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Edoardo Vesentini

Semiflows and semigroups

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Semiflows and semigroups

Memoria (*) di Edoardo Vesentini

Abstract. — Given a compact Hausdorff space K and a strongly continuous semigroup T of linear isometries of the Banach space of all complex-valued, continuous functions on K, the semiflow induced by T on K is investigated. In the particular case in which K is a compact, connected, differentiable manifold, a class of semigroups T preserving the differentiable structure of K is characterized.

Key words: Complex extreme point; Markov lattice operator; Cocycle; Banach-Stone theorem.

Riassunto. — *Sistemi dinamici e semigruppi*. Si considera un semigruppo fortemente continuo T di isometrie lineari dello spazio di Banach delle funzioni continue, a valori complessi, su uno spazio di Hausdorff compatto K, e si studia il legame fra T ed il sistema dinamico continuo indotto da T in K. Nel caso in cui K sia una varietà differenziabile, si caratterizza una classe di semigruppi T che lasciano invariante la struttura differenziabile di K.

A standard procedure in investigating a dynamical system (K, ϕ) acting on a set K consists in replacing (K, ϕ) by a semigroup T of linear operators defined in a Banach space intrinsically associated to the system. Historically, this approach was the main tool in the proof of the mean ergodic theorem, given by J. von Neumann in 1932, after B. O. Koopman had observed that a bijective map ϕ of a measure space K onto itself, preserving a probability measure μ, defines a unitary operator T on $L^2(K, \mu)$, whose action on any $f \in L^2(K, \mu)$ is given by $Tf = f \circ \phi$. This line of thought found its way into all chapters of measure theoretic dynamics and, later on, also into topological dynamics [5].

In all these instances, the investigation moves from a given dynamical system (K, ϕ) and builds around it a semigroup T of linear operators. This article deals with a sort of inverse problem, starting from a semigroup T and looking for a dynamical system (K, ϕ) whose associated semigroup is T. Of course, in such a general form the problem is so vague to be almost meaningless. What will be done here is much more specific and will only deal with a class of examples. Namely, starting with the Banach space $C(K)$ of all continuous, complex valued functions on a compact Hausdorff space K, we will see how a semigroup of linear isometries of $C(K)$ arises from a continuous cocycle on $C(K)$ and from a semiflow of surjective continuous maps of K onto itself.

In the particular case in which K is a compact, connected, differentiable manifold, we obtain a characterization of a class of semigroups of linear isometries of $C(K)$ which preserve the subspace $C^\infty(K)$ of all differentiable functions on K.

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1. Let K be a compact Hausdorff space. A continuous map $\psi : K \to K$ defines a bounded linear operator $A \in \mathcal{L}(C(K))$ on the Banach space $C(K)$ of all complex-valued continuous functions on K, endowed with the uniform norm. If $f \in C(K)$, the value of Af at a point $x \in K$ is given by

$$Af(x) = f(\psi(x)) . \tag{1}$$

The operator A is a Markov lattice operator (*i.e.* $|Af| = A|f|$ and $A1_K = 1_K$, where 1_K is the function equal to 1 at each point of K). Viceversa, for every Markov lattice operator $A \in \mathcal{L}(C(K))$, there is a unique continuous map $\psi : K \to K$ such that A is given by (1). The operator A is an isometry if, and only if, ψ is surjective. But, even if ψ is surjective, (1) is by no means the most general norm-preserving map of $C(K)$ into itself. Examples show, in fact, that there exist norm-preserving continuous maps of $C(K)$ into $C(K)$ which are not even linear and, *a fortiori*, are not represented by (1) [9].

A special feature of the operator A represented by (1) can be expressed in terms of extreme points. Let $B(K)$ be the open unit ball of $C(K)$, and let $\Gamma(K)$ be the set of all complex extreme points of the closure of $B(K)$. The set $\Gamma(K)$ consists of all functions $f \in C(K)$ such that $|f(x)| = 1$ for all $x \in K$ [9].

By (1), any Markov lattice operator A maps $\Gamma(K)$ into $\Gamma(K)$. Does this property – together with the surjectivity of ψ – suffice to characterize linear isometries among all continuous linear operators on $C(K)$?

An answer to this question is provided by the following result which was stated in [9] in a slightly weaker form than the one that is needed now, and which, – for this reason – will be re-obtained here.

Let H and K be two compact Hausdorff spaces and let $A \in \mathcal{L}(C(K), C(H))$ be such that $\|A\| \leq 1$ and

$$A\Gamma(K) \subset \Gamma(H) . \tag{2}$$

For any $y \in H$, the map $C(K) \ni f \mapsto Af(y)$ is a continuous linear form on $C(K)$. Hence, there is a unique regular Borel measure μ_y on K for which

$$Af(y) = \int f d\mu_y := \langle f, \mu_y \rangle .$$

According to the following lemma [9], the measure μ_y is concentrated at a single point of K.

LEMMA 1. *For every $y \in H$ there exist a complex constant $\omega(y)$, with $|\omega(y)| = 1$, and a point $x \in K$ such that*

$$\mu_y = \omega(y)\delta_x , \tag{3}$$

where δ_x is the measure with mass 1 *concentrated at the point* x; i.e.: $\langle f, \mu_y \rangle = \omega(y) f(x)$, *for all $f \in C(K)$.*

The fact that the support of μ_y is a singleton implies that the point $x \in K$ for which (3) holds is unique, and therefore $\omega(y)$ is unique. Choosing $f = 1_K$ one sees that $\omega(y)$ is a continuous function of $y \in H$. Letting $x = \psi(y)$, the continuity of the map $\psi : H \to K$ is an easy consequence of the Urysohn lemma, and can be established as in [9].

In conclusion the following theorem holds, that is essentially Theorem 1 of [9] (to which the uniqueness of ω and ψ is added).

THEOREM 1. *If $A \in \mathcal{L}(C(K), C(H))$ is such that $\|A\| \leq 1$ and (2) holds, there exist a unique function $\omega \in \Gamma(H)$ and a unique continuous map $\psi: H \to K$ such that*

$$Af = \omega \cdot (f \circ \psi), \tag{4}$$

i.e.: $Af(y) = \omega(y)\, f(\psi(y))$, for all $f \in C(K)$ and all $y \in H$. Furthermore, A is a linear isometry of $C(K)$ into $C(H)$ if, and only if, ψ is surjective.

The linear isometry A is surjective if, and only if, ψ is a homeomorphism. On the other hand, a surjective isometry A (which is linear by the Mazur-Ulam theorem) clearly satisfies condition (2). Hence, Theorem 1 implies the Banach-Stone theorem [3].

Assume from now on, $H = K$. It may be worth noticing that a weaker condition than

$$A\Gamma(K) \subset \Gamma(K) \tag{5}$$

suffices to imply that a lattice operator A is a Markov operator. If $A \in \mathcal{L}(C(K))$ is a lattice homomorphism, there exist a function $\gamma \in C(K)$ and a map $\chi : K \to K$ such that $Af(x) = \gamma(x) f(\chi(x))$ for all $f \in C(K)$ and all $x \in K$, and χ is continuous in every point $x \in K$ where $\gamma(x) \neq 0$ [5]. Since $|Af(x)| = |\gamma(x)|\,|f(\chi(x))|$ and $A|f|(x) = \gamma(x)|f(\chi(x))|$, then $|\gamma(x)| = \gamma(x)$ for all $x \in K$, *i.e.* $\gamma(K) \subset \boldsymbol{R}_+$. If $f \in \Gamma(K)$, then $|Af(x)| = \gamma(x)|f(\chi(x))| = \gamma(x)$. Hence, if $Af \in \Gamma(K)$, then $\gamma = 1_K$. Therefore χ is continuous, $Af = f \circ \chi$ for all $f \in C(K)$, and A is Markov lattice homomorphism. But then, if $f \in \Gamma(K)$, $|Af(x)| = 1\ \forall x \in K$, *i.e.* $Af \in \Gamma(K)$. That proves

LEMMA 2. *If $A \in \mathcal{L}(C(K))$ is a lattice homomorphism for which $A\,\Gamma(K) \cap \Gamma(K) \neq \emptyset$, then A is a Markov lattice homomorphism.*

Therefore (5) holds.

2. Let $T: \boldsymbol{R}_+ \to \mathcal{L}(C(K))$ be a strongly continuous semigroup of linear isometries of $C(K)$ such that

$$T(t)\,\Gamma(K) \subset \Gamma(K) \qquad \forall t \geq 0. \tag{6}$$

By Theorem 1, for every $t \in \boldsymbol{R}_+$ there exist a unique continuous surjective map $\phi_t: K \to K$ and a unique function $\alpha_t \in \Gamma(K)$ such that

$$T(t)f = \alpha_t \cdot (f \circ \phi_t) \qquad \forall t \in \boldsymbol{R}_+,\ \forall f \in C(K), \tag{7}$$

i.e. $T(t)f(x) = \alpha_t f(\phi_t(x))$, for all $t \in \boldsymbol{R}_+$, $f \in C(K)$, $x \in K$.

The fact that T is a semigroup implies that $\alpha_0 = 1$, ϕ_0 is the identity, and

$$\alpha_{t_1 + t_2} \cdot (f \circ \phi_{t_1 + t_2}) = \alpha_{t_1} \cdot (\alpha_{t_2} \circ \phi_{t_1}) \cdot (f \circ \phi_{t_1} \circ \phi_{t_2}) \tag{8}$$

for all t_1, t_2 in $\boldsymbol{R}_+$ and all $f \in C(K)$. Choosing $f = 1_K$, then $T(t)1_K = \alpha_t$, and

$$\alpha_{t_1 + t_2} = \alpha_{t_1} \cdot (\alpha_{t_2} \circ \phi_{t_1}), \tag{9}$$

showing that the function $\alpha\colon \boldsymbol{R}_+ \times K \to \boldsymbol{T} = \{\zeta \in \boldsymbol{C}\colon |\zeta| = 1\}$ mapping (t, x) to $\alpha_t(x)$ is a continuous cocycle.

Since T is strongly continuous, for all $x \in K$

$$\lim_{t \downarrow 0} \|\alpha_t - 1\| = \lim_{t \downarrow 0} \|T(t)\, 1_K - 1_K\| = 0$$

(where $\| \, \|$ denotes the uniform norm on $C(K)$). In view of (9), (8) yields $f \circ \phi_{t_1 + t_2} = f \circ \phi_{t_1} \circ \phi_{t_2}$, for all $f \in C(K)$, *i.e.*: $\phi_{t_1 + t_2} = \phi_{t_1} \circ \phi_{t_2}$, $\forall t_1, t_2 \in \boldsymbol{R}_+$. For $t \geqslant 0$, let $S(t) \in \mathcal{L}(C(K))$ be defined by $S(t) = \overline{\alpha_t}\, T(t)$, so that

$$S(t) f = f \circ \phi_t \,, \tag{10}$$

for all $t \geqslant 0$ and all $f \in C(K)$, and $S\colon t \mapsto S(t)$ is a semigroup. For all $x \in K$,

$$\lim_{t \downarrow 0} (S(t)f(x) - f(x)) = \lim_{t \downarrow 0} (\overline{\alpha_t(x)}\; T(t)f(x) - f(x)) = 0\,.$$

Since K is compact, then

$$\lim_{t \downarrow 0} \|S(t)f - f\| = 0\,.$$

Thus, $S : \boldsymbol{R}_+ \to \mathcal{L}(C(K))$ is a strongly continuous Markov lattice semigroup, and, as a consequence, its infinitesimal generator $D : \mathcal{D}(D) \subset C(K) \to C(K)$ is a derivation [1].

Let $X : \mathcal{D}(X) \subset C(K) \to C(K)$ be the infinitesimal generator of T. Let $\mathcal{D}(X) \cap \mathcal{D}(D) \neq \{0\}$, and let $f \not\equiv 0$ be contained in $\mathcal{D}(X) \cap \mathcal{D}(D)$. Since, for all $x \in K$ and all $t > 0$,

$$(1/t)(T(t)f - f)(x) = ((\alpha_t(x) - 1)/t)\, S(t)f(x) + (1/t)(S(t)f - f)(x)\,,$$

and since $\lim_{t \downarrow 0} S(t)f(x) = f(x)$, if $f(x) \neq 0$ then $\lim_{t \downarrow 0} ((\alpha_t(x) - 1)/t)$ exists, and

$$Xf(x) = \lim_{t \downarrow 0} ((\alpha_t(x) - 1)/t) f(x) + Df(x)\,. \tag{11}$$

THEOREM 2. *If $\mathcal{D}(X) \cap \mathcal{D}(D)$ contains some f with $f(x) \neq 0$ at all points $x \in K$ (or, equivalently, if $1_K \in \mathcal{D}(X) \cap \mathcal{D}(D)$), then $\mathcal{D}(X) = \mathcal{D}(D)$, and there is a continuous function $\beta\colon K \to \boldsymbol{R}$ such that*

$$X = i\beta I + D\,. \tag{12}$$

Furthermore

$$\alpha_t(x) = e^{i\int_0^t \beta(\phi_s(x))\,ds} \qquad \forall t \in \boldsymbol{R}_+\,. \tag{13}$$

PROOF. In view of what has been seen before, $\lim_{t \downarrow 0} ((\alpha_t(x) - 1)/t)$ exists for all $x \in K$. Since $|\alpha_t(x)| = 1$, the limit is purely imaginary. By (11), setting

$$i\beta(x) = \lim_{t \downarrow 0} ((\alpha_t(x) - 1)/t) \tag{14}$$

yields a continuous function $\beta\colon K \to \boldsymbol{R}$ for which $Xf = i\beta f + Df$. If $g \in \mathcal{D}(D)$, replacing f by g in (11) shows that $g \in \mathcal{D}(X)$, whence $\mathcal{D}(D) \subset \mathcal{D}(X)$. The same kind of argument establishes the opposite inclusion, and therefore $\mathcal{D}(D) = \mathcal{D}(X)$.

By (9), for any $t \geqslant 0$

$$\lim_{h \downarrow 0} \frac{\alpha_{t+h}(x) - \alpha_t(x)}{h} = \alpha_t(x) \lim_{h \downarrow 0} \frac{\alpha_h(\phi_t(x)) - 1}{h} = i\alpha_t(x)\, \beta(\phi_t(x)).$$

Hence, for all $x \in K$ and all $t \geqslant 0$ the right derivative of $\alpha_t(x)$ exists and is continuous. Hence [2], $t \mapsto \alpha_t(x)$ is of class C^1 on $\mathbf{R}_+$, and

$$\frac{d}{dt}\, \alpha_t(x) = i\beta(\phi_t(x))\, \alpha_t(x).$$

This fact, together with the initial condition $\alpha_0(x) = 1$, yields (13) and completes the proof of the theorem.

By (9), for $t, s \in \mathbf{R}_+$, $T(s)\alpha_t = \alpha_s \cdot (\alpha_t \circ \phi_s) = \alpha_{t+s}$. Hence, again by [2], $\alpha_t \in \mathcal{D}(X)$ for all $t \in \mathbf{R}_+$ if, and only if, $t \mapsto \alpha_t(x)$ is of class C^1 on $\mathbf{R}_+$. That proves

LEMMA 3. *If, and only if, $\alpha_t \in \mathcal{D}(X)$ for all $t \in \mathbf{R}_+$ there exists a continuous function $\beta\colon K \to \mathbf{R}$ for which* (13) *holds.*

The hypothesis of Theorem 2 is satisfied if $X \in \mathcal{L}(C(K))$; in which case $\mathcal{D}(D) = C(K)$, and, by the closed graph theorem, $D \in \mathcal{L}(C(K))$. But there are no non-trivial bounded derivations of $C(K)$ [8] (1). As a consequence ϕ_t is the identity for all $t \geqslant 0$. By (13), $\alpha_t(x) = e^{it\beta(x)}$, and the following lemma holds.

LEMMA 4. *If $X \in \mathcal{L}(C(K))$, then $X = i\beta I$, and $T(t) = e^{it\beta(x)} I$ for all $t \in \mathbf{R}_+$.*

In particular T is the restriction to $\mathbf{R}_+$ of a strongly continuous group of isometries of $C(K)$.

Let now $S\colon \mathbf{R}_+ \to \mathcal{L}(C(K))$ be a strongly continuous Markov lattice semigroup of linear isometries of $C(K)$, let D be its infinitesimal generator and let ϕ be the continuous semiflow uniquely associated to S by (10). If $\beta\colon K \to \mathbf{R}$ is a continuous function, the bounded perturbation X of D given by (12) generates a strongly continuous semigroup T. Defining α_t using (13), then

$$\alpha_{t_1+t_2}(x) = e^{i\int_0^{t_1+t_2} \beta(\phi_s(x))\,ds} = e^{i\int_0^{t_1} \beta(\phi_s(x))\,ds}\, e^{i\int_{t_1}^{t_1+t_2} \beta(\phi_s(x))\,ds} =$$

$$= e^{i\int_0^{t_1} \beta(\phi_s(x))\,ds}\, e^{i\int_0^{t_2} \beta(\phi_{t_1+s}(x))\,ds} = e^{i\int_0^{t_1} \beta(\phi_s(x))\,ds}\, e^{i\int_0^{t_2} \beta(\phi_s(\phi_{t_1}(x)))\,ds} = \alpha_{t_1}(x)\, \alpha_{t_2}(\phi_{t_1}(x)).$$

As a consequence, the semigroup generated by the right hand side of (12) and the semigroup defined by the right hand side of (7) have the same infinitesimal generator. That proves

(1) Here is a direct proof of this fact, following essentially an argument given in [6] in a different context. Choose any $r \in (0, 1)$. Any function $f \in C(K)$ can be written

$$f = f(x)\, 1_K + (f - f(x)\, 1_K)^r (f - f(x)\, 1_K)^{1-r},$$

where x is any point of K. If D is any bounded derivative of $C(K)$, then

$$(Df)(x) = [(f - f(x)\, 1_K)^r D((f - f(x)\, 1_K)^{1-r}) + (f - f(x)\, 1_K)^{1-r} D((f - f(x)\, 1_K)^r)](x) = 0,$$

for all $x \in K$, whence $Df = 0$.

Lemma 5. *If a derivation D defines a strongly continuous Markov lattice semigroup of linear isometries of $C(K)$, every bounded perturbation* (12) *of D defined by a real-valued continuous function β on K is the infinitesimal generator of a strongly continuous semigroup of linear isometries of $C(K)$.*

This lemma will be instrumental in characterizing all bounded perturbations of X that generate semigroups of linear isometries of $C(K)$.

The infinitesimal generators of strongly continuous semigroups of linear isometries of a complex Banach space $\mathcal{E}$ have been characterized in [10] in terms of conservative operators. Let $Z: \mathcal{D}(Z) \subset \mathcal{E} \to \mathcal{E}$ be a closed, densely defined, linear operator. It was shown in [10] that Z generates a strongly continuous semigroup of linear isometries of $\mathcal{E}$ if, and only if, Z is m-dissipative and, furthermore, conservative (*i.e.*, for every $z \in \mathcal{D}(Z)$, there is some continuous linear form ν on $\mathcal{E}$ such that $\langle z, \nu\rangle = \|z\|$, $\|\nu\| = 1$ and $\Re\langle Zz, \nu\rangle = 0$).

Going back to the infinitesimal generator X of the semigroup T of linear isometries of $C(K)$, let $Y = L + X$ be a perturbation of X by an operator $L \in \mathcal{L}(C(K))$, and assume that also the strongly continuous semigroup generated by Y consists of linear isometries of $C(K)$.

For every $f \in \mathcal{D}(X) = \mathcal{D}(Y)$, there is some continuous linear form λ on $C(K)$ such that

$$\langle f, \lambda\rangle = \|f\|, \qquad \|\lambda\| = 1 \tag{15}$$

and $\Re\langle Xf, \lambda\rangle = 0$. Since, by the Lumer-Phillips theorem [7] applied to Y, $\Re\langle Yf, \lambda\rangle \leqslant 0$, then $\Re\langle Lf, \lambda\rangle \leqslant 0$. Thus, the bounded linear operator L generates a contraction semigroup of $C(K)$, and the latter inequality holds for all continuous linear forms λ on $C(K)$ satisfying (15). Applying now the result of [10] to Y, one sees that there is some continuous linear form μ on $C(K)$ such that $\langle f, \mu\rangle = \|f\|$, $\|\mu\| = 1$ and $\Re\langle Yf, \mu\rangle = 0$, *i.e.*: $\Re\langle Lf, \mu\rangle + \Re\langle Xf, \mu\rangle = 0$. The Lumer-Phillips theorem applied to X implies that $\Re\langle Lf, \mu\rangle \geqslant 0$, and, in conclusion, $\Re\langle Lf, \mu\rangle = 0$. The bounded operator L, being m-dissipative and conservative, generates a uniformly continuous group of linear isometries of $C(K)$. Lemma 5 and Lemma 4 yield

Theorem 3. *If X generates a strongly continuous semigroup T of linear isometries of $C(K)$, every bounded perturbation of X generating a semigroup of the same kind is expressed by $i\beta I + X$, where $\beta : K \to \boldsymbol{R}$ is any continuous function.*

3. Sufficient conditions for the existence of a cocycle of class C^1 will now be investigated in the case in which K is a compact, connected, n-dimensional differentiable manifold of class C^∞.

Let $C^\infty(K) \subset C(K)$ be the space of all complex-valued C^∞ functions on K. Let $\omega: K \to \boldsymbol{T}$ and $\psi : K \to K$ be continuous functions. For any $f \in C(K)$ consider the function

$$K \ni x \mapsto \omega(x) f(\psi(x)) . \tag{16}$$

LEMMA 6. *If the function* (16) *is* C^∞ *whenever* f *is* C^∞, *then both* ω *and* ψ *are* C^∞.

PROOF. Choosing $f = 1_K$ shows that ω is of class C^∞. The differentiability of ψ follows when one chooses local coordinates on K as test functions f.

Let $A \in \mathcal{L}(C(K))$ be defined by (4), where $\omega: K \to \boldsymbol{T}$ and $\psi: K \to K$ are continuous.

COROLLARY 1. *The functions* ω *and* ψ *are of class* C^∞ *if, and only if,* $AC^\infty(K) \subset \subset C^\infty(K)$.

Let D be a (non-identically vanishing) C^∞ vector field on K, and let $\phi : \boldsymbol{R} \times K \to K$ be the C^∞ flow defined by D. The derivation D is the infinitesimal generator of the strongly continuous Markov lattice group $S: \boldsymbol{R} \to \mathcal{L}(C(K))$ defined by (10) for all $t \in \boldsymbol{R}, f \in C(K), x \in K$. Furthermore, $C^\infty(K)$ is the space of all differentiable vectors of the group S. Thus, $C^\infty(K)$ is a core of D and $S(t)\, C^\infty(K) \subset C^\infty(K)$ $\forall t \in \boldsymbol{R}$. Viceversa, let $S: \boldsymbol{R} \to \mathcal{L}(C(K))$ be a strongly continuous Markov lattice group satisfying this latter inclusion. Then all $f \in C^\infty(K)$ are differentiable vectors of the derivation D generating S, and therefore $D^p f \in C^\infty(K)$ for all $p = 1, 2, \ldots$. That shows that D is a C^∞ vector field on K.

LEMMA 7. *If* $\alpha : \boldsymbol{R} \times K \to \boldsymbol{T}$ *is a continuous cocycle such that* $\alpha_t \in C^\infty(K)$ *for all* $t \in \boldsymbol{R}$, *the function* $t \mapsto \alpha_t(x)$ *is of class* C^∞ *on* $\boldsymbol{R}$ *for all* $x \in K$.

PROOF. For $t_0 \in \boldsymbol{R}$ and $r > 0$, let $\sigma: \mathbf{R} \to [0, 1]$ be a C^∞ function such that

$$\begin{array}{ll} \sigma(t) = 1 & \text{if } |t - t_0| \leqslant r, \\ 0 < \sigma(t) < 1 & \text{if } r < |t - t_0| < 2r, \\ \sigma(t) = 0 & \text{if } |t - t_0| \geqslant 2r. \end{array}$$

In view of (9),

$$\int_{-\infty}^{+\infty} \alpha_{t+s}(x)\, \sigma(s)\, ds = \alpha_t(x) \int_{-\infty}^{+\infty} \alpha_s(\phi_t(x))\, \sigma(s)\, ds,$$

i.e.

$$\int_{-\infty}^{+\infty} \alpha_s(x)\, \sigma(s-t)\, ds = \alpha_t(x) \int_{-\infty}^{+\infty} \alpha_s(\phi_t(x))\, \sigma(s)\, ds.$$

Given $t_o \in \boldsymbol{R}$, there exist $r > 0$ and a neighbourhood U of t_0 in $\boldsymbol{R}$ for which

$$\int_{-\infty}^{+\infty} \alpha_s(\phi_t(x))\, \sigma(s)\, ds \neq 0$$

whenever $t \in U$. Differentiation with respect to $t \in U$ shows that the function $t \mapsto \alpha_t$ is of class C^1 for all $x \in K$, and that

$$-\int_{-\infty}^{+\infty} \alpha_s(x)\, \dot\sigma(s-t)\, ds = \alpha_t(x) \int_{-\infty}^{+\infty} (D\alpha_s)_{\phi_t(x)}\, \sigma(s)\, ds + \frac{\partial \alpha_t(x)}{\partial t} \int_{-\infty}^{+\infty} \alpha_s(\phi_t(x))\, \sigma(s)\, ds.$$

Iterating this computation, one shows that $t \mapsto \alpha_t(x)$ is of class C^∞ for all $x \in K$ and completes the proof of the lemma. In conclusion, the following theorem holds, which provides a characterization of a class of strongly continuous groups of isometries of $C(K)$, which preserve $\Gamma(K)$ and the differentiable structure of K.

THEOREM 4. *If the strongly continuous group* $T: \boldsymbol{R} \to \mathcal{L}(C(K))$ *of linear isometries of* $C(K)$ *satisfies the condition*: $T(t)C^\infty(K) \subset C^\infty(K)$ *for all* $t \in \boldsymbol{R}$, *then the infinitesimal generator* X *of* T *is given by* (12), *where* D *is a* C^∞ *vector field on* K, *and* $\beta: K \to \boldsymbol{R}$ *is the* C^∞ *function defined by* (14). *Furthermore,* $C^\infty(K)$ *is a core of* X.

Viceversa, if $\alpha_t \in \Gamma(K)$ *is a continuous cocycle such that* $\alpha_t \in C^\infty(K)$, *and if* D *is a* C^∞ *vector field on* K, *then* $T(t)C^\infty(K) \subset C^\infty(K)$ *for all* $t \in \boldsymbol{R}$.

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Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 PISA