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OLGA A. OLEINIK, TATIANA A. SHAPOSHNIKOVA

On the homogenization of the Poisson equation in partially perforated domains with arbitrary density of cavities and mixed type conditions on their boundary

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Analisi matematica. — *On the homogenization of the Poisson equation in partially perforated domains with arbitrary density of cavities and mixed type conditions on their boundary.* Nota (*) di Olga A. Oleinik e Tatiana A. Shaposhnikova, presentata dal Socio O. A. Oleinik.

Abstract. — In this paper we study the behavior of solutions of the boundary value problem for the Poisson equation in a partially perforated domain with arbitrary density of cavities and mixed type conditions on their boundary. The corresponding spectral problem is also considered. A short communication of similar results can be found in [1].

Key words: Homogenization; Poisson equation; Perforated domains; Mixed type conditions; Spectral problem.

Riassunto. — *Sull'omogeneizzazione dell'equazione di Poisson in domini parzialmente perforati con arbitraria densità delle cavità e condizioni di tipo misto sul loro contorno.* In questa *Nota* viene studiato il comportamento delle soluzioni del problema ai limiti per l'equazione di Poisson in un dominio parzialmente perforato con arbitrarie densità delle cavità e condizioni di tipo misto sul loro contorno. Viene anche considerato il corrispondente problema spettrale. Una breve comunicazione di simili risultati si trova in [1].

Introduction

Homogenization problems in a partially perforated domain with the Dirichlet, Neumann and mixed conditions on the boundary of cavities were considered in [2-10].

Boundary value problems in perforated domains were studied in [11, 12], and also in monographs [13-18]. In these books one can find an extensive bibliography for this subject. Note also that monograph [18] is one of the first investigations on the problems of homogenization in perforated domains.

1. – Let Ω be a bounded domain in R_x^n with a smooth boundary $\partial\Omega$, $Q = \{x \in R_x^n,\ 0 < x_j < 1, j = 1, \dots, n\}$, G_0 is a domain in Q such that $\overline{G_0} \subset Q$ and $\overline{G_0}$ is diffeomorphic to a ball. We denote

$$\gamma = \Omega \cap \{x : x_1 = 0\} \neq \emptyset, \quad \Omega^+ = \Omega \cap \{x : x_1 > 0\}, \quad \Omega^- = \Omega \cap \{x : x_1 < 0\},$$

$$G_\varepsilon = \bigcup_{z \in Z} (a_\varepsilon G_0 + \varepsilon z), \quad a_\varepsilon G_0 \subset \varepsilon Q,$$

where ε is a small positive parameter, a_ε is a positive number which depends on ε and $a_\varepsilon \to 0$ as $\varepsilon \to 0$, Z is the set of vectors z with integer components.

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We set

$$\Omega_\varepsilon^+ = \Omega^+ \backslash \overline{G_\varepsilon}, \qquad Y_\varepsilon = \varepsilon Q \backslash \overline{a_\varepsilon G_0}, \qquad S_0 = \partial G_0, \qquad \Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega^- \cup \gamma,$$
$$S_\varepsilon = \partial\Omega_\varepsilon \cap \Omega, \qquad \Gamma_\varepsilon = \partial\Omega \cap \partial\Omega_\varepsilon, \qquad \alpha B = \{x: \alpha^{-1}x \in B\},$$
$$\langle u \rangle_\omega = |\omega|^{-1} \int_\omega u\, dx, \text{ where } |\omega| \text{ is the volume of the domain } \omega.$$

In the partially perforated domain Ω_ε we consider the next boundary value problem:

$$(1) \qquad \begin{cases} \Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon, \\ \partial u_\varepsilon / \partial \nu + b u_\varepsilon = 0 & \text{on } S_\varepsilon, \end{cases}$$

where ν is a unit exterior normal vector to S_ε. For simplicity we assume that $b = \text{const} > 0$, $f \in L_2(\Omega)$. For the existence and uniqueness of solutions to problem (1) see [26]. As usual we denote by $H_1(\Omega, \Gamma_0)$ the space of functions which is obtained by completion of the set of infinitely differentiable in $\overline{\Omega}$ functions $u(x)$ equal to zero in a neighborhood of Γ_0, by the norm $H_1(\Omega)$:

$$\|u\|^2_{H_1(\Omega)} = \int_\Omega (u^2 + |\nabla u|^2)\, dx, \quad \text{where } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

We consider a weak solution $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ of the problem (1) and study the behavior of u_ε as $\varepsilon \to 0$.

We need some auxiliary results.

LEMMA 1. If $u \in H_1(Y_\varepsilon)$, $\langle u \rangle_{Y_\varepsilon} = 0$, then

$$(2) \qquad \|u\|_{L_2(Y_\varepsilon)} \leqslant K_1 \varepsilon \|\nabla u\|_{L_2(Y_\varepsilon)},$$

where all constants K_j here and in what follows do not depend on ε.

LEMMA 2. If $u \in H_1(Y_\varepsilon)$, then

$$(3) \qquad \|u\|^2_{L_2(a_\varepsilon S_0)} \leqslant K_2 \{ a_\varepsilon^{n-1} \varepsilon^{-n} \|u\|^2_{L_2(Y_\varepsilon)} + a_\varepsilon \|\nabla u\|^2_{L_2(Y_\varepsilon)} \},$$

if $n \geqslant 3$, and

$$(4) \qquad \|u\|^2_{L_2(a_\varepsilon S_0)} \leqslant K_3 \left\{ a_\varepsilon \varepsilon^{-2} \|u\|^2_{L_2(Y_\varepsilon)} + a_\varepsilon \left| \ln \frac{\varepsilon}{2a_\varepsilon} \right| \|\nabla u\|^2_{L_2(Y_\varepsilon)} \right\},$$

if $n = 2$.

Proofs of these lemmas can be found in [8].

REMARK 1. Let $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$. We consider the set Y_0 of cells $Y_\varepsilon + \varepsilon z$, $z \in Z$, which intersect the boundary $\partial\Omega$. This means $Y_\varepsilon + \varepsilon z \cap \partial\Omega \neq \emptyset$. We consider the function

$$\tilde{u} = \begin{cases} u, & \text{if } x \in \Omega_\varepsilon, \\ 0, & \text{if } x \in Y_0 \backslash \Omega. \end{cases}$$

It is easy to see that $\tilde{u} \in H_1(\Omega_\varepsilon \cup Y_0)$ and we can use Lemma 2 for every cell from

Y_0. Summing over all cells, which belong to $\Omega_\varepsilon \cup Y_0$ we obtain the estimates

(5) $$\|u\|^2_{L_2(S_\varepsilon)} \leq K_2 \{a_\varepsilon^{n-1} \varepsilon^{-n} \|u\|^2_{L_2(\Omega_\varepsilon^+)} + a_\varepsilon \|\nabla u\|^2_{L_2(\Omega_\varepsilon^+)}\},$$

if $n \geq 3$, and

(6) $$\|u\|^2_{L_2(S_\varepsilon)} \leq K_3 \left\{a_\varepsilon \varepsilon^{-2} \|u\|^2_{L_2(\Omega_\varepsilon^+)} + a_\varepsilon \left| \ln \frac{\varepsilon}{2a_\varepsilon} \right| \|\nabla u\|^2_{L_2(\Omega_\varepsilon^+)}\right\},$$

if $n = 2$.

LEMMA 3. If $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$, then

(7) $$\|u\|_{L_2(\Omega_\varepsilon^+)} \leq K_4 \varepsilon^{n/2} \{a_\varepsilon^{(1-n)/2} \|u\|_{L_2(S_\varepsilon)} + a_\varepsilon^{(2-n)/2} \|\nabla u\|_{L_2(\Omega_\varepsilon^+)}\},$$

if $n \geq 3$, and

(8) $$\|u\|_{L_2(\Omega_\varepsilon^+)} \leq K_5 \varepsilon \left\{a_\varepsilon^{-1/2} \|u\|_{L_2(S_\varepsilon)} + \sqrt{\left| \ln \frac{\varepsilon}{2a_\varepsilon} \right|} \|\nabla u\|_{L_2(\Omega_\varepsilon^+)}\right\},$$

if $n = 2$.

We shall give the proof of Lemma 3 in the appendix.

2. – Let $a_\varepsilon^{1-n} \varepsilon^n \to 0$ as $\varepsilon \to 0$, $f \in L_2(\Omega)$ and $n \geq 2$.

Let us introduce the function $v \in H_2(\Omega^-)$ as a weak solution of the problem

(9) $$\Delta v = f \quad \text{in } \Omega^-, \qquad v = 0 \quad \text{on } \partial\Omega^-.$$

Proof of the existence and uniqueness of a weak solution $v \in H_1(\Omega^-)$ of the boundary value problem (9) is a consequence of the Lax-Milgram theorem. It is proved in [20] that $v \in H_2(\Omega^-)$. Now we define a function w_ε as a weak solution from the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ of the problem:

(10) $$\begin{cases} \Delta w_\varepsilon = 0, & x \in \Omega^- \cup \Omega_\varepsilon^+, \\ \dfrac{\partial w_\varepsilon}{\partial \nu} + b w_\varepsilon = 0, & x \in S_\varepsilon, \\ w_\varepsilon = 0, & x \in \Gamma_\varepsilon, \\ [w_\varepsilon]|_\gamma = 0, & \\ \left[\dfrac{\partial w_\varepsilon}{\partial x_1}\right]\Big|_\gamma = \dfrac{\partial v}{\partial x_1}\Big|_{x_1 = -0}, & \end{cases}$$

where $[\varphi]|_{P \in \gamma} = \varphi|_{P+0} - \varphi|_{P-0}$ for any point $P \in \gamma$ and any function φ.

The existence and uniqueness theorem for the problem (10) can be obtained from the Lax-Milgram theorem. Taking in the integral identity for the problem (10) the solution w_ε as a test-function we obtain the equality

(11) $$\int_{\Omega_\varepsilon^+ \cup \Omega^-} |\nabla_x w_\varepsilon|^2 \, dx + b \int_{S_\varepsilon} w_\varepsilon^2 \, ds_x = - \int_\gamma w_\varepsilon \frac{\partial v}{\partial x_1}\Big|_{x_1 = -0} d\hat{x},$$

where $\hat{x} = (x_2, \ldots, x_n)$. By virtue of the Friedrichs inequality and the imbedding theorem for $w_\varepsilon \in H_1(\Omega^-, \partial\Omega^- \cap \partial\Omega)$, we have

(12) $$\|w_\varepsilon\|^2_{L_2(\gamma)} \leqslant K_6(\|w_\varepsilon\|^2_{L_2(\Omega^-)} + \|\nabla_x w_\varepsilon\|^2_{L_2(\Omega^-)}) \leqslant K_7\|\nabla_x w_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} .$$

From (11) and (12) we deduce

(13) $$\|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant K_8 , \quad \|w_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_9 .$$

From Lemma 3 and inequalities (13) we obtain the estimate

(14) $$\|w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{10} M(\varepsilon, n) ,$$

where $M(\varepsilon, n) = a_\varepsilon^{(1-n)/2}\varepsilon^{n/2}$. Let $\widetilde{w}_\varepsilon$ be an extension of w_ε on $G_\varepsilon \cap \Omega$ such that

$$\|\widetilde{w}_\varepsilon\|_{L_2(\Omega^+)} \leqslant K_{11}\|w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} , \quad \|\nabla_x \widetilde{w}_\varepsilon\|_{L_2(\Omega^+)} \leqslant K_{12}\|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} .$$

The construction of such a function $\widetilde{w}_\varepsilon$ is given in [13]. Then using the imbedding theorem, we obtain the estimate

(15) $$\|w_\varepsilon\|_{L_2(\gamma)} \leqslant K_{13} M^{1/2}(\varepsilon, n) .$$

Now we prove for the function w_ε the inequality

(16) $$\|w_\varepsilon\|_{L_2(\Omega^-)} \leqslant K_{14}\|w_\varepsilon\|_{L_2(\partial\Omega^-)} = K_{14}\|w_\varepsilon\|_{L_2(\gamma)} .$$

Indeed, let $V_\varepsilon \in H_2(\Omega^-)$ be a solution of the problem

(17) $$\Delta V_\varepsilon = w_\varepsilon , \quad x \in \Omega^- ; \quad V_\varepsilon = 0 , \quad x \in \partial\Omega^- .$$

It is obvious that the following relation is valid

$$\int_{\Omega^-} (w_\varepsilon \Delta V_\varepsilon - V_\varepsilon \Delta w_\varepsilon)\, dx = \int_{\partial\Omega^-} \left(w_\varepsilon \frac{\partial V_\varepsilon}{\partial \nu} - V_\varepsilon \frac{\partial w_\varepsilon}{\partial \nu} \right) ds .$$

From this equality we deduce the estimate

(18) $$\|w_\varepsilon\|^2_{L_2(\Omega^-)} \leqslant \left\| \frac{\partial V_\varepsilon}{\partial \nu} \right\|_{L_2(\partial\Omega^-)} \|w_\varepsilon\|_{L_2(\gamma)} .$$

We prove that for V_ε the following inequality is valid

(19) $$\|V_\varepsilon\|_{H_2(\Omega^-)} \leqslant K_{15}\|w_\varepsilon\|_{L_2(\Omega^-)} .$$

For this let us introduce the mapping $I_\varepsilon : H_2(\Omega^-) \to L_2(\Omega^-)$ such that

$$I_\varepsilon(V_\varepsilon) = w_\varepsilon ,$$

where V_ε is a solution of the problem (17).

Taking into account that we have the uniqueness theorem in the space $H_1(\Omega^-)$ for the problem (17) we can conclude that I_ε is a one-to-one correspondence. In addition, it is easy to see that the following estimate is valid,

$$\|w_\varepsilon\|_{L_2(\Omega^-)} \leqslant K_{16}\|V_\varepsilon\|_{H_2(\Omega^-)} .$$

Therefore, by the Banach theorem [19] the estimate (19) is valid.

By virtue of the imbedding theorem we obtain

(20)
$$\left\| \frac{\partial V_\varepsilon}{\partial \nu} \right\|_{L_2(\partial\Omega^-)} \leqslant K_{17} \|V_\varepsilon\|_{H_2(\Omega^-)} .$$

From inequalities (18)-(20) we get the estimate (16).

Thus, taking into account (15) and (16) we deduce

$$\|w_\varepsilon\|_{L_2(\Omega^-)} \leqslant K_{18} M^{1/2}(\varepsilon, n) .$$

From (11) and (15) we obtain

$$\|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant K_{19} M^{1/2}(\varepsilon, n) .$$

Thus we have

LEMMA 4. Let w_ε be a weak solution of problem (10), $w_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$. Then

(21)
$$\begin{cases} \|w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{20} M(\varepsilon, n) , \\ \|w_\varepsilon\|_{L_2(\Omega^-)} + \|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant K_{21} M^{1/2}(\varepsilon, n) . \end{cases}$$

We set

$$f^+ = \begin{cases} f, & x \in \Omega^+ , \\ 0, & x \in \Omega^- . \end{cases}$$

We introduce the function $v_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ as a weak solution of the problem

(22)
$$\begin{cases} \Delta v_\varepsilon = f^+ , & x \in \Omega_\varepsilon ; \quad v_\varepsilon = 0 , \quad x \in \Gamma_\varepsilon ; \\ \dfrac{\partial v_\varepsilon}{\partial \nu} + b v_\varepsilon = 0 , & x \in S_\varepsilon . \end{cases}$$

The existence theorem in the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ for the problem (22) can be deduced from [26]. Now we derive estimates for the solution v_ε.

Using the integral identity for problem (22) and the Friedrichs inequality for the functions of the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ [13], we obtain

(23)
$$\|\nabla_x v_\varepsilon\|_{L_2(\Omega_\varepsilon)} + \|v_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{22} .$$

From Lemma 3 and inequality (23) we have the estimate

(24)
$$\|v_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{23} M(\varepsilon, n) .$$

From the estimate (24), the Friedrichs inequality and the integral identity for v_ε we get

(25)
$$\|v_\varepsilon\|_{L_2(\Omega^-)} + \|\nabla_x v_\varepsilon\|_{L_2(\Omega_\varepsilon)} \leqslant K_{24} M^{1/2}(\varepsilon, n) .$$

Thus we have

LEMMA 5. Let $v_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ be a weak solution of the problem (22). Then estimates (24), (25) are valid.

By virtue of the uniqueness theorem for a weak solution of problem (1) we have the representation

$$\begin{cases} u_\varepsilon = v_\varepsilon + w_\varepsilon + v & \text{in } \Omega^- , \\ u_\varepsilon = w_\varepsilon + v_\varepsilon & \text{in } \Omega_\varepsilon^+ . \end{cases} \tag{26}$$

Therefore, from Lemmas 5 and 6 and representation (26) we obtain for the case $a_\varepsilon^{1-n}\varepsilon^n \to 0$ as $\varepsilon \to 0$

THEOREM 1. Let $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ be a weak solution of problem (1), $v \in H_2(\Omega^-)$ be a weak solution of problem (9) and $a_\varepsilon^{1-n}\varepsilon^n \to 0$ as $\varepsilon \to 0$, ($n \geq 2$). Then the following estimates are valid

$$\begin{cases} \|u_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leq K_{25} M(\varepsilon, n) , \\ \|u_\varepsilon - v\|_{H_1(\Omega^-)} + \|\nabla_x u_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leq K_{26} \sqrt{M(\varepsilon, n)} , \end{cases}$$

where $M(\varepsilon, n) = \sqrt{a_\varepsilon^{1-n}\varepsilon^n}$.

3. – Let $a_\varepsilon^{1-n}\varepsilon^n \to \infty$ as $\varepsilon \to 0$.

We define function v_0 as a smooth solution of the boundary value problem

$$\Delta v_0 = f \quad \text{in } \Omega , \qquad v_0 = 0 \quad \text{on } \partial\Omega , \tag{27}$$

where $f \in C^\alpha(\Omega)$, $\alpha > 0$.

We set $w_\varepsilon = u_\varepsilon - v_0$. According to the definition of the functions u_ε and v_0, $w_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ is a weak solution of the problem

$$\begin{cases} \Delta w_\varepsilon = 0 & \text{in } \Omega , \\ w_\varepsilon = 0 & \text{on } \Gamma_\varepsilon , \\ \dfrac{\partial w_\varepsilon}{\partial \nu} + b w_\varepsilon = -\left(\dfrac{\partial v_0}{\partial \nu} + b v_0 \right) & \text{on } S_\varepsilon . \end{cases} \tag{28}$$

Using the integral identity for problem (28) and taking w_ε as a test-function, we obtain the equality

$$\int_{\Omega_\varepsilon} |\nabla_x w_\varepsilon|^2 \, dx + b \int_{S_\varepsilon} w_\varepsilon^2 \, ds_x = - \int_{S_\varepsilon} \left(\frac{\partial v_0}{\partial \nu} + b v_0 \right) w_\varepsilon \, ds_x . \tag{29}$$

Taking into account Remark 1 and the Friedrichs inequality for space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$, we get

$$\begin{aligned} \|w_\varepsilon\|_{L_2(S_\varepsilon)} &\leq K_{27} ([M(\varepsilon, n)]^{-1} \|w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \sqrt{a_\varepsilon} \|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon^+)}) \leq \\ &\leq K_{28} ([M(\varepsilon, n)]^{-1} + \sqrt{a_\varepsilon}) \|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon)} , \end{aligned} \tag{30}$$

if $n \geq 3$, and

$$\|w_\varepsilon\|_{L_2(S_\varepsilon)} \leq K_{29} \left([M(\varepsilon, n)]^{-1} + \sqrt{a_\varepsilon \left| \ln \frac{\varepsilon}{2a_\varepsilon} \right|} \right) \|\nabla_x w_\varepsilon\|_{L_2(\Omega_\varepsilon)} , \tag{31}$$

if $n = 2$.

Therefore, from (29) and inequalities (30), (31) we deduce

$$\|w_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leqslant K_{30}(\sqrt{a_\varepsilon} + [M(\varepsilon, n)]^{-1}),$$

if $n \geqslant 3$,

$$\|w_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leqslant K_{31}\left(\sqrt{a_\varepsilon \left| \ln \frac{\varepsilon}{2a_\varepsilon} \right|} + [M(\varepsilon, n)]^{-1}\right),$$

if $n = 2$.

THEOREM 2. Let $f \in L_2(\Omega)$ and Ω be a domain in R_x^n with a smooth boundary $\partial\Omega$, u_ε be a weak solution of problem (1), v_0 be a smooth solution of problem (27); $a_\varepsilon^{1-n}\varepsilon^n \to \infty$ as $\varepsilon \to 0$. Then the following estimates are valid

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \leqslant K_{32}(\sqrt{a_\varepsilon} + [M(\varepsilon, n)]^{-1}),$$

if $n \geqslant 3$, and

$$\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)} \leqslant K_{33}\left(\sqrt{a_\varepsilon \left| \ln \frac{\varepsilon}{2a_\varepsilon} \right|} + [M(\varepsilon, n)]^{-1}\right),$$

if $n = 2$.

4. – Now we assume that $a_\varepsilon^{1-n}\varepsilon^n \to C_0$ as $\varepsilon \to 0$ and $C_0 = \text{const} > 0$.

We introduce the functions $\theta_\varepsilon(x)$ as the solution of the problem

$$(32) \qquad \begin{cases} \Delta\theta_\varepsilon = \mu_\varepsilon \quad \text{in } Y_\varepsilon, & \dfrac{\partial\theta_\varepsilon}{\partial\nu} = -b \quad \text{on } a_\varepsilon S_0, \\ \langle\theta_\varepsilon\rangle_{Y_\varepsilon} = 0, \quad \theta_\varepsilon & \text{is } \varepsilon\text{-periodic function}, \end{cases}$$

where $\mu_\varepsilon = \text{const}$ which is defined from the solvability condition of problem (32), that is

$$\mu_\varepsilon \operatorname{meas} Y_\varepsilon = -b \operatorname{meas}(a_\varepsilon S_0).$$

From here we have

$$(33) \quad \mu_\varepsilon = -\frac{b}{C_0}\operatorname{meas} S_0 - \frac{b(a_\varepsilon\varepsilon^{-1})^n \operatorname{meas} S_0 \operatorname{meas} G_0}{C_0(1-(a_\varepsilon\varepsilon^{-1})^n \operatorname{meas} G_0)} -$$

$$-\frac{(a_\varepsilon^{n-1}\varepsilon^{-n} - C_0^{-1})b\operatorname{meas} S_0}{1-(a_\varepsilon\varepsilon^{-1})^n\operatorname{meas} G_0} = -\frac{b}{C_0}\operatorname{meas} S_0 + A_\varepsilon(a_\varepsilon^{1-n}\varepsilon^n - C_0) + B_\varepsilon(a_\varepsilon\varepsilon^{-1})^n,$$

where $|A_\varepsilon| \leqslant A_0$, $|B_\varepsilon| \leqslant B_0$ and A_0, B_0 are *constants*, which do not depend on ε.

Note that $a_\varepsilon\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$ since $(a_\varepsilon\varepsilon^{-1})^n \sim C_0^{n-1}a_\varepsilon$ as $\varepsilon \to 0$.

We define also the function $N_j^\varepsilon(y)$ ($y = x\varepsilon^{-1}$; $j = 1, \dots, n$) as a solution of the problem

$$(34) \qquad \begin{cases} \Delta_y N_j^\varepsilon = 0 \quad \text{in } \varepsilon^{-1}Y_\varepsilon, & \dfrac{\partial N_j^\varepsilon}{\partial\nu} = -\nu_j \quad \text{on } \varepsilon^{-1}a_\varepsilon S_0, \\ \langle N_j^\varepsilon\rangle_{\varepsilon^{-1}Y_\varepsilon} = 0, \quad N_j^\varepsilon & \text{is 1-periodic function}. \end{cases}$$

In addition we introduce the function $u_0(x)$ as a smooth solution in $\overline{\Omega^+}$ and $\overline{\Omega^-}$ of the problem

$$(35)\qquad \begin{cases} \Delta_x u_0 = f \quad \text{in } \Omega^-, \qquad \Delta_x u_0 + \mu_0 u_0 = f \quad \text{in } \Omega^+, \\ u_0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\mu_0 = -(b \operatorname{meas} S_0)/C_0$.

Problems of this type were considered in papers [21-23]. In the case of the boundary value problem

$$\Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \qquad \Omega = \{x: 0 < x_j < 1, j = 2, \dots, n, -1 < x_1 < 1\}$$

with the boundary conditions

$u_\varepsilon = 0$ for $x_1 = -1$ and for $x_1 = 1$, u_ε is a 1-periodic function in $\hat{x} = (x_2, \dots, x_n)$

the results, obtained above, are valid. For this problem the solution u_0, corresponding to the problem (35), exists and has the regularity properties which we need below. It follows from theorems proved in [24].

Using the integral identity for problem (32) and also Lemma 1 and Lemma 2, we obtain

$$\|\nabla_x \theta_\varepsilon\|^2_{L_2(Y_\varepsilon)} \leqslant K_{34} a_\varepsilon^{(n-1)/2} \|\theta_\varepsilon\|_{L_2(a_\varepsilon S_0)} \leqslant$$
$$\leqslant K_{35}(a_\varepsilon^{n-1}\varepsilon^{-n/2+1} + a_\varepsilon^{n/2}) \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{36} a_\varepsilon^{n/2} \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)},$$

since $a_\varepsilon^{n-1}\varepsilon^{-n/2+1} < a_\varepsilon^{n/2}$ for small ε, if $n \geqslant 3$, and

$$\|\nabla_x \theta_\varepsilon\|^2_{L_2(Y_\varepsilon)} \leqslant K_{37}(a_\varepsilon + a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)}) \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{38} a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)} \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)},$$

if $n = 2$.

From here and from Lemma 1 we get the following estimates

$$(36)\qquad \begin{cases} \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{39} a_\varepsilon^{n/2}, \quad \|\theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{40}\varepsilon a_\varepsilon^{n/2}, \\ \text{if } n \geqslant 3, \text{ and} \\ \|\nabla_x \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{41} a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \quad \|\theta_\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{42}\varepsilon a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \\ \text{if } n = 2. \end{cases}$$

From Lemma 2 and (36) we deduce

$$(37)\qquad \begin{cases} \|\theta_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{43} a^{n/2} \varepsilon^{-n/2+1}, \\ \|\nabla_x \theta_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{44}(a_\varepsilon \varepsilon^{-1})^{n/2}, \quad \|\theta_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{45} a_\varepsilon^{(n+1)/2} \varepsilon^{-n/2}, \\ \text{if } n \geqslant 3, \text{ and} \\ \|\theta_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{46} a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \\ \|\nabla_x \theta_\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{47}(a_\varepsilon/2\varepsilon) \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \quad \|\theta_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{48} a_\varepsilon^2 \varepsilon^{-1} \ln(\varepsilon/2a_\varepsilon), \\ \text{if } n = 2. \end{cases}$$

Thus we have

LEMMA 6. Let $a_\varepsilon^{1-n}\varepsilon^n \to C_0$ as $\varepsilon \to 0$ and $C_0 = \text{const} > 0$, and let $\theta_\varepsilon(x)$ be a solution of problem (32). Then estimates (36) and (37) are valid.

For the solution N_j^ε we have the following propositions. They are proved in [8].

$$\text{(38)}\qquad \begin{cases} \|N_j^\varepsilon\|_{L_2(Y_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{49} a_\varepsilon^{n/2}, \\ \|N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{50} (a_\varepsilon \varepsilon^{-1})^{n/2}, \end{cases}$$

if $n \geqslant 3$, and

$$\text{(39)}\qquad \begin{cases} \|N_j^\varepsilon\|_{L_2(Y_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(Y_\varepsilon)} \leqslant K_{51} a_\varepsilon \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \\ \|N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{52} (2a_\varepsilon/\varepsilon) \sqrt{\ln(\varepsilon/2a_\varepsilon)}, \end{cases}$$

if $n = 2$.

Now we define the function $\varphi_\varepsilon(x_1) \in C^\infty(R^1_{x_1})$, $\varphi_\varepsilon = 0$ for $x_1 \leqslant a_0\varepsilon$, $\varphi_\varepsilon = 1$ for $x_1 \geqslant 2a_0\varepsilon$, $0 \leqslant \varphi_\varepsilon \leqslant 1$, $|\dot\varphi_\varepsilon| \leqslant b_0\varepsilon^{-1}$, $|\ddot\varphi_\varepsilon| \leqslant b_1\varepsilon^{-2}$ and the *constant* a_0 is chosen in such a way that $\varphi_\varepsilon = 1$ for $x \in S_\varepsilon$.

We set

$$u_\varepsilon^1 = (1 + \varphi_\varepsilon\theta_\varepsilon)u_0 + \varepsilon N_j^\varepsilon \varphi_\varepsilon \frac{\partial u_0}{\partial x_j}, \qquad x \in \Omega_\varepsilon^+ \cap \Omega^-.$$

Here and in the following we use the usual convention of repeated indices. It is easy to see that $g_\varepsilon = u_\varepsilon^1 - u_\varepsilon$ is a weak solution of the problem

$$\begin{aligned} \Delta g_\varepsilon = {} & A_\varepsilon(a_\varepsilon^{1-n}\varepsilon^n - C_0)u_0\varphi_\varepsilon + \mu_\varepsilon(\varphi_\varepsilon - 1)u_0 + B_\varepsilon(a_\varepsilon\varepsilon^{-1})^n u_0\varphi_\varepsilon + \ddot\varphi_\varepsilon\theta_\varepsilon u_0 + \\ & + 2\dot\varphi_\varepsilon u_0 \frac{\partial\theta_\varepsilon}{\partial x_1} + 2\dot\varphi_\varepsilon\theta_\varepsilon\frac{\partial u_0}{\partial x_1} + 2\varphi_\varepsilon(\nabla_x\theta_\varepsilon, \nabla_x u_0) + \varphi_\varepsilon\theta_\varepsilon\Delta u_0 + 2\dot\varphi_\varepsilon\frac{\partial N_j^\varepsilon}{\partial y_1}\frac{\partial u_0}{\partial x_j} + \\ & + \varphi_\varepsilon\frac{\partial N_j^\varepsilon}{\partial y_p}\frac{\partial^2 u_0}{\partial x_j\,\partial x_p} + \varepsilon\ddot\varphi_\varepsilon N_j^\varepsilon\frac{\partial u_0}{\partial x_j} + \varepsilon\dot\varphi_\varepsilon N_j^\varepsilon\frac{\partial^2 u_0}{\partial x_j\,\partial x_1} + \frac{\partial}{\partial x_k}\left(\varepsilon N_j^\varepsilon\varphi_\varepsilon\frac{\partial^2 u_0}{\partial x_k\,\partial x_j}\right), \end{aligned}$$

in Ω_ε^+ where the derivatives in the last term are considered as distributions,

$$\Delta g_\varepsilon = 0, \qquad x \in \Omega_\varepsilon, \qquad [g_\varepsilon]|_\gamma = \left[\frac{\partial g_\varepsilon}{\partial x_1}\right]\Big|_\gamma = 0,$$

$$g_\varepsilon = \varepsilon N_j^\varepsilon\varphi_\varepsilon\frac{\partial u_0}{\partial x_1}, \qquad x \in \Gamma_\varepsilon,$$

$$\frac{\partial g_\varepsilon}{\partial \nu} + bg_\varepsilon = \theta_\varepsilon\left(\frac{\partial u_0}{\partial\nu} + bu_0\right) + \varepsilon N_j^\varepsilon\left(\frac{\partial}{\partial\nu}\left(\frac{\partial u_0}{\partial x_j}\right) + b\frac{\partial u_0}{\partial x_j}\right), \qquad x \in S_\varepsilon.$$

We represent the solution g_ε in the form

$$g_\varepsilon = g_{1,\varepsilon} + g_{2,\varepsilon},$$

where $g_{1,\varepsilon}$ is a weak solution in the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ of the problem

$$(40)\qquad \begin{cases} \Delta g_{1,\varepsilon} = F_\varepsilon^+ + \dfrac{\partial F_{\varepsilon,k}}{\partial x_k} \quad \text{in } \Omega_\varepsilon^+ , \\ \Delta g_{1,\varepsilon} = 0 \quad \text{in } \Omega^- , \quad g_{1,\varepsilon} = 0 \quad \text{on } \Gamma_\varepsilon , \\ \dfrac{\partial g_{1,\varepsilon}}{\partial \nu} + b g_{1,\varepsilon} = F_{\varepsilon,k}\nu_k + \kappa_\varepsilon \quad \text{on } S_\varepsilon , \end{cases}$$

where

$$F_\varepsilon^+ = A_\varepsilon (a_\varepsilon^{1-n}\varepsilon^n - C_0) u_0 \varphi_\varepsilon + \mu_0 (\varphi_\varepsilon - 1) u_0 + \ddot{\varphi}_\varepsilon \theta_\varepsilon u_0 + B_\varepsilon (a_\varepsilon \varepsilon^{-1})^n u_0 \varphi_\varepsilon +$$
$$+ 2\dot{\varphi}_\varepsilon \frac{\partial \theta_\varepsilon}{\partial x_1} u_0 + 2\dot{\varphi}_\varepsilon \theta_\varepsilon \frac{\partial u_0}{\partial x_1} + 2\varphi_\varepsilon (\nabla_x \theta_\varepsilon, \nabla_x u_0) + \varphi_\varepsilon \theta_\varepsilon \Delta u_0 +$$
$$+ 2\dot{\varphi}_\varepsilon \frac{\partial N_j^\varepsilon}{\partial y_1} \frac{\partial u_0}{\partial x_j} + \varphi_\varepsilon \frac{\partial N_j^\varepsilon}{\partial y_p} \frac{\partial^2 u_0}{\partial x_j \partial x_p} + \varepsilon \ddot{\varphi}_\varepsilon N_j^\varepsilon \frac{\partial u_0}{\partial x_j} + \varepsilon \dot{\varphi}_\varepsilon N_j^\varepsilon \frac{\partial^2 u_0}{\partial x_j \partial x_1} ,$$

$$F_{\varepsilon,k} = \varepsilon N_j^\varepsilon \varphi_\varepsilon \frac{\partial^2 u_0}{\partial x_j \partial x_k} ,$$

$$\kappa_\varepsilon = \theta_\varepsilon \left(\frac{\partial u_0}{\partial \nu} + b u_0 \right) + \varepsilon b N_j^\varepsilon \frac{\partial u_0}{\partial x_j} .$$

The function $g_{2,\varepsilon}$ is defined as a weak solution in the space $H_1(\Omega_\varepsilon)$ of the problem

$$(41)\qquad \begin{cases} \Delta g_{2,\varepsilon} = 0 \quad \text{in } \Omega^- \cup \Omega_\varepsilon^+ , \\ \dfrac{\partial g_{2,\varepsilon}}{\partial \nu} + b g_{2,\varepsilon} = 0 \quad \text{on } S_\varepsilon , \\ g_{2,\varepsilon} = \varepsilon N_j^\varepsilon \dfrac{\partial u_0}{\partial x_j} \varphi_\varepsilon \quad \text{on } \Gamma_\varepsilon . \end{cases}$$

Now we will obtain estimates for $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$. For this we represent the right hand side of (40) in the form

$$F_\varepsilon^+ = \sum_{i=1}^{4} f_\varepsilon^i ,$$

where

$$f_{1,\varepsilon} = \varphi_\varepsilon \Bigg[A_\varepsilon (a_\varepsilon^{1-n}\varepsilon^n - C_0) u_0 + 2(\nabla_x \theta_x, \nabla_x u_0) +$$
$$+ B_x (a_\varepsilon \varepsilon^{-1})^n u_0 + \theta_\varepsilon \Delta u_0 + \frac{\partial N_j^\varepsilon}{\partial y_p} \frac{\partial^2 u_0}{\partial x_j \partial x_p} \Bigg] ,$$

$$f_{2,\varepsilon} = \dot{\varphi}_\varepsilon \left(2 \frac{\partial \theta_\varepsilon}{\partial x_1} u_0 + 2\theta_\varepsilon \frac{\partial u_0}{\partial x_1} + 2 \frac{\partial N_j^\varepsilon}{\partial y_1} \frac{\partial u_0}{\partial x_j} + \varepsilon N_j^\varepsilon \frac{\partial^2 u_0}{\partial x_j \partial x_1} \right) ,$$

$$f_{3,\varepsilon} = \ddot{\varphi}_\varepsilon \left(\theta_\varepsilon u_0 + \varepsilon N_j^\varepsilon \frac{\partial u_0}{\partial x_j} \right),$$

$$f_{4,\varepsilon} = \mu_0 (\varphi_\varepsilon - 1) u_0 .$$

From Lemma 6 and Lemma 7 we have

$$\|f_{1,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{53} [(a_\varepsilon^{1-n} - C_0) + (a_\varepsilon \varepsilon^{-1})^{n/2}], \tag{42}$$

if $n \geqslant 3$, and

$$\|f_{1,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{54} [(a_\varepsilon^{1-n} \varepsilon^2 - C_0) + a_\varepsilon \varepsilon^{-1} \sqrt{\ln(\varepsilon/2a_\varepsilon)}], \tag{43}$$

if $n = 2$.

Here the smoothness of the function u_0 is used.

We set

$$\Pi_\varepsilon = \Omega_\varepsilon^+ \cap \{x \in R_x^n : a_0 \varepsilon < x_1 < 2a_0 \varepsilon\} .$$

It is easy to see that

$$\begin{aligned} \|f_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} &= \|f_{2,\varepsilon}\|_{L_2(\Pi_\varepsilon)} \leqslant \\ &\leqslant K_{55} \varepsilon^{-1} \left[\|\nabla_x \theta_\varepsilon\|_{L_2(\Pi_\varepsilon)} + \|\theta_\varepsilon\|_{L_2(\Pi_\varepsilon)} + \sum_{j=1}^n (\|\nabla_y N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} + \varepsilon \|N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)}) \right]. \end{aligned} \tag{44}$$

Using estimates (36)-(39) we obtain the following inequalities

$$\begin{cases} \|\nabla_x \theta_\varepsilon\|_{L_2(\Pi_\varepsilon)} \leqslant K_{56} \sqrt{\varepsilon} (a_\varepsilon \varepsilon^{-1})^{n/2} , \\ \|\theta\|_{L_2(\Pi_\varepsilon)} \leqslant K_{57} \varepsilon \sqrt{\varepsilon} (a_\varepsilon \varepsilon^{-1})^{n/2} , \\ \|N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} \leqslant K_{58} \varepsilon^{1/2} (a_\varepsilon \varepsilon^{-1})^{n/2} , \end{cases} \tag{45}$$

if $n \geqslant 3$, and

$$\begin{cases} \|\nabla_x \theta\|_{L_2(\Pi_\varepsilon)} \leqslant K_{59} a_\varepsilon \varepsilon^{-1/2} \sqrt{\ln(\varepsilon/2a_\varepsilon)} , \\ \|\theta_\varepsilon\|_{L_2(\Pi_\varepsilon)} \leqslant K_{60} a_\varepsilon \sqrt{\varepsilon \ln(\varepsilon/2a_\varepsilon)} , \\ \|N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} \leqslant K_{61} \dfrac{a_\varepsilon}{\sqrt{\varepsilon}} \sqrt{\ln(\varepsilon/2a_\varepsilon)} , \end{cases} \tag{46}$$

if $n = 2$.

From estimates (44)-(46) we deduce

$$\begin{cases} \|f_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{62} \varepsilon^{-1/2} (a_\varepsilon \varepsilon^{-1})^{n/2} , & \text{if } n \geqslant 3 , \quad \text{and} \\ \|f_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{63} a_\varepsilon \varepsilon^{-3/2} \sqrt{\ln(\varepsilon/2a_\varepsilon)} , & \text{if } n = 2 . \end{cases} \tag{47}$$

Taking into account that $a_\varepsilon^{n/2} \varepsilon^{-n/2} / C_0^{-1/2} a_\varepsilon^{1/2} \to 1$ as $\varepsilon \to 0$ and therefore $a_\varepsilon^{n/2} \varepsilon^{-(n+1)/2} / C_0^{-1/2} (a_\varepsilon \varepsilon^{-1})^{1/2} \to 1$ as $\varepsilon \to 0$, we conclude that the right-hand sides in inequalities (47) tend to zero as $\varepsilon \to 0$.

Thus we have

$$(48)\qquad \begin{cases} \|f_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{64}\sqrt{a_\varepsilon \varepsilon^{-1}}\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|f_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{65}\sqrt{\dfrac{a_\varepsilon}{\varepsilon}\ln(\varepsilon/2a_\varepsilon)}\,, & \text{if } n = 2\,. \end{cases}$$

Similarly we get the following estimates

$$\|f_{3,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{66}\,\varepsilon^{-1/2}(a_\varepsilon \varepsilon^{-1})^{n/2}\,,$$

if $n \geqslant 3$, and

$$\|f_{3,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{67}\,a_\varepsilon\,\varepsilon^{-3/2}\sqrt{\ln(\varepsilon/2a_\varepsilon)}\,,$$

if $n = 2$.

Therefore we have

$$(49)\qquad \begin{cases} \|f_{3,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{68}\sqrt{a_\varepsilon \varepsilon^{-1}}\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|f_{3,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{69}\sqrt{(a_\varepsilon/\varepsilon)\ln(\varepsilon/2a_\varepsilon)}\,, & \text{if } n = 2\,. \end{cases}$$

Taking into account the definition of the function φ_ε we obtain the following estimate

$$(50)\qquad \|f_{4,\varepsilon}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{70}\sqrt{\varepsilon}\,.$$

From estimates (42), (43) and (47)-(50) we deduce that

$$(51)\qquad \begin{cases} \|F_\varepsilon^+\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{71}[(a_\varepsilon^{1-n}\varepsilon^n - C_0) + \sqrt{a_\varepsilon \varepsilon^{-1}}]\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|F_\varepsilon^+\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{72}[(a_\varepsilon^{-1}\varepsilon^2 - C_0) + \sqrt{a_\varepsilon \varepsilon^{-1}\ln(\varepsilon/2a_\varepsilon)}]\,, & \text{if } n = 2\,. \end{cases}$$

From Lemma 7 we derive

$$(52)\qquad \begin{cases} \|F_{\varepsilon,4}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{73}\,\varepsilon(a_\varepsilon \varepsilon^{-1})^{n/2}\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|F_{\varepsilon,4}\|_{L_2(\Omega_\varepsilon^+)} \leqslant K_{74}\,a_\varepsilon\sqrt{\ln(\varepsilon/2a_\varepsilon)}\,, & \text{if } n = 2\,. \end{cases}$$

From Lemma 2 and Lemma 7 we obtain the following inequalities

$$(53)\qquad \begin{cases} \|N_j^\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{75}(a_\varepsilon \varepsilon^{-1})^{n/2}\sqrt{a_\varepsilon}\,\varepsilon^{-1}\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|N_j^\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{76}\dfrac{a_\varepsilon}{\varepsilon^2}\sqrt{a_\varepsilon}\ln(\varepsilon/2a_\varepsilon)\,, & \text{if } n = 2\,. \end{cases}$$

From the definition of the κ_ε we obtain the estimate

$$(54)\qquad \|\kappa_\varepsilon\|_{L_2(S_\varepsilon)} \leqslant K_{77}\left(\|\theta_\varepsilon\|_{L_2(S_\varepsilon)} + \varepsilon\sum_{j=1}^{n}\|N_j^\varepsilon\|_{L_2(S_\varepsilon)}\right).$$

Therefore, from inequality (54) and estimates (37), (53) we deduce

$$
(55)\qquad \begin{cases} \|\kappa_\varepsilon\|_{L_2(S_\varepsilon)} \leq K_{78} \sqrt{a_\varepsilon}\,(a_\varepsilon \varepsilon^{-1})^{n/2}, \quad \text{if } n \geq 3, \quad \text{and} \\ \|\kappa_\varepsilon\|_{L_2(S_\varepsilon)} \leq K_{79} \dfrac{a_\varepsilon \sqrt{a_\varepsilon}}{\varepsilon} \ln(\varepsilon/2a_\varepsilon), \quad \text{if } n = 2. \end{cases}
$$

Using the integral identity for a weak solution of space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ of problem (40) we get the equality

$$
(56)\qquad \int_{\Omega_\varepsilon} |\nabla_x g_{1,\varepsilon}|^2\, dx + b \int_{S_\varepsilon} g_{1,\varepsilon}^2\, ds_x =
$$

$$
= -\int_{\Omega_\varepsilon} F_\varepsilon^+ g_{1,\varepsilon}\, dx + \sum_{k=1}^{n} \int_{\Omega_\varepsilon^+} F_{\varepsilon,k} \frac{\partial g_{1,\varepsilon}}{\partial x_k}\, dx + \int_{S_\varepsilon} \kappa_\varepsilon g_{1,\varepsilon}\, ds_x .
$$

Using Friedrichs inequality for functions of space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ equality (56) and also the elementary inequality $ab \leq \delta a^2 + \delta^{-1} b^2$, $(a, b, \delta > 0)$ we obtain the estimate

$$
(57)\qquad \int_{\Omega_\varepsilon} |\nabla_x g_{1,\varepsilon}|^2\, dx + \int_{S_\varepsilon} g_{1,\varepsilon}^2\, ds_x \leq K_{80} \left(\|F_\varepsilon^+\|^2_{L_2(\Omega_\varepsilon^+)} + \sum_{k=1}^{n} \|F_{\varepsilon,k}\|^2_{L_2(\Omega_\varepsilon^+)} + \|\kappa_\varepsilon\|^2_{L_2(S_\varepsilon)} \right).
$$

Therefore, from inequalities (51), (52), (55), (57) we conclude that

$$
(58)\qquad \begin{cases} \|g_{1,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leq K_{81} [(a_\varepsilon^{1-n} \varepsilon^n - C_0) + \sqrt{a_\varepsilon \varepsilon^{-1}}], \quad \text{if } n \geq 3, \quad \text{and} \\ \|g_{1,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leq K_{82} \left[(a_\varepsilon^{-1} \varepsilon^2 - C_0) + \sqrt{\dfrac{a_\varepsilon}{\varepsilon} \ln(\varepsilon/2a_\varepsilon)} \right], \quad \text{if } n = 2. \end{cases}
$$

Thus we have

LEMMA 8. Let $g_{1,\varepsilon}$ be a weak solution of problem (40) and $a_\varepsilon^{1-n} \varepsilon^n \to C_0$ as $\varepsilon \to 0$, $C_0 = \text{const} > 0$. Then for $g_{1,\varepsilon}$ estimates (58) are valid.

Now we obtain the estimate for the solution of problem (41). We set

$$
V_{2,\varepsilon} = g_{2,\varepsilon} - \varepsilon \varphi_\varepsilon N_j^\varepsilon \frac{\partial u_0}{\partial x_j} .
$$

Then it is easy to see that $V_{2,\varepsilon} \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ and $V_{2,\varepsilon}$ is a weak solution of the problem

$$
(59)\qquad \begin{cases} \Delta V_{2,\varepsilon} = -\varepsilon \Delta \left(\varphi_\varepsilon N_j^\varepsilon \dfrac{\partial u_0}{\partial x_j} \right) \quad \text{in } \Omega_\varepsilon, \\ V_{2,\varepsilon} = 0 \quad \text{on } \Gamma_\varepsilon, \\ \dfrac{\partial V_{2,\varepsilon}}{\partial \nu} + b V_{2,\varepsilon} = -\varepsilon \dfrac{\partial}{\partial \nu} \left(N_j^\varepsilon \dfrac{\partial u_0}{\partial x_j} \right) - b \varepsilon N_j^\varepsilon \dfrac{\partial u_0}{\partial x_j} \quad \text{on } S_\varepsilon . \end{cases}
$$

From the integral identity for problem (59) we deduce the equality

$$(60)\qquad \int_{\Omega_\varepsilon} |\nabla_x V_{2,\varepsilon}|^2\, dx + b\int_{S_\varepsilon} V_{2,\varepsilon}^{\;2}\, ds_x =$$

$$= -\varepsilon \int_{\Omega_\varepsilon^+} \left(\nabla_x\left(\varphi_\varepsilon N_j^\varepsilon \frac{\partial u_0}{\partial x_j}\right), \nabla_x V_{2,\varepsilon}\right) dx - b\varepsilon \int_{S_\varepsilon} N_j^\varepsilon V_{2,\varepsilon} \frac{\partial u_0}{\partial x_j}\, ds_x\,.$$

It is easy to see that from equality (60) one can get inequalities

$$(61)\qquad \|\nabla_x V_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon)} + \|V_{2,\varepsilon}\|_{L_2(S_\varepsilon)} \leqslant$$

$$\leqslant K_{83}\,\varepsilon \sum_{j=1}^{n} \left\{\left\|\nabla_\varepsilon\left(\varphi_\varepsilon N_j^\varepsilon \frac{\partial u_0}{\partial x_j}\right)\right\|_{L_2(\Omega_\varepsilon^+)} + \|N_j^\varepsilon\|_{L_2(S_\varepsilon)}\right\} \leqslant$$

$$\leqslant K_{84}\,\varepsilon \sum_{j=1}^{n} \{\varepsilon^{-1}\|N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} + \varepsilon^{-1}\|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \|N_j^\varepsilon\|_{L_2(S_\varepsilon)}\} =$$

$$= K_{84} \sum_{j=1}^{n} \{\|N_j^\varepsilon\|_{L_2(\Pi_\varepsilon)} + \|\nabla_y N_j^\varepsilon\|_{L_2(\Omega_\varepsilon^+)} + \varepsilon\|N_j^\varepsilon\|_{L_2(S_\varepsilon)}\}\,.$$

From estimates (38), (39), (45), (46), (61) and the Friedrichs inequality for $V_{2,\varepsilon}$ we obtain

$$\|V_{2,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leqslant K_{85}\,(a_\varepsilon\,\varepsilon^{-1})^{n/2}\,,$$

if $n \geqslant 3$, and

$$\|V_{2,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leqslant K_{86}\,\frac{a_\varepsilon}{\varepsilon}\sqrt{\ln(\varepsilon/2a_\varepsilon)}\,,$$

if $n = 2$.

From these estimates we deduce that

$$(62)\qquad \begin{cases} \|g_{2,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leqslant K_{87}\,(a_\varepsilon\,\varepsilon^{-1})^{n/2}\,, & \text{if } n \geqslant 3\,, \quad \text{and} \\ \|g_{2,\varepsilon}\|_{H_1(\Omega_\varepsilon)} \leqslant K_{88}\,\dfrac{a_\varepsilon}{\varepsilon}\sqrt{\ln(\varepsilon/2a_\varepsilon)}\,, & \text{if } n = 2\,. \end{cases}$$

Thus we have

LEMMA 9. *Let $g_{2,\varepsilon} \in H_1(\Omega_\varepsilon)$ be a weak solution of problem (41) and $a_\varepsilon^{1-n}\varepsilon^n \to C_0$ as $\varepsilon \to 0$, $C_0 = \text{const} > 0$. Then estimates (62) are valid.*

THEOREM 3. *Let u_ε be a weak solution of problem (1), $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$, $u_0 \in C^2(\overline{\Omega^-})$, $u_0 \in C^2(\overline{\Omega^+})$ be the solution of problem (35) and let $a_\varepsilon^{1-n}\varepsilon^n \to C_0$ as $\varepsilon \to 0$, $C_0 = \text{const}$. Then*

$$\|u_\varepsilon - u_0\|_{H_1(\Omega_\varepsilon)} \leqslant K_{89}\{(a_\varepsilon^{1-n}\varepsilon^n - C_0) + \sqrt{a_\varepsilon\,\varepsilon^{-1}}\}\,,$$

if $n \geq 3$, and

$$\|u_\varepsilon - u_0\|_{H_1(\Omega_\varepsilon)} \leq K_{90}\left\{(a_\varepsilon^{-1}\varepsilon^2 - C_0) + \sqrt{\frac{a_\varepsilon}{\varepsilon}\ln(a_\varepsilon/2a_\varepsilon)}\right\}$$

if $n = 2$.

5. – The spectral problem, corresponding to the boundary-value problem (1) can be considered in the same way as in [4,5], using the theorem from [13,25] about the spectrum of a sequence of singularly perturbed operators.

On the basis of Theorem 1 we have

THEOREM 4. Let $\{\lambda_\varepsilon^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$(63)\qquad \begin{cases} \Delta u_\varepsilon^m + \lambda_\varepsilon^m u_\varepsilon^m = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon^m}{\partial \nu} + b u_\varepsilon^m = 0 \quad \text{on } S_\varepsilon, & u_\varepsilon^m = 0 \quad \text{on } \Gamma_\varepsilon, \end{cases}$$

where $a_\varepsilon^{1-n}\varepsilon^n \to 0$ as $\varepsilon \to 0$ and let $\{\lambda^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\Delta u^m + \lambda^m u^m = 0 \quad \text{in } \Omega^-, \qquad u^m = 0 \quad \text{on } \Omega^-,$$

and every eigenvalue is counted as many times as its multiplicity. Then

$$\left|\frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m}\right| \leq C_1\sqrt{M(\varepsilon, n)},$$

where $M(\varepsilon, n) = a_\varepsilon^{(1-n)/2}\varepsilon^{n/2}$, C_1 is a *constant* independent of ε.

From Theorem 2 we obtain

THEOREM 5. Let $\{\lambda_\varepsilon^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem (63) and let $a_\varepsilon^{1-n}\varepsilon^n \to +\infty$ as $\varepsilon \to 0$, $\{\lambda^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\Delta u^m + \lambda^m u^m = 0 \quad \text{in } \Omega, \qquad u^m = 0 \quad \text{on } \partial\Omega,$$

and every eigenvalue is counted as many times as its multiplicity. Then

$$\left|\frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m}\right| \leq C_2\{\sqrt{a_\varepsilon} + [M(\varepsilon, n)]^{-1}\},$$

if $n \geq 3$, and

$$\left|\frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m}\right| \leq C_3\{\sqrt{a_\varepsilon \ln(\varepsilon/2a_\varepsilon)} + [M(\varepsilon, n)]^{-1}\},$$

if $n = 2$, where $M(\varepsilon, n)$ was defined in Theorem 4, and C_2, C_3 are *constants* independent of ε.

On the basis of Theorem 3 we have

THEOREM 6. Let $\{\lambda_\varepsilon^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem (63) and let $a_\varepsilon^{1-n}\varepsilon^n \to C_0$ as $\varepsilon \to 0$, $C_0 = \text{Const} > 0$, $\{\lambda^m\}$ be a nondecreasing sequence of eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta u^m + \lambda^m u^m = 0 & \text{in } \Omega^-, \\ \Delta u^m + \mu_0 u^m + \lambda^m u^m = 0 & \text{in } \Omega^+, \\ u^m = 0 & \text{on } \partial\Omega, \end{cases}$$

and every eigenvalue is counted as many times as its multiplicity. Then

$$\left| \frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m} \right| \leq C_4 \{ (a_\varepsilon^{1-n} \varepsilon^n - C_0) + \sqrt{a_\varepsilon \varepsilon^{-1}} \},$$

if $n \geq 3$, and

$$\left| \frac{1}{\lambda_\varepsilon^m} - \frac{1}{\lambda^m} \right| \leq C_5 \left\{ (a_\varepsilon^{-1} \varepsilon^2 - C_0) + \sqrt{\frac{a_\varepsilon}{\varepsilon} \ln(\varepsilon/2a_\varepsilon)} \right\},$$

if $n = 2$.

Appendix

Proof of Lemma 3. Let us extend the function $u(x)$ for $x \in R^n \setminus \Omega$ setting $u = 0$ in $R^n \setminus \Omega$. It is easy to see that such a function $u \in H_1(R^n \setminus G_\varepsilon)$. Consider the cell Y_ε. For simplicity we assume that G_0 is a ball with radius $\varrho < 1$ whose center coincides with the center of Q, $\varepsilon(1 - 1/\sqrt{2}) > a_\varepsilon \varrho$. Then the function u is defined in $T_{\varepsilon/\sqrt{2}} \setminus a_\varepsilon G_0$, where T_σ is the ball of radius σ with its center coinciding with the center of εQ. Let $P \in a_\varepsilon S_0$, $\bar{P} \in r S_1$, $a_\varepsilon \varrho < r \leq \varepsilon/\sqrt{2}$ and P, $\bar{P}$ lie on the same radius-vector. Then for $n \geq 3$ we have

$$(64) \quad u^2(\bar{P}) \leq 2u^2(P) + 2 \int_{a_\varepsilon \varrho}^{\varepsilon/\sqrt{2}} r^{1-n}\, dr \int_{a_\varepsilon \varrho}^{\varepsilon/\sqrt{2}} \left| \frac{\partial u}{\partial r} \right|^2 r^{n-1}\, dr \leq$$

$$\leq 2u^2(P) + \frac{2(a_\varepsilon \varrho)^{2-n}}{n-2} \int_{a_\varepsilon \varrho}^{\varepsilon/\sqrt{2}} \left| \frac{\partial u}{\partial r} \right|^2 r^{n-1}\, dr.$$

Multiplying (64) by $J|_{r = a_\varepsilon \varrho} = a_\varepsilon^{n-1} \varrho^{n-1} \Phi(\phi_1, \ldots, \phi_{n-1})$, where $J = r^{n-1} \Phi(\phi_1, \ldots, \phi_{n-1})$ is the Jacobian for the spherical coordinates, and integrating it with respect to $\phi_1, \ldots, \phi_{n-1}$, we obtain

$$(65) \quad a_\varepsilon^{n-1} \varrho^{n-1} \int_{S_1} u^2(\bar{P})\, d\phi_1 \ldots d\phi_{n-1} \leq 2 \int_{a_\varepsilon S_0} u^2(P)\, ds_x +$$

$$+ 2a_\varepsilon \varrho \int_{T_{\varepsilon/\sqrt{2}} \setminus T_{a_\varepsilon \varrho}} \left| \frac{\partial u}{\partial r} \right|^2 r^{n-1} \Phi(\phi_1, \ldots, \phi_{n-1})\, dr\, d\phi_1 \ldots d\phi_{n-1},$$

where S_1 is a sphere of radius 1.

Then multiplying both sides of inequality (65) by r^{n-1} and integrating it with respect to $\overline{P}$ over $r \in (a_\varepsilon \varrho, \varepsilon/\sqrt{2})$, we deduce the estimate

$$a_\varepsilon^{n-1} \varrho^{n-1} \int_{T_{\varepsilon/\sqrt{2}} \setminus T_{a_\varepsilon \varrho}} u^2 \, dx \leq K_0 \left\{ \varepsilon^n \int_{a_\varepsilon S_0} u^2 \, ds_x + a_\varepsilon \varepsilon^n \int_{T_{\varepsilon/\sqrt{2}} \setminus T_{a_\varepsilon \varrho}} |\nabla_x u|^2 \, dx \right\}.$$

From that inequality we conclude

$$\|u\|^2_{L_2(T_{\varepsilon/\sqrt{2}} \setminus T_{a_\varepsilon \varrho})} \leq \overline{K} \{ a_\varepsilon^{1-n} \varepsilon^n \|u\|^2_{L_2(a_\varepsilon S_0)} + a_\varepsilon^{2-n} \varepsilon^n \|\nabla_x u\|^2_{L_2(T_{\varepsilon/\sqrt{2}} \setminus T_{a_\varepsilon \varrho})} \}. \tag{66}$$

Thus, we have an estimate of the form (7) for cell Y_ε. In the same way we can get an estimate of all form (7) for any cell $Y_\varepsilon + \varepsilon z$ (z is a vector with integer components). Summing up the inequalities of the form (66) over all cells of the form $Y_\varepsilon + \varepsilon z$, we get (7). In a similar way we can get estimate (8).

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Department of Mechanics and Mathematics
Moscow State University
119899 MOSCA (Russia)