Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Philippe Clément, Giuseppe Da Prato

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **7** (1996), n.3, p. 147–153.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1996_9_7_3_147_0>

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Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1996.

Rend. Mat. Acc. Lincei
s. 9, v. 7:147-153 (1996)

Analisi matematica. — *Some results on stochastic convolutions arising in Volterra equations perturbed by noise.* Nota di PHILIPPE CLÉMENT e GIUSEPPE DA PRATO, presentata (*) dal Corrisp. G. Da Prato.

ABSTRACT. — Regularity of stochastic convolutions corresponding to a Volterra equation, perturbed by a white noise, is studied. Under suitable assumptions, hölderianity of the corresponding trajectories is proved.

KEY WORDS: Stochastic convolution; Volterra equations; Completely positive kernels.

RIASSUNTO. — *Sulle convoluzioni stocastiche relative a equazioni di Volterra perturbate da un rumore.* Si dimostra la regolarità della convoluzione stocastica relativa a equazioni di Volterra perturbate da un rumore bianco. Sotto opportune ipotesi viene provata l'hölderianità delle traiettorie corrispondenti.

1. INTRODUCTION

Let H be a separable Hilbert space and let $\{e_k\}$ be a complete orthonormal system in H. We are concerned with a stochastic version of a linear Volterra equation in H of the general form:

$$u(t) = \int_0^t a(t-\tau)Au(\tau)\,d\tau + x + g(t), \qquad x \in H, \tag{1.1}$$

where A is a linear operator in H, a is a locally integrable kernel, and g is an H-valued mapping.

This equation has been treated by many people in connection with applications to problems in mathematical physics, such as viscoelasticity and heat conduction in materials with memory. We refer to J. Prüss [3] for a recent survey.

We shall assume that problem (1.1) is well posed, and we shall denote by $S(t)$, $t \geq 0$ the corresponding resolvent operator. We recall that $S(\cdot)x$ is the solution to (1.1) corresponding to $g = 0$.

In order to take into account random fluctuations, it is natural to consider equation (1.1) with a very irregular exterior force: $g(t) = W(t)$, where W is a cylindrical Wiener process, or *white noise*, defined in a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We shall take W of the form

$$\langle W(t), h\rangle = \sum_{k=1}^{\infty} \langle h, e_k\rangle \beta_k(t), \qquad h \in H, \tag{1.2}$$

where $\{\beta_k\}$ is a sequence of real valued, mutually independent, Wiener processes.

For as the kernel a is concerned, we shall assume, following Ph. Clément and J. A. Nohel [1], that a is completely positive, since completely positive kernels naturally arise

(*) Nella seduta del 19 aprile 1996.

in the applications, see [3]. We recall that $a \in L^1_{loc}(0, +\infty)$ is said to be *completely positive* if the solution $s(\alpha, \cdot)$, $\alpha > 0$ to the integral equation

$$s(t) + \alpha \int_0^t a(t-\tau)s(\tau)\,d\tau = 1\,, \qquad t \geqslant 0\,, \tag{1.3}$$

is nonnegative and nonincreasing for any $\alpha > 0$.

Thus, we arrive to the problem

$$X(t) = \int_0^t a(t-\tau)AX(\tau)\,d\tau + x + W(t)\,, \qquad x \in H\,. \tag{1.4}$$

If $a \in W^{1,1}_{loc}(0, +\infty)$, we can write problem (1.4) as an integrodifferential equation

$$dX(t) = \left[a(0)AX(t) + \int_0^t a'(t-\tau)AX(\tau)\,d\tau\right]dt + dW(t)\,, \qquad X(0) = x \in H\,. \tag{1.5}$$

For the sake of simplicity we shall assume that A is self-adjoint, negative, and diagonal with respect to the basis $\{e_k\}$:

$$Ae_k = -\mu_k e_k\,, \qquad \mu_k > 0\,, \qquad k \in \mathbf{N}\,. \tag{1.6}$$

In this paper we will try to extend the semigroup approach of G. Da Prato and J. Zabczyk [2] to problem (1.4). By definition, a *mild* solution of (1.4) is a process $X(t)$, $t \geqslant 0$, adapted to the filtration $\mathcal{F}_t$, $t \geqslant 0$, such that

$$X(t) = S(t)x + \int_0^t S(t-\tau)\,dW(\tau)\,. \tag{1.7}$$

In Section 2, we shall give sufficient conditions in order that this formula be meaningful. Sections 3 and 4 are devoted to prove regularity properties of the stochastic convolution

$$W_{A,a}(t) = \int_0^t S(t-\tau)\,dW(\tau)\,. \tag{1.8}$$

We notice that, as it was shown in [2] in the case when $a = 1$, these regularity properties are important to solve nonlinear equations as for instance

$$X(t) = \int_0^t a(t-\tau)\big(AX(\tau) + F(X(\tau))\big)\,d\tau + x + W(t)\,, \qquad x \in H\,, \tag{1.9}$$

where $F\colon H \to H$ is a locally Lipschitz continuous mapping.

Applications of our results to linear and nonlinear heat equations with memory, will be the object of a future paper.

2. Stochastic convolution

We shall assume

Hypothesis 1. *(i) A is a self-adjoint negative operator. Moreover $Ae_k = -\mu_k e_k$, $k \in \mathbf{N}$, for some positive numbers μ_k, $k \in \mathbf{N}$.*

(ii) a is completely positive.

(iii) We have

$$-\operatorname{Tr}(A^{-1}) = \sum_{k=1}^{\infty} (1/\mu_k) < +\infty .$$

Under these assumptions, it is easy to see that there exists the resolvent $S(t)$, $t \geq 0$ of the deterministic equation (1.1), which is determined by

$$S(t)e_k = s(\mu_k, t)e_k, \qquad k \in \mathbf{N}. \tag{2.1}$$

Moreover, the stochastic convolution $W_{A,a}$ is given formally by

$$W_{A,a}(t) = \sum_{k=1}^{\infty} \int_0^t s(\mu_k, t-\tau)e_k \, d\beta_k(\tau), \tag{2.2}$$

the integrals being intended in the Ito's sense. In order to prove that the above series is convergent in $L^2(\Omega)$, we first need a lemma.

Lemma 2.1. *Under Hypothesis 1 we have*

$$\sum_{k=1}^{\infty} \int_0^t s^2(\mu_k, \tau)\, d\tau < +\infty, \qquad \text{for all } t > 0, \tag{2.3}$$

where $s(\mu_k, \cdot)$ is the solution to (1.3) with $\alpha = \mu_k$.

Proof. A locally integrable function a is completely positive if and only if there exist $\kappa_0 \geq 0$ and $\kappa_1 \in L^1_{\mathrm{loc}}(0, +\infty)$, nonnegative nonincreasing, satisfying

$$\kappa_0 a(t) + (\kappa_1 * a)(t) = 1, \qquad t > 0, \tag{2.4}$$

where $*$ represents the convolution product. The pair (κ_0, κ_1) is uniquely determined by a, see [3]. Moreover the operator L in $L^1(0, T)$ defined by

$$\begin{cases} D(L) := \{u \in L^1(0,T) \colon \kappa_0 u + \kappa_1 * u \in W^{1,1}(0,T), (\kappa_0 u + \kappa_1 * u)(0) = 0\}, \\ Lu := \dfrac{d}{dt}(\kappa_0 u + \kappa_1 * u), \end{cases} \tag{2.5}$$

is m-accretive and densely defined in $L^1(0, T)$. Denoting by $w = w(\varrho, t)$ the function

$$w(\varrho, t) = (e^{-\varrho L}(1))(t), \qquad \varrho \geq 0, \qquad t \in [0, T], \tag{2.6}$$

one can show that $0 \leq w(\varrho, t) \leq 1$, for all $\varrho \geq 0$, $t \in [0, T]$, and

$$\int_0^T s(\mu_k, t)\, dt = \int_0^{+\infty} e^{-\varrho \mu_k}(\kappa_0 w(\varrho, T) + (\kappa_1 * w)(T))\, d\varrho . \tag{2.7}$$

We now can prove (2.3). Since $s(t, \mu) \in [0, 1]$, we have

$$\int_0^T s(\mu_k, t)^2\, dt \leq \int_0^T s(\mu_k, t)\, dt,$$

from which, recalling (2.7)

$$\int_0^T s(\mu_k, t)^2\, dt \leq \int_0^{+\infty} e^{-\varrho\mu_k}(\kappa_0 w(\varrho, T) + (\kappa_1 * w)(T))\, d\varrho\,.$$

Since $w(\varrho, t) \in [0, 1]$ we obtain

$$\int_0^T s(\mu_k, t)^2\, dt \leq \int_0^{+\infty} e^{-\varrho\mu_k}\left(\kappa_0 + \int_0^T \kappa_1(s)\, ds\right) d\varrho = (1/\mu_k)\left(\kappa_0 + \int_0^T \kappa_1(s)\, ds\right),$$

and the conclusion follows summing up on k and recalling Hypothesis 1-(*iii*). ■

Now we prove the main result of this section.

THEOREM 2.2. *Assume that Hypothesis* 1 *holds. Then for any* $t \geq 0$ *the series*:

$$\sum_{k=1}^{\infty} \int_0^t s(\mu_k, t - \tau)\, e_k\, d\beta_k(\tau),$$

is convergent in $L^2(\Omega)$ *to a Gaussian random variable* $W_{A,a}(t)$ *with mean* 0 *and covariance operator* Q_t *determined by*

$$Q_t e_k = \int_0^t s^2(\mu_k, \tau)\, d\tau\, e_k\,, \qquad k \in \mathbf{N}\,. \tag{2.8}$$

PROOF. Set

$$W_{A,a}^n(t) = \sum_{k=1}^{n} \int_0^t s(\mu_k, t - \tau)\, e_k\, d\beta_k(\tau), \qquad n \in \mathbf{N}\,.$$

If $n, p \in \mathbf{N}$ we have

$$|W_{A,a}^{n+p}(t) - W_{A,a}^n(t)|^2 = \sum_{k=n+1}^{n+p} \left| \int_0^t s(\mu_k, t - \tau)\, d\beta_k(\tau) \right|^2.$$

Taking expectation, we find

$$\mathbf{E}(|W_{A,a}^{n+p}(t) - W_{A,a}^n(t)|^2) = \sum_{k=n+1}^{n+p} \int_0^t s^2(\mu_k, t - \tau)\, d\tau\,.$$

In view of Lemma 2.1, this implies that the sequence $\{W_{A,a}^n(t)\}$ is convergent in $L^2(\Omega)$ to a random variable $W_{A,a}(t)$. Moreover $W_{A,a}(t)$ is Gaussian since $W_{A,a}^n(t)$ is. Also we have $\mathbf{E}(W_{A,a}(t)) = 0$ since $\mathbf{E}(W_{A,a}^n(t)) = 0$.

It remains to compute the covariance $\mathrm{Cov}\, W_{A,a}(t)$ of $W_{A,a}(t)$. Since, for any $x, y \in H$,

$$\langle \mathrm{Cov}\,(W_{A,a}(t))\cdot x, y\rangle = \boldsymbol{E}(\langle W_{A,a}(t), x\rangle\langle W_{A,a}(t), y\rangle) = \sum_{k=1}^{\infty} \int_0^t s^2(\mu_k, \tau)\, d\tau \langle x, e_k\rangle\langle y, e_k\rangle,$$

the conclusion follows. ■

EXAMPLE 2.3. Assume that $H = L^2(0, 1)$, and set

$$Au = D^2 u, \qquad \forall u \in H^2(0, 1) \cap H_0^1(0, 1). \tag{2.9}$$

Then Hypothesis 1-(i) holds with

$$e_k(\xi) = \sqrt{2/\pi}\, \sin k\xi, \qquad \xi \in [0, 1], \qquad k \in \mathbf{N},$$

and $\mu_k = \pi^2 k^2$, $k \in \mathbf{N}$. Let moreover

$$a(t) = e^{-t}, \qquad t \geq 0, \tag{2.10}$$

then one has, as easily checked,

$$s(\mu, t) = (1+\mu)^{-1}[1 + \mu e^{-(1+\mu)t}], \qquad t, \mu > 0. \tag{2.11}$$

Thus a is completely positive and Hypothesis 1 is fulfilled.

3. HÖLDERIANITY OF THE STOCHASTIC CONVOLUTION

We first prove the result.

PROPOSITION 3.1. *$W_{A,a}$ is mean square continuous.*

PROOF. If $t > \tau > 0$, we have

$$|W_{A,a}(t) - W_{A,a}(\tau)|^2 = |W_{A,a}(t)|^2 - |W_{A,a}(\tau)|^2 - 2\langle W_{A,a}(t) - W_{A,a}(\tau), W_{A,a}(\tau)\rangle.$$

Since $W_{A,a}(t) - W_{A,a}(\tau)$ and $W_{A,a}(\tau)$ are independent, we have

$$\boldsymbol{E}(|W_{A,a}(t) - W_{A,a}(\tau)|^2) = \boldsymbol{E}(|W_{A,a}(t)|^2) - \boldsymbol{E}(|W_{A,a}(\tau)|^2) = \sum_{k=1}^{\infty} \int_\tau^t s^2(\mu_k, \sigma)\, d\sigma, \tag{3.1}$$

and the conclusion follows from (2.3). ■

We now want to prove almost sure hölderianity of $W_{A,a}$. For this we need an additional assumption

HYPOTHESIS 2. *There exists $\theta \in]0, 1[$ and $C_\theta > 0$ such that, for all $0 < \tau < t$ we have*

$$\int_\tau^t s^2(\mu, \sigma)\, d\sigma \leq C_\theta \mu^{\theta - 1} |t - \tau|^\theta, \tag{3.2}$$

$$\int_0^\tau [s(\mu, \tau - \sigma) - s(\mu, t - \sigma)]^2\, d\sigma \leq C_\theta \mu^{\theta - 1} |t - \tau|^\theta \tag{3.3}$$

and

$$\sum_{k=1}^{\infty} \mu_k^{\theta - 1} < +\infty. \tag{3.4}$$

REMARK 3.2. If the kernel a is completely positive, inequalities (3.2) and (3.3) of Hypothesis 2 holds for any $\theta \in [0, 1]$.

PROPOSITION 3.3. *Under Hypotheses* 1 *and* 2, *for every positive number* $\alpha < \theta/2$, *the trajectories of* $W_{A,a}$ *are almost surely* α*-Hölder continuous.*

PROOF. By (3.1), taking into account (3.2) and (3.3), we have

$$E(|W_{A,a}(t) - W_{A,a}(\tau)|^2) \leq C_\theta \sum_{k=1}^{\infty} \mu_k^{\theta-1} |t-\tau|^\theta .$$

Since, by Theorem 2.2, $W_{A,a}(t) - W_{A,a}(\tau)$ is Gaussian, then for any $m \in \mathbf{N}$, there exists a constant $C_m > 0$ such that

$$E(|W_{A,a}(t) - W_{A,a}(\tau)|^{2m}) \leq C_m \left[C_\theta \sum_{k=1}^{\infty} \mu_k^{\theta-1} \right]^m |t-\tau|^{m\theta} .$$

Choosing m such that $m\theta > 1$ and applying the Kolmogorov test, see *e.g.* [2], we find that $W_{A,a}$ is α-Hölder continuous for $\alpha = \theta/2 - 1/(2m)$. The conclusion follows from the arbitrariness of m. ■

EXAMPLE 3.4. We use here notation from Example 2.3. We want to check that Hypothesis 2 is fulfilled. Since (3.3) obviously holds, it remains to prove (3.2).

Let $t > \tau > 0$, then from (2.11) we have

$$\int_\tau^t s^2(\mu, \sigma)\, d\sigma = (1+\mu)^{-2} [(t-\tau) + 2\mu(1+\mu)^{-1} (e^{-(1+\mu)\tau} - e^{-(1+\mu)t}) + \\ + \mu^2 \cdot (2(1+\mu))^{-1} (e^{-2(1+\mu)\tau} - e^{-2(1+\mu)t})] .$$

Let C_θ be such that

$$|e^{-\alpha} - e^{-\beta}| \leq C_\theta |\alpha - \beta|^\theta, \qquad \alpha, \beta \geq 0 .$$

Then we have

$$\int_\tau^t s^2(\mu, \sigma)\, d\sigma \leq (1+\mu)^{-2} [(t-\tau) + 2\mu(1+\mu)^{\theta-1} C_\theta |t-\tau|^\theta + \\ + \mu^2 (2(1+\mu)^{1-\theta})^{-1} 2^\theta C_\theta |t-\tau|^\theta] .$$

Thus (3.2) is fulfilled.

4. HÖLDERIANITY OF THE STOCHASTIC CONVOLUTION IN SPACES OF CONTINUOUS FUNCTIONS

We assume here that $H = L^2(\mathcal{O})$, where $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$. We set $W_{A,a}(t)(\xi) = W_{A,a}(t, \xi)$ and write the stochastic convolution as

$$W_{A,a}(t, \xi) = \sum_{k=1}^{\infty} \int_0^t s(\mu_k, t-\tau) e_k(\xi)\, d\beta_k(\tau) . \tag{4.1}$$

We want to prove that $W_{A,a}(t,\xi)$ is Hölder continuous in t and ξ. For this we need an additional hypothesis

HYPOTHESIS 3. *There exists* $M > 0$ *such that*

$$\begin{cases} |e_k(\xi)| \leqslant M, & k \in \mathbf{N}, \quad \xi \in \mathcal{O}, \\ |\nabla e_k(\xi)| \leqslant M\mu_k^{1/2}, & k \in \mathbf{N}, \quad \xi \in \mathcal{O}. \end{cases} \tag{4.2}$$

Note that if Hypothesis 3 holds then, by interpolation, for all $\theta \in]0, 1[$ there exists $M_\theta > 0$ such that

$$|e_k(\xi) - e_k(\eta)| \leqslant M_\theta \mu_k^{\theta/2} |\xi - \eta|^\theta, \quad k \in \mathbf{N}. \tag{4.3}$$

THEOREM 4.1. *Under Hypotheses* 1, 2, *and* 3, *the trajectories of* $W_{A,a}(t,\xi)$ *are almost surely* α*-Hölder continuous in* (t,ξ) *for any* $\alpha \in]0, 1/4[$.

PROOF. We first note that, arguing as in the proof of Lemma 2.1, we find

$$\sum_{k=1}^{\infty} \int_0^t \mu_k^\theta s^2(\mu_k, \sigma)\, d\sigma < +\infty. \tag{4.4}$$

It follows that there exists $N_\theta > 0$ such that

$$|W_{A,a}(t,\xi) - W_{A,a}(t,\eta)| \leqslant N_\theta |\xi - \eta|^\theta, \quad \theta \in]0, 1[. \tag{4.5}$$

Moreover, arguing as in the proof of Proposition 3.1 we find that there exists $N_{1,\theta} > 0$ such that

$$\mathbf{E}|W_{A,a}(t,\xi) - W_{A,a}(\tau,\xi)|^2 \leqslant N_{1,\theta}|t-\tau|^\theta, \quad \theta \in]0, 1[, \quad t, \tau > 0. \tag{4.6}$$

By (4.5) and (4.6) it follows that, for some constant $N_{2,\theta}$

$$\mathbf{E}|W_{A,a}(t,\xi) - W_{A,a}(\tau,\eta)|^2 \leqslant N_{2,\theta}[|\xi-\eta|^2 + |t-\tau|^2]^{\theta/2}, \quad \theta \in]0, 1[. \tag{4.7}$$

By the Kolmogorov's test, see [2], we arrive at the conclusion. ■

REMARK 4.2. It is easy to see that the functions $\{e_k\}$ defined in Example 2.3 fulfill Hypothesis 3. Moreover in Example 2.3, the kernel $a(t) = e^{-t}$ can be replaced for instance by any locally integrable, positive, decreasing and log convex function.

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Ph. Clément: Technical University Delft
DELFT (Olanda)

G. Da Prato: Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 PISA