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Curry algebras N_1

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Logica matematica. — *Curry algebras* N_1. Nota di JAIR MINORO ABE, presentata (*) dal Socio A. Bressan.

ABSTRACT. — In [6] da Costa has introduced a new hierarchy N_i, $1 \leqslant i \leqslant w$ of logics that are both paraconsistent and paracomplete. Such logics are now known as non-alethic logics. In this article we present an algebraic version of the logics N_i and study some of their properties.

KEY WORDS: Algebraic logic; Paraconsistent logic; Paracomplete logic; Non-alethic logic.

RIASSUNTO. — *Le algebre «Curry»* N_1. Nell'articolo [6] da Costa ha introdotto una nuova gerarchia N_i, $1 \leqslant i \leqslant w$, di logiche che sono al tempo stesso paraconsistenti e paracomplete. Tali logiche sono adesso conosciute come logiche nonaletiche. In questo articolo presentiamo una versione algebrica della logica N_i e studiamo alcune proprietà.

1. INTRODUCTION

In recent years, a number of different kinds of logic have been proposed with the aim of avoiding the property that from a contradiction anything may be deduced. Roughly speaking, these logics (called *paraconsistent* logics) allow formulas of the form $A \& \neg A$ to be applied in a non-trivial manner in deductions. One such type of logic are the logics C_n of da Costa (see, *e.g.* [4, 5]). Their «duals», in a precise sense, are the logics known as *paracomplete* logics (systems P_n, see, *e.g.* [7]). A logic is called *paracomplete* if, according to it, a proposition and its negation can be both false.

In [6], da Costa describes a new hierarchy N_i, $1 \leqslant i \leqslant w$ of logics which are simultaneously paraconsistent and paracomplete. These logics were dubbed *non-alethic* by F. M. Quesada.

The aim of the present *Note* is to present an algebraic version of the logic N_1, developing some ideas of the authors and to study some of the main properties of this algebraic version of N_1.

2. THE CALCULUS N_1

We now present N_1 formally. We begin with a language L consisting of a denumerable number of sentential variables closed as usual under $\neg$ (negation), $\rightarrow$ (implication), $\vee$ (disjunction), and & (conjunction); the symbol $\leftrightarrow$ (for equivalence) is introduced as usual, and we have three new defined symbols: A^0 is an abbreviation for $\neg(A \& \neg A)$, A^* is an abbreviation for $A \vee \neg A$, and $-A$ is an abbreviation for $\neg A \& A^0$ (called strong negation). Capital latin letters are metalinguistic schematic variables.

(*) Nella seduta dell'11 maggio 1996.

The postulates (axiom schemata and primitive rules of inference) of N_1 are those of classical positive logic plus the following:

(I) $A^* \& B^0 \to ((A \to B) \to (A \to \neg B) \to \neg A)$,

(II) $A^0 \& B^0 \to (A \to B)^0 \& (A \& B)^0 \& (A \vee B)^0 \& (\neg A)^0$,

(III) $A^* \& B^* \to (A \to B)^* \& (A \& B)^* \& (A \vee B)^* \& (\neg A)^*$,

(IV) $A^0 \to (A \to \neg\neg A) \& (A \to (\neg A \to B))$,

(V) $A^* \to (\neg\neg A \to A)$,

(VI) $A^0 \vee A^*$.

The concepts of *proof*, of *deduction*, etc. are defined as in Kleene [8].

DEFINITION 2.1. *In* N_1: $A \leqslant B =_{\text{def.}} \vdash A \to B$, $A \equiv B =_{\text{def.}} A \leqslant B$ *and* $B \leqslant A$.

THEOREM 2.2. $\leqslant$ *is a quasi-order, and* $\equiv$ *is an equivalence relation.*

THEOREM 2.3. *In* N_1, $\neg$ *is not compatible with the equivalence relation* $\equiv$, *and we have* $\vdash A \wedge -A \to B$.

THEOREM 2.4. *Adding the principle of the excluded middle,* $A \vee \neg A$, *to* N_1, *we get* C_1.

THEOREM 2.5. *Adding the principle of contradiction,* $\neg(A \& \neg A)$ *to* N_1, *we get* P_1.

THEOREM 2.6. *Adjoining to* N_1 *the schemata* $A \vee \neg A$ *and* $\neg(A \& \neg A)$, *we obtain the classical propositional calculus.*

THEOREM 2.7. *The strong negation possesses all properties of the classical negation; for instance*

$$\vdash (A \to B) \to ((A \to -B) \to -A), \quad \vdash A \to (-A \to B), \quad \vdash A \& -A \to B,$$

$$\vdash A \to --A, \quad \vdash --A \to A.$$

3. THE CURRY ALGEBRAS N_1

The algebraic structures considered here are those seen in [2, 3].

From the algebraic point of view, N_1 is a classical implicative lattice. By Theorem 2.3 it follows that this lattice has a first element. Due to (I)-(VI), we concluded that in this lattice there is an operator, denoted by $'$, possessing some properties of the Boolean complement. Summarizing, N_1 is a Curry algebra N_1.

DEFINITION 3.1. *A* Curry algebra N_1 *is a classical implicative lattice* $\langle S, \equiv, \wedge, \vee, ' \rangle$ *with greatest and smallest elements* (*not necessarily unique*), 1 *and* 0, *and with an operator* $'$ *satisfying the following properties, where* $p^0 = (p \wedge p')'$ *and* $p^* = p \vee p'$:

1) $p^0 \wedge q^* \leqslant ((p \to q) \to ((p \to q') \to p'))$,
2) $p^0 \wedge q^0 \leqslant (p \to q)^0 \wedge (p \wedge q)^0 \wedge (p \vee q)^0 \wedge (q')^0$,
3) $p^* \wedge q^* \leqslant (p \to q)^* \wedge (p \wedge q)^* \wedge (p \vee q)^* \wedge (q')^*$,
4) $p^0 \leqslant (p \to p'') \wedge (p \to (p' \to q))$,
5) $p^* \leqslant p'' \to p$,
6) $p^0 \vee q^* \equiv 1$.

THEOREM 3.2. *Adjoining to a Curry algebra* N_1 *the postulate* $(p \wedge p')' \equiv 1$, *we obtain a* CP_1*-algebra* (for these algebras see [1]), *and adjoining the postulate* $p \vee p' \equiv 1$ *we get a* C_1*-algebra* (for these algebras see [2]). *Moreover if we add both postulates we obtain a Boolean algebra.*

THEOREM 3.3. *A Curry algebra* N_1 *is distributive and has a greatest element.*

DEFINITION 3.4. *Let* p *be an element of a Curry algebra* N_1. *We put* $-p = p' \wedge p^0$.

THEOREM 3.5. *In a Curry algebra* N_1, $-p$ *is a Boolean complement of* p, *so* $p \vee -p \equiv 1$ *and* $p \wedge -p \equiv 0$.

THEOREM 3.6. *In a Curry algebra* N_1, *the structure composed by the underlying set and by operations* $\wedge$, $\vee$, $-$, *is a Boolean algebra.*

DEFINITION 3.7. *Let* $\langle S, \equiv, \to, \wedge, \vee, ' \rangle$ *be a Curry algebra* N_1, *and* $\langle S, \equiv, \wedge, \vee, - \rangle$ *be the Boolean algebra obtained as in the above theorem. Any Boolean algebra that is isomorphic to the quotient algebra of* $\langle S, \equiv, \wedge, \vee, - \rangle$ *by* $\equiv$ *is called* Boolean algebra associated with the Curry algebra N_1.

THEOREM 3.8 (Representation Theorem). *Any Curry algebra* N_1 *is associated with a field of sets. Moreover, any Curry algebra* N_1 *is associated with the field of sets simultaneously open and closed of a totally disconnected compact Hausdorff space.*

4. THE COMPLETENESS OF THE LOGIC N_1

It is easy to introduce the concepts of *filter*, *ultrafilter*, and *homomorphisms* between Curry algebras N_1. All usual properties from classical algebra are as expected: for instance, the shell of a homomorphism is a filter.

We would like to mention only the following results.

THEOREM 4.1 (Soundness). *If* A *is a provable formula of the logic* N_1, *then* $h(A) \equiv 1$ *for any homomorphism* h *from the set of all formulas of the logic* N_1 *into any Curry algebra* N_1.

PROOF. By induction on the length of proofs.

THEOREM 4.2. *Let U be an ultrafilter in F (the set of all formulas of N_1). Then, there is a homomorphism h from F into 2 (where 2 = {0, 1} is the two-element Boolean algebra) such that the shell of h is U.*

THEOREM 4.3 (Completeness). *Let F be the set of all formulas of N_1, and $A \in F$. Let us suppose that $h(A) \equiv 1$ for any homomorphism h from F into an arbitrary Curry algebra N_1. Then, A is a provable formula of N_1.*

PROOF. Similar to the classical case, taking into account the previous theorem.

The algebraic treatment for the logics N_i, $1 \leqslant i \leqslant w$ does not offer difficulties. Besides the results presented here we would like to emphasize the importance of certain «pre-algebraical» structures (in the sense that the fundamental relation of the structure is an equivalence relation instead of equality). They were called Curry algebras not only as a hommage to the american logician H. B. Curry, but because he is one of defendants of the use of pre-structures (see [3]). They are the central tool to deal algebraically with the majority of non-classical logics.

We hope to say something more about this in forthcoming papers.

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