such that $mA - nE$ is very ample, and $H^i(X_C, \mathcal{O}_{X_C}(r(mA - nE))) = 0$, for i, $r > 0$. We shall postpone for now the proof of the following lemma:

LEMMA 3.1. *Let $\Lambda \subset \mathbb{P}H^0(X_C, \mathcal{O}_{X_C}(mA - nE))$ be a general pencil, where m, $n \gg 0$ and $\eta = n/m$. Then $\forall t \in \Lambda$ the following properties hold: (i) the surface $S_t = \mathrm{div}(t)$ is re-*

duced; (ii) S_t is irreducible; (iii) the curve $S_t \cap E$ is irreducible; (iv) $S_t \cap E$ is reduced.

Let us grant the lemma for now, and let $\Lambda \subset \mathbb{P}H^0(X_C, \mathcal{O}_{X_C}(mA - nE))$ be as in the statement. Since $\delta(C, A) > 0$, for all $t \in \Lambda$ the line bundle $\mathcal{O}_{S_t \cap E}(E)$ is ample. Let us recall the following:

FACT (see [9, § III.4]). Let S be a projective surface, and $C \subset S$ an effective Cartier divisor, such that $\mathcal{O}_C(C)$ is ample. Then for $r \gg 0$ the line bundle $\mathcal{O}_S(rC)$ is globally generated, and the projective morphism $\phi_{rC}: S \to S' := \phi_{rC}(S) \subset \mathbb{P}^N$ is an isomorphism onto its image in the neighbourhood of C, and $\phi_{rC}(C) \subset S'$ is ample. Therefore, ϕ_{rC} contracts a curve $D \subset S$, disjoint from C, and is an embedding away from D.

If $r > 0$ is as above, we shall say that rE is *almost very ample*, and that D is the *null curve* of rE (or of E). Let now $\mathcal{S} \subset \mathbb{P}^1 \times X_C$ be the incidence correspondence of the pencil Λ. After discarding finitely many points $p_1, \dots, p_k \in \Lambda \cong \mathbb{P}^1$, and letting $U = \Lambda \backslash \{p_i\}$, we may assume:

(*i*) for a suitable $r \gg 0$, the line bundles $\mathcal{O}_{S_t}(rE)$ are globally generated, for all $t \in \Lambda$;

(*ii*) $N_t = h^0(S_t, \mathcal{O}_{S_t}(rE)) - 1$ is constant on U, and therefore the sheaf $\pi_* \mathcal{O}_{\mathcal{S}}(rE)$ restricts to a vector bundle of rank $N + 1$ on U.

(*iii*) there exists a frame of $\pi_* \mathcal{O}_{\mathcal{S}}(rE)$ on U, *i.e.*, sections $e_0, \dots, e_N$ restricting to a basis $e_0(t), \dots, e_N(t)$ of $H^0(S_t, \mathcal{O}_{S_t}(rE))$, for all $t \in U$.

For each $t \in \Lambda$, we shall denote by $D_t \subset S_t$ the null curve of $\mathcal{O}_{S_t}(rE)$. Letting $\mathcal{S}_U$ be the restriction of $\mathcal{S}$ to U, we then have a morphism $\psi: \mathcal{S}_U \to \mathbb{P}^1 \times \mathbb{P}^N$, which is an embedding away from the union of the curves D_t. It follows that the curves D_t form a flat divisor $\mathcal{D}$ on U, and we set $\widetilde{\mathcal{L}} := \mathcal{O}_{\mathcal{S}_U}(\mathcal{D})$. Let $\mathcal{N}$ be a coherent extension of $\widetilde{\mathcal{L}}$ to $\mathcal{S}$, and set $\mathcal{L} := \mathcal{N}^{**}$; then $\mathcal{L}$ is a line bundle on $\mathcal{S}$, restricting to $\mathcal{O}_{S_t}(D_t)$ on S_t, for $t \in U$. Now the intersection numbers $(\mathcal{L} \cdot E)_t$ are constant, and therefore, being equal to zero on U, they vanish on all of Λ.

On the other hand, $h^0(S_t, \mathcal{O}_{S_t}(D_t)) = 1$, for $t \in U$. Thus, by semicontinuity, $h^0(S_t, \mathcal{L} \otimes \mathcal{O}_{S_t}) \geqslant 1$ for all $t \in \Lambda$.

LEMMA 3.2. *For every $t \in \Lambda$, any $D \in |\mathcal{L} \otimes \mathcal{O}_{S_t}|$ is contained in the null curve of $\mathcal{O}_{S_t}(rE)$.*

PROOF OF LEMMA 3.2. By construction, for each t the intersection curve $C_t = S_t \cap E$ is irreducible, and $C_t^2 > 0$. Let us write $D_t = aC_t + D_r$, where $a \geqslant 0$, and D_r has no components supported on C_t. Then $0 = D_t \cdot C_t = aC_t^2 + D_r \cdot C_t$, and this can vanish only if $a = 0$ and D_r is disjoint from E. ■

COROLLARY 3.1. *For all $t \in \Lambda$, $\dim |\mathcal{L} \otimes \mathcal{O}_{S_t}| = 0$. In particular, $\pi_* \mathcal{L}$ is a line bundle on Λ.*

PROOF. By construction, the null curve contracts under of $\phi_{\mathcal{O}_{S_t}}(rE)$, and therefore no effective divisor with support contained in it can move in a positive dimensional linear series. ■

For $t \in \Lambda$, let $D_t \in |\mathcal{L} \otimes \mathcal{O}_{S_t}|$ be the unique curve. The union of the curves D_t is then a surface, whose image in X is disjoint from C.

We are then reduced to proving Lemma 3.1.

PROOF OF LEMMA 3.1. Let us set $L = \mathcal{O}_{X_C}(mA - nE)$.

(i) Let $S \subset X_C$ be a smooth surface, such that $\mathcal{O}_{X_C}(S) \otimes L^{-1}$ is ample, and such that $H^i(X_C, L(-S)) = 0$, for $i = 0, 1$. Then we have an isomorphism $H^0(X_C, L) \to \to H^0(S, L|_S)$. Furthermore, the family Σ of effective line bundles N on X_C, for which $H^0(X_C, L \otimes N^{-1}) \neq 0$, is bounded, and therefore we can also assume that $H^i(X_C, M \otimes N^{-1}(-S)) = 0$, for $i = 0, 1$ and all M, $N \in \Sigma$. Hence $H^0(X_C, M \otimes \otimes N^{-1}) \cong H^0(S, M \otimes N^{-1}|_S)$, for all M, $N \in \Sigma$.

Let now $D = \sum_i n_i D_i \in |L|$, where the D_is are the (distinct) irreducible components of D, and where $n_i > 0$ for all i. By our choice of S, for each i, $H^0(X_C, \mathcal{O}_{X_C}(D_i - - S)) = 0$, and therefore $D_i|_S$ is well-defined.

CLAIM 3.2. *If D is not irreducible, then neither is $D|_S$.*

PROOF. We may as well assume that $D_i|_S$ is irreducible, for each i. Let us then suppose, for example, that $D_1|_S$ and $D_2|_S$ are supported on the same integral divisor $R \subset S$: there exist a, $b > 0$ such that $D_1|_S = aR$ and $D_2|_S = bR$. Suppose, for example, that $a \geq b$. Then, $D_1|_S = D_2|_S + (a-b)R$, and so $(a-b)R \in |D_1|_S - D_2|_S|$. By the above, $|D_1 - D_2| \cong |D_1|_S - D_2|_S|$, and therefore there exists $F \in |D_1 - D_2|$ that restricts to $(a-b)R$ on S. Hence, $D_1 \equiv D_2 + F$. On the other hand, $D_1 \neq D_2 + F$, but $D_1|_S = D_2|_S + F|_S$, against the fact, also following from our choice of S, that $H^0(X_C, \mathcal{O}_{X_C}(D_1)) \cong H^0(S, \mathcal{O}_S(D_1))$. ■

Hence, under the isomorphism $H^0(X_C, L) \to H^0(S, L|_S)$, the family Σ of reducible divisors in $|L|$ is contained in the family Σ_S of reducible divisors in $|L|_S|$. Now $\phi_L(S) \subset \mathbb{P}H^0(X_C, L)^*$ is not ruled by lines, and it is not the Veronese or the Steiner surface. Hence, by Lemma II.2.4 of [13], $\operatorname{codim}(\Sigma|_S) \geq 2$, and thus the same holds for Σ.

(ii) By (i), the general pencil $\Lambda \subset \mathbb{P}H^0(X_C, L)$ consists of irreducible surfaces. If $S \in |L|$ is irreducible but not reduced, then there exist $m > 1$ such that $(1/m)L$ is integral, and $V \in |(1/m)L|$, such that $S = mV$. Let $\Phi \subset |L|$ be the family of all such divisors; we then have $\dim(\Phi) \leq \sum'_{m>2} h^0((1/m)L)$, where $\sum'$ means that the sum is taken over those values of m, for which $(1/m)L$ is integral. By replacing L with rL, we may then write

$$\dim(\Phi) \leq \sum_{m>r}' h^0(L^{\otimes(r/m)}) + \sum_{r \geq m \geq 2}' h^0(L^{\otimes(r/m)}).$$

One can easily see that the family of line bundles of the form pL, with $p \in (0, 1)$ a rational number, is finite, and therefore

$$\sum_{m>r}' h^0(L^{\otimes(r/m)}) < N,$$

where N is some fixed integer, independent of r. Furthermore, if $r \geqslant m$ then $L^{\otimes(r/m)} - L$ is ample or trivial, and thus by assumption we have $h^i(X_C, L^{\otimes(r/m)}) = 0$. Hence, for $r \geqslant m$ one has $h^0(X_C, L^{\otimes(r/m)}) = \chi(X_C, L^{\otimes(r/m)}) = r^3L^3/(6m^3) + r^2L^2/(4m^2) + (c_1^2 + c_2)rL/(12m) + c_1c_2/24$. By summing over m from 2 to r, the three latter terms grow at most like r^2. The first term, on the other hand, is bounded by $r^3L^3/12\left(\sum_{m\geqslant 2} m^{-3} < 1/2\right)$, and hence is slower than $\dim|rL|$, which goes like $r^3L^3/6$. Thus, $\operatorname{codim}(\Phi, |rL|) \to \infty$.

(*iii*) We may assume that $H^1(X_C, L(-E)) = 0$. Hence, section restriction $H^0(L) \to H^0(E, L|_E)$ is onto; projectively, it corresponds to the projection from $\mathbb{P}H^0(L)$ with vertex $\mathbb{P}H^0(X_C, L(-E))$. Since $\phi_L(E)$ is not ruled by lines, by invoking again Lemma II.2.4 of [13] we conclude that we need to discard the cone with vertex

$$\Lambda = \mathbb{P}H^0(X_C, L(-E))$$

over a subvariety in $\mathbb{P}H^0(E, L|_E)$ of codimension $\geqslant 2$.

(*iv*) Let $\Xi_r \subset |rL|_E|$ be the family of those irreducible divisors that are not reduced. We want to avoid the cone over Ξ_r in $|L|$, with vertex $|L(-E)|$. Arguing as in (*ii*), one sees that $\operatorname{codim}(\Xi_r, |rL|_E|) \to \infty$, as $r \to \infty$. ■

Corollary 3.2. *Suppose that* $\mathrm{NS}(X) \cong \mathbb{Z}$. *Then C is A-ample if and only if it is A-big.*

Corollary 3.3. *Let X be a smooth projective threefold, $\phi\colon \mathcal{E} \to \mathcal{F}$ a morphism of vector bundles on X, with* $\operatorname{rank}(\mathcal{E}) = r$, $\operatorname{rank}(\mathcal{F}) = r + 1$, *and suppose that $\mathcal{E}^* \otimes \mathcal{F}$ is ample, and either one of $\mathcal{E}$ and $\mathcal{F}$ has a filtration whose quotients are algebraically trivial line bundles. Suppose that $C = X_{r-1}(\phi)$ is smooth. Then C is Seshadri-ample.*

Proof. Suppose, say, that $\mathcal{E}$ has a filtration whose quotients are algebraically trivial line bundles. We know from Proposition 2.3 that C is $\det(\mathcal{F})$-big. Hence we need to check that C meets every surface in X. This follows from Proposition 3.5 of [8]. ■

As a partial converse, we have:

Proposition 3.1. *Let X be a smooth projective threefold, and $\mathcal{E}$ a rank-2 vector bundle on X, such that* $\det(\mathcal{E})$ *is ample. Suppose that for some section $s \in H^0(X, \mathcal{E})$, the zero locus $C = Z(s)$ is smooth and* $\det(\mathcal{E})$*-big. Then $\mathcal{E}$ is big and nef, and for $r \gg 0$ the linear series $|\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(r)|$ is base point free. Let $\phi\colon \mathbb{P}\mathcal{E} \to \mathbb{P}^N$ be the corresponding morphism. If C is* $\det(\mathcal{E})$*-ample, but $\mathcal{E}$ is not ample, then there exists a curve $D \subset \mathbb{P}\mathcal{E}$, such that ϕ is an isomorphism onto its image away from D, and $\phi_*(D) = 0$.*

PROOF. Being connected (Remark 1.1) and non-singular, C is irreducible. Hence, it is $\det(\mathcal{E})$-big if and only if $\delta(C,\det(\mathcal{E}))>0$. Arguing as in the Proof of Proposition 2.3, this is equivalent to $\varepsilon(C,\det(\mathcal{E}))>1$, that is to the condition that $\det(\mathcal{E})(-E)$ be ample on $X_C=\mathrm{Bl}_C(X)$. Now $X_C\subset\mathbb{P}\mathcal{E}$ is divisor, and it is easily seen that $\mathcal{O}_{\mathbb{P}\mathcal{E}}(X_C)\cong\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)$. It follows that on X_C we have $\det(\mathcal{E})(-E)\cong\mathcal{O}_{\mathbb{P}\mathcal{E}^*}(1)\otimes\mathcal{O}_{X_C}\cong\mathcal{O}_{X_C}(X_C)$, and therefore if C is $\det(\mathcal{E})$-big then X_C has ample normal bundle in $\mathbb{P}\mathcal{E}$. Hence, for $r\gg0$ the linear series $|rX_C|$ has no base points, and furthermore the morphism $\phi\colon\mathbb{P}\mathcal{E}\to\mathbb{P}^N$ that it determines is an isomorphism onto its image in a neighbourhood on X_C. Thus, $\mathcal{E}$ is big and nef.

Suppose next that C is $\det(\mathcal{E})$-ample, but that $\mathcal{E}$ is not ample. Then ϕ cannot be a closed embedding, and therefore it contracts some subvariety $Y\subset\mathbb{P}\mathcal{E}$ disjoint from X_C. In particular, Y cannot contain any fiber of π, and it is disjoint from $\pi^{-1}(C)=E\subset X_C$, and thus $\dim(\pi(Y))=\dim(Y)$, and $\pi(Y)\cap C=\emptyset$. On the other hand, being $\det(\mathcal{E})$-ample, C cannot miss surface in X, by Lemma 2.2. Hence, $\dim(Y)\leqslant1$. ■

COROLLARY 3.4. *Let $C\subset J$ be a nonsingular curve of genus three, sitting in its Jacobian by the Abel-Jacobi map. Then C is Θ-ample.*

PROOF. We already know from Corollary 2.2 that C is Θ-big. Thus, we need to show that C meets every surface $S\subset J$. But the normal bundle $N_{C/J}$ is ample [12, § I.4], and on the other hand J is a homogeneous variety. Hence the statement follows from [5, § 12.2.4]. ■

From the argument of the proof of Theorem 3.1, one can also draw the following:

COROLLARY 3.5. *Suppose that $C\subset X$ is A-big, but not A-ample. Then there exists a rational map $\psi\colon X_C\to\mathbb{P}^k$, for some k, with the following properties: i) ψ is generically an isomorphism onto its image along E (*i.e.*, there exists an open subset $V\subset X_C$, such that $V\cap E\neq\emptyset$, and $V\cong\psi(V)$); ii) let $S\subset X$ be the surface, disjoint from E_C, described by Theorem* 3.1; *then* $\dim(\psi(S))\leqslant1$; *iii) for a suitable hyperplane $H\subset\mathbb{P}^k$, and for suitable $S_1,\dots,S_k\in|mA-nE|$, we have $\psi^*H=E+\sum_1^l S_i$.*

4. In this section, we relate A-bigness and A-ampleness to the ampleness of the normal bundle and to the cohomological dimension of the complement of C in X.

THEOREM 4.1. *Let (X,A) be a smooth polarized projective threefold, $C\subset X$ a nonsingular curve. If C is A-big, then the normal bundle $N=N_{C/X}$ is ample.*

PROOF. Suppose that N is not ample. Then either $\deg(N)<0$, in which case C is obviously not A-big, or else there exist $m>0$ and a line bundle $L\subset\mathrm{Sym}^m(N)$, such that $\deg(L)/m\geqslant\deg(N)$ [9]. Let $\gamma(N)=\sup\{\deg(L)/m\,|\,L\subset\mathrm{Sym}^m(N)$ a line bundle$\}$. Then $\gamma(N)\geqslant\deg N$. It follows from Proposition 3.2 of [15] that $\varepsilon(C)\leqslant A\cdot C/\gamma(N)$, and so $\delta(C)\leqslant0$, a contradiction. ■

COROLLARY 4.1. *Let $C \subset X$ be A-big. Then C does not contract under any non-trivial morphism.*

COROLLARY 4.2. *If X is homogeneous, then C is A-ample if and only if it is A-big.*

PROOF. Argue as in the Proof of Corollary 3.4. ■

In the following, $\mathrm{cd}(U)$ and $\mathrm{q}(U)$ are the cohomological invariants defined for example in [9]. The former is the cohomological dimension of the scheme U.

THEOREM 4.2. *Let (X, A) be a smooth polarized threefold, $C \subset X$ a nonsingular curve $U = X \setminus C$. Then:*

(*i*) *If C is A-ample,* $\mathrm{cd}(U) = 1$;

(*ii*) *If C is A-big,* $\mathrm{q}(U) = 1$.

PROOF. (*i*) Since U obviously contains complete curves, it is clear that $\mathrm{cd}(U) \geqslant 1$. On the other hand, to show that $\mathrm{cd}(U) \leqslant 1$ it is enough to prove that

$$H^i(U, \mathcal{O}_U(-mA)) = 0\,,$$

for $i > 1$ and $m \gg 0$ [9, § III.3]. Pick m, $n > 0$ as in Definition 2.1 of [9]. For $S \in |mA - nE|$ general, the line bundle $\mathcal{O}_S(E)$ is ample, and therefore $S \setminus E \cong S \cap U$ is an affine surface. Let then L be a line bundle on U, and look at the exact sequence $0 \to L(-S) \to L \to L|_{S \cap U} \to 0$. Taking cohomology, we get the isomorphisms $H^i(U, L(-S)) \cong H^i(U, L)$, for all $i > 1$. On the other hand $\mathcal{O}_U(S) \cong \mathcal{O}_U(mA)$, and therefore $H^i(U, \mathcal{O}_U(mA))$ is independent of m. Hence, we may as well show that $H^i(U, \mathcal{O}_U(mA)) = 0$, for $i > 1$ and $m \gg 0$. Let us then consider the local cohomology sequence

$$H^i(X, \mathcal{O}_X(mA)) \to H^i(U, \mathcal{O}_U(mA)) \to H^{i+1}_C(X, \mathcal{O}_X(mA))\,.$$

Since for $i > 0$ and $m \gg 0$ we have $H^i(X, \mathcal{O}_X(mA)) = 0$, we are reduced to showing that $H^k_C(X, \mathcal{O}_X(mA)) = 0$, for $k \geqslant 3$ and $m \gg 0$. By the formal duality theorem [9]

$$H^k_C(X, \mathcal{O}_X(mA))^* \cong H^{3-k}_C(\widehat{X}, \widehat{\omega_X(-mA)})\,,$$

where $\widehat{X}$ and $\widehat{\omega_X(-mA)}$ denote completions along C as scheme and as sheaf, respectively. Thus we need to show that $H^0(\widehat{X}, \widehat{\omega_X(-mA)}) = 0$. Now

$$H^0(\widehat{X}, \widehat{\omega_X(-mA)}) = \varprojlim H^0(C_l, \omega_X(-mA) \otimes \mathcal{O}_{C_l})\,,$$

where $C_l \subset X$ is defined by the ideal sheaf $\mathfrak{I}^l_C$, $\mathfrak{I}_C$ being the ideal of C. Choose m large enough so that $\omega_C^{-1}(mA)$ is ample, and consider then the exact sequence

$$0 \to \frac{\mathfrak{I}^l}{\mathfrak{I}^{l+1}} \otimes \omega_X(-mA) \to \omega_X(-mA) \otimes \mathcal{O}_{C_{l+1}} \to \omega_X(-mA) \otimes \mathcal{O}_{C_l} \to 0\,.$$

Since $H^0(C, \mathfrak{I}^l/\mathfrak{I}^{l+1} \otimes \omega_X(-mA)) \cong H^1(C, S^l N \otimes \det(N)(mA))^* = 0$ for all $l \geqslant 1$

by the Griffiths' vanishing theorem [18, ch. V] we have inclusions

$$H^0(C_{l+1}, \omega_X(-mA) \otimes \mathcal{O}_{C_{l+1}}) \hookrightarrow H^0(C_l, \omega_X(-mA) \otimes \mathcal{O}_{C_l}) \hookrightarrow \dots$$

and the last term in this chain is $H^0(C, \omega_X(-mA) \otimes \mathcal{O}_C) = 0$.

(*ii*) One possible argument is entirely similar to the one for (i); alternatively, one can appeal to page 107 of [9] in view of Theorem 4.1 above. ■

Corollary 4.3. *Let $\mathcal{E}$ be an ample globally generated rank-2 vector bundle on the smooth projective threefold X, and $s \in H^0(X, \mathcal{E})$ a regular section, such that $C = Z(s)$ is smooth. Then* $\mathrm{cd}(X \backslash C) = 1$.

Corollary 4.4. *Let (X, A) be a smooth polarized projective threefold, $C \subset X$ a smooth connected curve. If C is A-big (resp., A-ample), then C is* G2 *(resp.,* G3*) in X.*

Proof. By [9, § 6], if C has ample normal bundle then it is in G2 in X. On the other hand, if C is A-big then it has ample normal bundle, by Theorem 4.1. This implies the first statement. Next, if C is A-ample, then in particular it is A-big, and therefore is G2, and furthermore $\mathrm{cd}(X \backslash C)$ by Theorem 4.2. By a result of Speiser [20], it then follows that C is G3 in X. ■

5. We now illustrate by two simple examples the dependence of A-ampleness and A-bigness on the polarization.

Example 5.1. The first example continues Example 3.1. Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $s \geqslant 10$, and fix $P \subset S$. For some $a > 0$, choose a general smooth curve $C \in |\mathcal{O}_S(aH)|$, such that $P \notin C$. Denote by $g: X_P \to \mathbb{P}^3$ the blow-up of P, and by $F = g^{-1}(P)$ the exceptional divisor of g. We shall identify C with its inverse image in X_P. As we have seen in Example 3.1, C is H_x-big for $x > 0$ sufficiently small, where $H_x = H - xF$. On the other hand, if a is sufficiently large with respect to s, then $\delta(C, H_x) < 0$ when $x \sim 1^-$. To see this, let $f: X = \mathrm{Bl}_C(X_P) \to X_P$, the exceptional divisor, and observe to start with that $\varepsilon(C, H_x) = \min\{\varepsilon_1(C, H_x), \varepsilon_2(C, H_x)\}$, where $\varepsilon_1(C, H_x) = \sup\{\eta \in \mathbb{Q} \mid (H_x - \eta E)|_E \text{is ample}\}$, and $\varepsilon_2(C, H_x) = \sup\{\eta \in \mathbb{Q} \mid \widetilde{D} \cdot (H_x - \eta E) \geqslant 0$, $\forall$ irreducible curves $D \subset X_P$, $D \neq C\}$ [15]. Since E and F are disjoint $\varepsilon_1(C, H_x) = \varepsilon_1(C, H)$, and therefore all the depencence of $\varepsilon(C, H_x)$ on x comes from $\varepsilon_2(C, H_x)$. By adjunction, $\mathcal{O}_S(H) = \mathcal{O}_S((s-3)H) \otimes \omega_S^{-1}$. Since $H^2 = s > 3^2$, by the argument in Steps 1 and 2 in the proof of Theorem 2.1 in [10], for all $k \gg 0$ there is $M \in |\mathcal{O}_S(kH)|$ such that $\mathrm{mult}_P(M) \geqslant 3k$. By generality of our choices, furthermore, we may assume that no component of M is supported on C. By definition, $\varepsilon_2(C, H_x) \leqslant H_x \cdot \widetilde{M} / E \cdot \widetilde{M}$; since $H_x \cdot \widetilde{M} = H \cdot M - x\,\mathrm{mult}_P(M) \leqslant ks - 3kx$, we then have $\varepsilon_2(C, H_x) \leqslant (s - 3x)/as$. If $\delta(C, H_x) > 0$, then $\varepsilon_2(C, H_x) \geqslant H_x \cdot C / \deg(N) = 1/(a+s)$; therefore, $(s - 3x)/as > 1/(a+s)$. Hence, $s^2/3(a+s) > x$; by taking a sufficiently large, then, we may force x to be arbitrarily small. Therefore, for every $y \in (0, 1)$ there is curve $C \subset X_P$ which is H_x-big for $x \sim 0^+$ but satisfies $\delta(C, H_y) < 0$.

Example 5.2. As the previous example shows, given two polarizations A and B on

X, a curve $C \subset X$ may be A-big and yet fail to be B-big. The second example shows that this also holds for ampleness. Let X be a smooth projective threefold, and L, $M \subset X$ smooth very ample surfaces, such that $C = L \cap M$ is smooth and irreducible. By Corollary 3.3, C is $(L + M)$-ample; we may then ask whether C is also L-ample. Clearly, we may assume that the numerical classes of L and N are independent. Pick $D \in |\mathcal{O}_L(L)|$, and let $\widetilde{D} \subset X_C$ be its proper transform. Then $D \cdot L = L^3$, and $E_C \cdot \widetilde{D} = L^2 \cdot M$. Hence, $\varepsilon_2(C, L) \leqslant D \cdot L / E_C \cdot \widetilde{D} = L^3 / L^2 \cdot M$. On the other hand, if C is L-ample, then $\varepsilon_2(C, L) > C \cdot L / \deg(N)$, where N is the normal bundle. We have $C \cdot L = L^2 \cdot M$ and $\deg(N) = L^2 \cdot M + L \cdot M^2$, and therefore if C is L-ample then $L^3/L^2 \cdot M > > L^2 \cdot M / (L^2 \cdot M + L \cdot M^2)$. Suppose now $X = \mathbb{P}^1 \times \mathbb{P}^2$, with projections $p_i : X \to \mathbb{P}^i$, for $i = 1, 2$. Set $H_i = p_i^* \mathcal{O}_{\mathbb{P}^i}(1)$, for $i = 1, 2$. Pick x, $y \in \mathbb{Q}^+$, with $x \neq y$, and choose $a > 0$ an integer such that $a(H_1 + xH_2)$ and $a(H_1 + yH_2)$ are both integral (very ample) divisors. Choose $L \in |a(H_1 + xH_2)|$ and $M \in |a(H_1 + yH_2)|$ smooth and such that $C = L \cap M$ is smooth. Then C is $H_1 + ((x + y)/2) H_2$-ample. If C is L-ample, then by the above we should have $3x^2/(x^2 + 2xy) > (x^2 + 2xy)/(x^2 + 4xy + y^2)$. Setting $t = y/x$, we then have $0 > t^2 - 8t - 2$, and this is impossible for t large enough.

References

[1] E. Arbarello - M. Cornalba - P. Griffiths - J. Harris, *The Geometry of Algebraic Curves*. Vol. I, Springer-Verlag, 1985.

[2] F. Bogomolov, *Unstable vector bundles and curves on surfaces*. Proc. Int. Congr. Mathem., Helsinki 1978, 517-524.

[3] D. Eisenbud - A. Van de Ven, *On the normal bundle of smooth rational space curves*. Math. Ann., 256, 1981, 453-463.

[4] T. Fujita, *On the hyperplane principle of Lefshetz*. J. Math. Soc. Japan, 32, 1980, 153-165.

[5] W. Fulton, *Intersection Theory*. Springer-Verlag, 1984.

[6] W. Fulton - R. Lazarsfeld, *Positivity and excess intersection*. in : *Enumerative and Classical Algebraic Geometry*. Prog. in Math., 24, (Nice 1981) Birkhauser, 1982, 97-105.

[7] W. Fulton - R. Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*. Acta Math., 146, 1981, 271-283.

[8] W. Fulton - R. Lazarsfeld, *Positive polynomials for ample vector bundles*. Ann of Math., 118, 1983, 35-60.

[9] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*. LNM, 156, Springer-Verlag, 1970.

[10] Y. Kawamata - K. Matsuda - K. Matsuki, *Introduction to the minimal model problem*. In: T. Oda (ed.), *Algebraic Geometry*. Sendai 1985, Adv. St. in Pure Math., vol. 10, North-Holland, 1987, 283-360.

[11] R. Lazarsfeld, *Some applications of the theory of ample vector bundles*. In: S. Greco - R. Strano (eds.), *Complete Intersections*. Arcireale 1983, LNM 1092, Springer-Verlag, 1984, 29-61.

[12] R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*. In: M. Cornalba *et al.* (eds.), Proceedings of the Intern. Centre Theor. Phys. College on *Riemann Surfaces* (Trieste 1987). World Scientific Press, 1989, 500-559.

[13] A. Lopez, *Noether-Lefshetz theory and the Picard group of projective surface*. PhD Thesis, Brown University, 1988.

[14] R. Paoletti, *Free pencils on divisors*. Mathematische Annalen, to appear.

[15] R. Paoletti, *Seshadri constants, gonality of space curves and restriction of stable bundles*. J. Diff. Geom., 40, 1972, 475-504.

[16] K. Paranjape - S. Ramanan, *On the canonical ring of a curve.* In: *Algebraic Geometry and Commutative Algebra, in Honor of Masayoshi Nagata.* Kinokuniya, Tokio 1988, vol. II, 503-516.
[17] F. Serrano, *Extension of morphisms defined on a divisor.* Math. Ann., 277, 1987, 395-413.
[18] B. Shiffman - A. J. Sommese, *Vanishing theorems on complex manifolds.* Progr. Math., vol. 56, Birkhauser, Boston 1985.
[19] A. J. Sommese, *On manifolds that cannot be ample divisors.* Math. Ann., 221, 1987, 55-72.
[20] R. D. Speiser, *Cohomological dimension of abelian varieties.* Thesis, Cornell University, 1970.

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