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Asymptotic analysis of surface waves due to high-frequency disturbances

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Analisi matematica. — *Asymptotic analysis of surface waves due to high-frequency disturbances.* Nota (*) di NIKOLAY KUZNETSOV e VLADIMIR MAZ'YA, presentata dal Socio E. Magenes.

ABSTRACT. — The present paper is devoted to the asymptotic analysis of the linear unsteady surface waves. We study two problems concerned with high-frequency surface and submerged disturbances. The two-scale asymptotic series are obtained for the velocity potential. The principal terms in the asymptotics of some hydrodynamical characteristics of the wave motion (the free surface elevation, the energy, etc.) are described.

KEY WORDS: Surface waves theory; Asymptotic expansions; Cauchy-Poisson problem; Two-scaled asymptotic series.

RIASSUNTO. — *Analisi asintotica delle onde di superficie relative a fenomeni di disturbo di alta frequenza*. Il lavoro è dedicato all'analisi asintotica delle onde lineari di superficie instabile. Si studiano due problemi relativi le superfici di alta frequenza e i relativi fenomeni di disturbo. Le serie asintotiche sono ottenute per il potenziale di velocità. Vengono poi descritti i termini principali nelle parti asintotiche di alcune caratteristiche idrodinamiche del moto delle onde.

1. INTRODUCTION

In the present paper the effect of high-frequency oscillations of surface and submerged disturbances on hydrodynamical characteristics of the wave motion is considered. We study two initial-boundary value problems in the linearized theory of surface waves. By using the dimensionless period of the oscillations as a small parameter, we construct complete asymptotic expansions of the velocity potential. The principal terms in these expansions admit a physical interpretation.

There is an extensive literature on non-stationary linear problems of the surface wave theory (see *e.g.* [6-9 and bibliography cited therein]). In particular, solvability and uniqueness theorems were obtained by Garipov [3], Friedman and Shinbrot [1, 2] and Hamdache [5].

In Section 2 we investigate waves due to the pressure

$$p(x, z, t, \varepsilon) = \kappa(t/\varepsilon)\, \mathcal{P}(x, z)$$

applied to the horizontal free surface ($y = 0$) of a fluid, which is at rest at the initial moment $t = 0$. Here κ is a 1-periodic function. We suppose that the frequency ε^{-1} is high in comparison with an inverse of characteristic time $(g/h)^{1/2}$, where g is the gravity acceleration and h is a characteristic length. We show that the velocity potential ϕ is given by the asymptotic expansion

$$(1.1) \qquad \phi(P, t, \varepsilon) \sim \sum_{m=0}^{\infty} \varepsilon^m \alpha_m w_m(P, t) + \sum_{m=1}^{\infty} \varepsilon^{2m-1} \beta_m(t/\varepsilon)\, v_m(P)$$

(*) Pervenuta all'Accademia il 17 settembre 1996.

where β_m are certain 1-periodic functions, α_m are constants expressed in terms of β_m, and v_m, w_m are harmonic functions, which do not depend on ε. The functions v_m are subjected to the Dirichlet condition on the free surface, and the Neumann condition on the rigid surfaces, whereas w_m are solutions of the Cauchy-Poisson problem. Both sequences can be found recurrently.

Analysis of the principal term in (1.1) shows that up to the magnitude $O(\varepsilon)$ the waves are the same as those, resulting from the initial elevation of the free surface

$$[\langle\kappa\rangle - \kappa(0)]\,\mathcal{P}(x,z).$$

Here $\langle\kappa\rangle$ is the mean value of periodic function. Moreover, if $\langle\kappa\rangle = \kappa(0)$, then the wave pattern is stationary up to a term $O(\varepsilon)$. However, apart of this slow wave motion there is a high-frequency motion of amplitude $O(\varepsilon)$ with zero mean value, which gives a finite contribution to the force-vector applied to a submerged body (see the first term in the second sum in (1.1)).

In Section 3 we consider a pulsating source in the fluid and oscillations of the bottom or of a submerged body. In the case of the source the fluid motion also proves to be composed of two motions up to $O(\varepsilon)$. The principal term of the velocity potential is

$$[\kappa(t/\varepsilon) - \langle\kappa\rangle]\,v_0(P,P_0) + \langle\kappa\rangle w_0(P,P_0,t).$$

Here $v_0(P,P_0)$ is the waveless potential of the stationary unit source at the point P_0, and $\langle\kappa\rangle\, w_0$ is the potential of the wave motion due to the source of mean strength, appearing at $t=0$.

Thus, the force applied to a submerged body is a sum of two components. The first one is the large high-frequency force

$$\varepsilon^{-1}\kappa'(t/\varepsilon)\int_S v_0\,\boldsymbol{n}\,dS.$$

Here S is the body surface, and $\boldsymbol{n}$ is the normal directed into the fluid. The second component of the force, which slowly varies in time, is given by

$$\langle\kappa\rangle\int_S (\partial w_0/\partial t)\,\boldsymbol{n}\,dS.$$

However, the force impulse during the time interval t has the asymptotics:

$$S(t,\varepsilon) = [\kappa(t/\varepsilon) - \langle\kappa\rangle]\int_S v_0\,\boldsymbol{n}\,dS + \langle\kappa\rangle\int_S w_0\,\boldsymbol{n}\,dS + O(\varepsilon).$$

We see that two terms in the formula for the force, which have different orders, give contributions of the same order into $S(t,\varepsilon)$.

For each problem we derive a formal asymptotic series, discuss asymptotic formulae for hydrodynamical characteristics and prove estimates for the remainder term.

We present our results for the three-dimensional case although the same argument and similar asymptotic formulae are true for two-dimensional problems as well.

2. Waves due to a high-frequency surface pressure

2.1. *Statement of the problem.*

Let the fluid occupy a region W, bounded from above by the free surface $F = \partial \boldsymbol{R}^3_-$. The boundary $\partial W \backslash F$ is the union of the bottom B and the surface S of a totally immersed body D. We assume that B and S are disjoint. It is possible that either B or S is empty. The bottom is a smooth surface dividing $\boldsymbol{R}^3_- = \{P = (x, y, z) \colon y < 0\}$ into two unbounded regions and it coincides with the plane $y = -H$ at infinity ($H > 0$). The surface S is smooth, connected and closed. We suppose that if $S \cup B \neq \emptyset$, then W contains a layer $\{P\colon -h < y < 0\}$, $0 < h \leqslant H$.

The problem on waves due to a surface pressure p, applied to the fluid of constant density ϱ, which is initially at rest, is stated as follows. We seek a velocity potential ϕ satisfying

$$\nabla^2 \phi = 0 \quad \text{in } W, \tag{2.1}$$

$$\phi_{tt} + \phi_y = -p_t \quad \text{on } F, \tag{2.2}$$

$$\partial\phi/\partial n = 0 \quad \text{on } S \cup B, \tag{2.3}$$

for $t \geqslant 0$ and

$$\phi = 0, \tag{2.4}$$

$$\phi_t = 0, \tag{2.5}$$

on F when $t = 0$. Here dimensionless variables are introduced, and the characteristic length h, the time $(h/g)^{1/2}$, the pressure $\varrho g h$ and the characteristic value $(gh^3)^{1/2}$ for the velocity potential are used for scaling.

We assume that

$$p(x, z, t, \varepsilon) = \kappa(t/\varepsilon)\, \mathcal{P}(x, z)$$

where κ is a function of unit period, and $\mathcal{P}$ is a sufficiently smooth function decaying at infinity. Our aim is to construct an asymptotic expansion for ϕ, valid when $\varepsilon \ll 1$. (This means that the period of pressure oscillations is small in comparison with the characteristic time $(h/g)^{1/2}$).

2.2. *Formal asymptotic expansion.*

We seek the potential in the form of the two time-scaled asymptotic series:

$$\phi(P, t, \varepsilon) \sim \sum_{m=0}^{\infty} \varepsilon^m [\varphi_m(P, t/\varepsilon) + \psi_m(P, t)] \tag{2.6}$$

where φ_m are 1-periodic functions with respect to the second argument. The functions $\varphi_m(P, \tau)$ and $\psi_m(P, t)$ must decay as $|P| \to \infty$.

By inserting (2.6) into (2.1)-(2.5), and equating the coefficients at the same degrees of ε one obtains for $m = 0, 1, \ldots$

$$\nabla^2(\varphi_m + \psi_m) = 0 \quad \text{in } W, \tag{2.7}$$

$$(\partial^2 \varphi_{m+2}/\partial \tau^2) + (\partial \varphi_m/\partial y) + (\partial^2 \psi_m/\partial t^2) + (\partial \psi_m/\partial y) = 0 \quad \text{on } F, \tag{2.8}$$

$$\partial(\varphi_m + \psi_m)/\partial n = 0 \quad \text{on } S \cup B, \tag{2.9}$$

for $t,\ \tau \geqslant 0$ and

(2.10) $$\varphi_m + \psi_m = 0\,,$$

(2.11) $$(\partial\varphi_{m+1}/\partial\tau) + (\partial\psi_m/\partial t) = 0\,,$$

on F when $t = \tau = 0$.

Furthermore, we get

(2.12) $$\partial^2\varphi_0/\partial\tau^2 = 0 \quad \text{when } \tau \geqslant 0\,, \qquad \partial\varphi_0/\partial\tau = 0 \quad \text{when } \tau = 0 \quad \text{on } F$$

and

(2.13) $$\partial^2\varphi_1/\partial\tau^2 = -\kappa'(\tau)\,\mathcal{P}(x,z) \quad \text{on } F \quad \text{when } \tau \geqslant 0\,.$$

We separately equate the functions, depending on τ and t in (2.7)-(2.9), to zero.

From (2.12) we find

$$\varphi_0 = C_0 = \text{const} \quad \text{on } F \quad \text{for } \tau \geqslant 0\,.$$

Here $C_0 = 0$, since φ_0 vanishes at infinity. Moreover, φ_0 satisfies the Laplace equation in W and the homogeneous Neumann condition on $S \cup B$. Hence,

(2.14) $$\varphi_0 = 0 \quad \text{on } W \quad \text{for } \tau \geqslant 0\,.$$

It follows from (2.8), that

(2.15) $$\partial^2\varphi_m/\partial\tau^2 + \partial\varphi_{m-2}/\partial y = 0 \quad \text{on } F \quad \text{when } \tau \geqslant 0\,.$$

Since φ_m is periodic in τ and decays as $|P| \to \infty$, then by (2.14) and (2.15)

(2.16) $$\varphi_{2k} = 0 \quad \text{on } W \quad \text{when } \tau \geqslant 0\,, \qquad k = 1, 2, \dots\,.$$

Now we seek $\varphi_{2k-1}(P,\tau)$ in the form $\beta_k(\tau)v_k(P)$ for $k \geqslant 1$. By (2.13) we obtain

(2.17) $$\beta_1'' = \kappa' \quad \text{when} \quad \tau \geqslant 0, v_1 = -\mathcal{P} \quad \text{on } F\,.$$

Taking into account (2.7) and (2.9), we arrive at the boundary value problem:

(2.18) $$\begin{cases} \nabla^2 v_1 = 0 \quad \text{in } W\,, \quad v_1 = -\mathcal{P} \quad \text{on } F\,, \\ \partial v_1/\partial n = 0 \quad \text{on } S \cup B\,. \end{cases}$$

The periodic solution of the first equation (2.17) is given by

(2.19) $$\beta_1(\tau) = \int_0^\tau (\kappa(\mu) - \langle\kappa\rangle)\,d\mu + c\,.$$

Here

$$\langle\kappa\rangle = \int_0^1 \kappa(\mu)\,d\mu$$

and c is an arbitrary constant. Another form of the solution is

$$\beta_1(\tau) = (-i/2\pi) \sum_{n \neq 0} (\kappa_n/n) e^{2\pi i n \tau} + c_1 , \tag{2.20}$$

where κ_n are the Fourier coefficients of κ, and c_1 is a constant.

From (2.15) we get for odd $m = 2k - 1$:

$$\beta_k'' = \beta_{k-1} \quad \text{when } \tau \geqslant 0 , \quad v_k = -\partial v_{k-1}/\partial y \quad \text{on } F , \quad k = 2, 3, \ldots . \tag{2.21}$$

Combining the second of these relations with (2.7) and (2.9), we get the boundary value problem:

$$\begin{cases} \nabla^2 v_k = 0 & \text{in } W, \quad \partial v_k/\partial n = 0 \quad \text{on } S \cup B , \\ v_k = -\partial v_{k-1}/\partial y & \text{on } F , \quad k = 2, 3, \ldots . \end{cases} \tag{2.22}$$

In order to obtain a periodic solution of the first equation (2.21) for $k = 2$ one has to put $c_1 = 0$ in (2.20). Then

$$\beta_2(\tau) = i(2\pi)^{-3} \sum_{n \neq 0} (\kappa_n/n^3) e^{2\pi i n \tau} + c_2 ,$$

where c_2 is an arbitrary constant, which vanishes on the next step. Proceeding in the same manner, we find

$$\beta_k(\tau) = i(-1)^k (2\pi)^{1-2k} \sum_{n \neq 0} (\kappa_n/n^{2k-1}) e^{2\pi i n \tau} , \qquad k = 1, 2, \ldots . \tag{2.23}$$

Thus, the formulae (2.14), (2.16), (2.18), (2.22) and (2.23) give a complete description of the first term in the square brackets in (2.6) for all values of m. The second series in (1.1) is obtained.

We put

$$\alpha_m = -i^m (2\pi)^{-m} \sum_{n \neq 0} (\kappa_n/n^m), \qquad m = 0, 1, \ldots .$$

Then,

$$\alpha_m = \begin{cases} -\beta_{k+1}'(0) & \text{for } m = 2k \ (k = 0, 1, \ldots), \\ -\beta_k(0) & \text{for } m = 2k - 1 \ (k = 1, 2, \ldots). \end{cases}$$

Seeking ψ_m in the form $\alpha_m w_m$, we get from (2.7)-(2.11) the initial-boundary value problem for $m = 0, 1, \ldots$

$$\nabla^2 w_m = 0 \quad \text{in } W , \tag{2.24}$$

$$(\partial^2 w_m/\partial t^2) + (\partial w_m/\partial y) = 0 \quad \text{on } F , \tag{2.25}$$

$$\partial w_m/\partial n = 0 \quad \text{on } S \cup B , \tag{2.26}$$

for $t \geqslant 0$, with the initial conditions on F when $t = 0$

$$w_m = \begin{cases} 0 & \text{for } m = 2k \ (k = 0, 1, \ldots), \\ v_k & \text{for } m = 2k - 1 \ (k = 1, 2, \ldots). \end{cases} \tag{2.27}$$

$$\partial w_m/\partial t = \begin{cases} v_{k+1} & \text{for } m = 2k \ (k = 0, 1, \ldots), \\ 0 & \text{for } m = 2k - 1 \ (k = 1, 2, \ldots). \end{cases} \tag{2.28}$$

Here the above formulae for φ_m are taken into account. Thus, the formal derivation of the asymptotic expansion (1.1) is complete.

2.3. *Some hydrodynamical conclusions.*

We consider the initial terms in (1.1) in more detail. By (2.19) and by the definition of α_m, we have

$$(2.29)\qquad \phi(P,t,\varepsilon) = (\langle\kappa\rangle - \kappa(0))\, w_0(P,t) + \\ + \varepsilon\left\{\int_0^{t/\varepsilon} (\kappa(\mu) - \langle\kappa\rangle)\, d\mu + \int_0^1 \mu(\kappa(\mu) - \langle\kappa\rangle)\, d\mu\right\} v_1(P) + \dots .$$

Here v_1 is the solution of (2.18) and w_0 satisfies

$$\nabla^2 w_0 = 0 \quad \text{in } W, \quad (\partial^2 w_0/\partial t^2) + (\partial w_0/\partial y) = 0 \quad \text{on } F, \quad \partial w_0/\partial n = 0 \quad \text{on } S \cup B,$$

for $t \geqslant 0$ and

$$w_0 = 0, \qquad \partial w_0/\partial t = -\mathcal{P} \quad \text{on } F \quad \text{when } t = 0 .$$

Hence, we can interpret w_0 as the potential of the wave motion due to the initial elevation of the free surface $\mathcal{P}(x,z)$.

Now, we derive asymptotic formulae for such hydrodynamical characteristics of the wave motion as the free surface elevation, the force and the moment applied to the submerged body D, the impulse of the force during the time interval t and the energy.

The elevation of the free surface is given by

$$(2.30)\qquad \eta(x,z,t) = -\{\phi_t(x,0,z,t) + p(x,z,t)\},$$

where p is the surface pressure. The force applied to D is equal to

$$(2.31)\qquad \boldsymbol{F}(t) = \int_S \phi_t(P,t)\, \boldsymbol{n}\, ds .$$

The moment with respect to a point P_0 can be calculated similarly:

$$(2.32)\qquad \boldsymbol{M}(t) = \int_S \phi_t(P,t)\, \boldsymbol{r} \times \boldsymbol{n}\, ds .$$

Here $\boldsymbol{r}$ is the radius vector, directed from P_0 to a point on S. The impulse of the force during the time interval t is

$$\boldsymbol{S}(t) = \int_0^t \boldsymbol{F}(\mu)\, d\mu .$$

The energy of the wave motion is expressed by

$$(2.33)\qquad E(t) = (1/2)\left\{\int_W |\nabla\phi|^2\, dx\, dy\, dz + \int_F \eta^2\, dx\, dz\right\}.$$

Now, we turn to the asymptotic expressions for these characteristics. By differentiating (1.1) and using (2.19), we find the asymptotic formula for the hydrodynamical pressure $p = -\phi_t$ in the fluid

$$p(P, t, \varepsilon) = -[\langle\kappa\rangle - \kappa(0)](\partial w_0/\partial t)(P, t) - [\kappa(t/\varepsilon) - \langle\kappa\rangle]v_1(P) + O(\varepsilon). \tag{2.34}$$

Substituting this into (2.30), we arrive at the asymptotic formula for the surface elevation:

$$\eta(x, z, t, \varepsilon) = [\kappa(0) - \langle\kappa\rangle](\partial w_0/\partial t)(x, 0, z, t) - \langle\kappa\rangle \mathcal{P}(x, z) + O(\varepsilon). \tag{2.35}$$

According to (2.31) we have the following asymptotic formula for the force:

$$\boldsymbol{F}(t, \varepsilon) = [\langle\kappa\rangle - \kappa(0)]\int_S (\partial w_0/\partial t)\,\boldsymbol{n}\,dS + [\kappa(t/\varepsilon) - \langle\kappa\rangle]\int_S v_1\boldsymbol{n}\,dS + O(\varepsilon).$$

Replacing $\boldsymbol{r}$ by $\boldsymbol{r} \times \boldsymbol{n}$ here, we get the asymptotics for the moment of the force. Thus, the principal parts of the force and the moment are sums of slow and rapidly oscillating terms.

While we need v_1 to calculate the principal term of the force, the analogous term of the force impulse is expressed by w_0 only:

$$\boldsymbol{S}(t, \varepsilon) = [\langle\kappa\rangle - \kappa(0)]\int_S w_0\boldsymbol{n}\,dS + O(\varepsilon).$$

This results from the fact that v_1 is involved only in the rapidly oscillating part of the force with zero mean value.

By (2.33) we get the asymptotic representation of the wave energy

$$E(t, \varepsilon) = (1/2)(\langle\kappa\rangle - \kappa(0))^2\int_W |\nabla w_0|^2\,dx\,dy\,dz +$$
$$+(1/2)\int_F [(\langle\kappa\rangle - \kappa(0))(\partial w_0/\partial t) + \langle\kappa\rangle\mathcal{P}]^2\,dx\,dz + O(\varepsilon).$$

We have

$$\int_W |\nabla w_0|^2\,dx\,dy\,dz + \int_F (\partial w_0/\partial t)^2\,dx\,dz = \int_F \mathcal{P}^2\,dx\,dz$$

because of the conservation law (see [8, p. 196]). Hence,

$$E(t, \varepsilon) = (1/2)\{(\langle\kappa\rangle - \kappa(0))^2 + \langle\kappa\rangle^2\}\int_F \mathcal{P}^2\,dx\,dz + \tag{2.36}$$
$$+\langle\kappa\rangle(\langle\kappa\rangle - \kappa(0))\int_F \mathcal{P}(\partial w_0/\partial t)\,dx\,dz + O(\varepsilon).$$

Denote the non-oscillating terms in the asymptotic formulae (2.29) and (2.35) by $\widehat{\phi}(P,t)$ and $\widehat{\eta}(x,z,t)$. By (2.24)-(2.28) these functions satisfy the initial-boundary value problem:

$$\nabla^2 \widehat{\phi} = 0 \quad \text{in } W,$$
$$\widehat{\phi}_t + \widehat{\eta} = -\langle \kappa \rangle \mathcal{P} \quad \text{on } F,$$
$$\partial \widehat{\phi} / \partial n = 0 \quad \text{on } S \cup B,$$

for $t \geqslant 0$ and

$$\widehat{\phi} = 0, \qquad \widehat{\eta} = -\kappa(0)\mathcal{P} \quad \text{on } F \quad \text{when } t = 0.$$

The problem describes the motion due to the surface pressure $\langle \kappa \rangle \mathcal{P}$ and the initial elevation $-\kappa(0)\mathcal{P}$. Since for an arbitrary $t \geqslant 0$

$$(1/\varepsilon) \int_t^{t+\varepsilon} \phi(\cdot, \lambda, \varepsilon)\, d\lambda - \widehat{\phi}(\cdot, t) = O(\varepsilon)$$

and

$$(1/\varepsilon) \int_t^{t+\varepsilon} \eta(\cdot, \lambda, \varepsilon)\, d\lambda - \widehat{\eta}(\cdot, t) = O(\varepsilon)$$

as $\varepsilon \to 0$, it is natural to call this motion averaged. We shall use the notation $\widehat{p}, \widehat{E}, \widehat{\boldsymbol{F}}, \widehat{\boldsymbol{M}}, \widehat{\boldsymbol{S}}$ for the pressure, the energy, the force, the moment and the impulse of the force connected with the averaged motion. According to (2.31), (2.32) and (2.34) the principal parts of $\boldsymbol{F} - \widehat{\boldsymbol{F}}$, $\boldsymbol{M} - \widehat{\boldsymbol{M}}$ and $p - \widehat{p}$ are not small as $\varepsilon \to 0$ and oscillate rapidly with zero mean values. On the other hand, $\phi - \widehat{\phi}$, $\eta - \widehat{\eta}$, $E - \widehat{E}$ and $\boldsymbol{S} - \widehat{\boldsymbol{S}}$ are of order ε.

In the next Section we consider a special case when the principal terms in the above asymptotic formulae can be evaluated explicitly.

2.4. *Waves in a layer of constant depth.*

Let the fluid occupy the layer of constant depth, *i.e.* $W = \{-h < y < 0\}$ and $D = \emptyset$. Then, by using the Fourier transform

$$\widetilde{\mathcal{P}}(\sigma) = \int_{\boldsymbol{R}^2} \mathcal{P}(x,z)\, e^{-i(x\xi + z\zeta)}\, dx\, dz, \qquad \sigma = (\xi, \zeta),$$

we find the solution of problem (2.1)-(2.5):

$$\phi(P,t,\varepsilon) = \frac{1}{(2\pi)^2 \varepsilon} \int_{\boldsymbol{R}^2} \widetilde{\mathcal{P}}(\sigma) \frac{\cosh|\sigma|(y+h)}{|\sigma|^{1/2}\cosh|\sigma|h}\, e^{i(x\xi + z\zeta)}\, d\xi\, d\zeta \cdot$$

$$\cdot \int_0^t \kappa'(\mu/\varepsilon) \sin\left((\mu - t)(|\sigma| \tanh|\sigma|h)^{1/2}\right) d\mu .$$

When $W = \boldsymbol{R}^3_-$, *i.e.* $S \cup B = \emptyset$, one has to omit $\tanh|\sigma|h$ and to replace

$$\frac{\cosh|\sigma|(y+h)}{\cosh|\sigma|h} \quad \text{by } \exp(|\sigma|y)$$

in the above formula.

The behaviour of the explicit expression for ϕ is not obvious as $\varepsilon \to 0$. Hence, the expansion (1.1) is preferable, since its terms have a simpler form:

$$w_m(P,t) = \frac{-1}{(2\pi)^2} \int_{\boldsymbol{R}^2} \widetilde{\mathcal{P}}(\sigma)(|\sigma|\tanh|\sigma|h)^{(m-1)/2} \cdot$$

$$\cdot \frac{\cosh|\sigma|(y+h)}{\cosh|\sigma|h} \sin\left(\frac{m\pi}{2} + t(|\sigma|\tanh|\sigma|h)^{1/2}\right) e^{i(x\xi + z\zeta)}\, d\xi\, d\zeta$$

where $m = 0, 1, \dots$ and

$$v_m(P) = \frac{(-1)^m}{(2\pi)^2} \int_{\boldsymbol{R}^2} \widetilde{\mathcal{P}}(\sigma)(|\sigma|\tanh|\sigma|h)^{m-1} \frac{\cosh|\sigma|(y+h)}{\cosh|\sigma|h}\, e^{i(x\xi+z\zeta)}\, d\xi\, d\zeta$$

where $m = 1, 2, \dots$.

In order to obtain similar formulae for the case $W = \boldsymbol{R}^3_-$ one has to make the same changes as above.

Now we consider the energy of the wave motion for the layer. First we note, that an application of the Parseval theorem gives the explicit formula for the unsteady integral in the principal term of the energy:

$$\int_F \mathcal{P} \frac{\partial w_0}{\partial t}\, dx\, dz = -(2\pi)^{-2} \int_{\boldsymbol{R}^2} |\widetilde{\mathcal{P}}(\sigma)^2| \cos\left(t(|\sigma|\tanh|\sigma|h)^{1/2}\right) d\xi\, d\zeta\,.$$

Example. Let $W = \boldsymbol{R}^3_-$ and $\mathcal{P}(x) = (a^2 + x^2)^{-1}$, where $a > 0$. Then,

$$\widetilde{\mathcal{P}}(\xi) = (\pi/a)e^{-a|\xi|}$$

and

$$\int_F \mathcal{P} \frac{\partial w_0}{\partial t}\, dx = -(2\pi)^{-1} \int_{-\infty}^{+\infty} |\mathcal{P}(\xi)|^2 \cos(t|\xi|^{1/2})\, d\xi\,.$$

Hence the last integral takes the form

$$2\pi/a^2 \int_0^\infty e^{-2a\xi} \cos(t\xi^{1/2})\, d\xi = (2\pi/a)^2 \int_0^\infty \mu e^{-2a\mu^2} \cos(t\mu)\, d\mu\,.$$

Hence, by formula 3.953.4 in [4], we have

$$\int_F \mathcal{P} \frac{\partial w_0}{\partial t}\, dx = -\frac{\pi}{2a}\left\{1 + \frac{i\pi^{1/2}}{\sqrt{8a}} \exp\left(-\frac{t^2}{8a}\right) \operatorname{erf}\left(\frac{it}{\sqrt{8a}}\right)\right\}.$$

On the other hand,

$$\int_F \mathcal{P}^2\, dx = 2\int_0^\infty (a^2 + x^2)^{-2}\, dx = \pi/(2a)\,.$$

Substituting these expressions into (2.36) we obtain

$$E(t, \varepsilon) = \pi\kappa^2(0)/(4a) +$$
$$+ (\kappa(0) - \langle\kappa\rangle)\langle\kappa\rangle(it/(4\sqrt{2}))(\pi/a)^{3/2} \exp(-t^2/(8a))\, \mathrm{erf}\,(it/\sqrt{8a}) + O(\varepsilon)\,.$$

Here the first term gives the energy at the initial moment and the second one decribes the evolution of the principal part of the energy in time.

By using (2.34) we arrive at the formula for the averaged pressure in the fluid

$$\text{(2.37)}\qquad \hat{p}(P, t) = (\kappa(0) - \langle\kappa\rangle)((y - a)/a(x^2 + (y-a)^2)^2) -$$
$$-(t\pi^{1/2}/2a)\,\mathrm{Im}\,[\exp(-t^2/4(a - y - ix))/(a - y - ix)^{3/2}]\,\mathrm{erf}\,(it/2(a - y - ix)^{1/2})\,.$$

The rapidly oscillating part of the pressure is also written explicitly

$$(\langle\kappa\rangle - \kappa(t/\varepsilon))(y-a)/a(x^2 + (y-a)^2)^2\,.$$

Hence,

$$\text{(2.38)}\qquad p(P, t, \varepsilon) =$$
$$= (\langle\kappa\rangle - \kappa(0))(t\pi^{1/2}/2a)\,\mathrm{Im}\,[\exp(-t^2/4(a - y - ix))/(a - y - ix)^{3/2}]\,\cdot$$
$$\cdot\,\mathrm{erf}\,(it/2(a - y - ix)^{1/2}) + (\kappa(0) - \kappa(t/\varepsilon))(y-a)/a(x^2 + (y-a)^2) + O(\varepsilon)\,.$$

By setting here $y = 0$ and using (2.35) we obtain the asymptotics of the surface elevation

$$\text{(2.39)}\qquad \eta(x, t, \varepsilon) = -\kappa(0)/(a^2 + x^2) + [\langle\kappa\rangle - \kappa(0)](t\pi^{1/2}/2a)\,\cdot$$
$$\cdot\,\mathrm{Im}\,[\exp(-t^2/4(a - ix))/(a - ix)^{3/2}]\,\mathrm{erf}\,(it/2(a - ix)^{1/2}) + O(\varepsilon)\,.$$

Here the second term in the right-hand side is the zero-order in ε time-dependent perturbation of the initial elevation.

Since

$$\frac{1}{(a^2 + x^2)^{N+1}} = N!\left(\frac{-1}{2a}\frac{\partial}{\partial a}\right)^N \frac{1}{a^2 + x^2}$$

then applying the operator

$$N!\left(\frac{-1}{2a}\frac{\partial}{\partial a}\right)^N$$

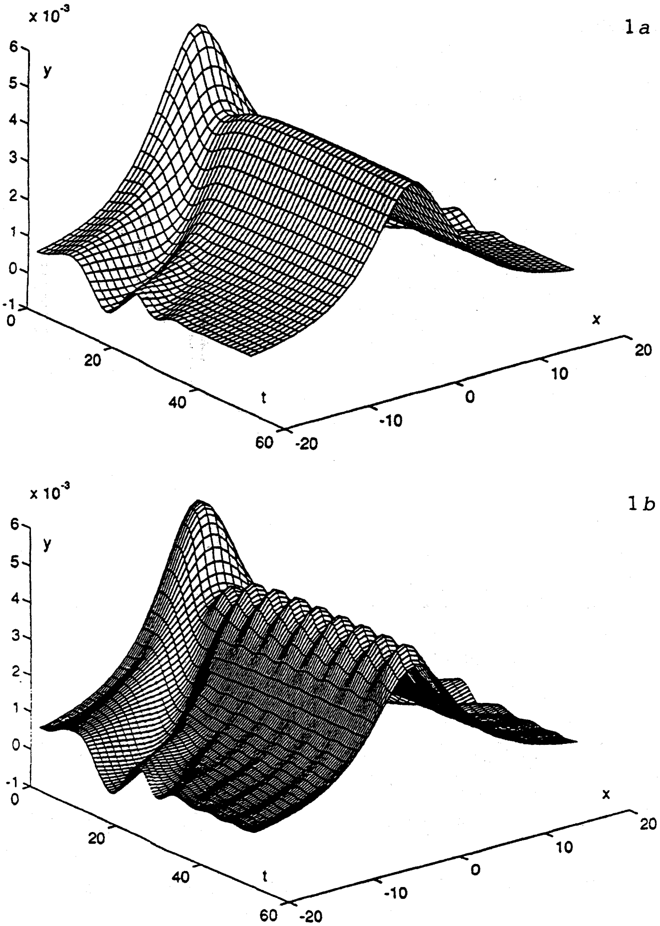


Fig. 1.

to the right-hand side in (2.39) we get a formula for the elevation, corresponding to $\mathcal{P}(x) = (a^2 + x^2)^{-N-1}$.

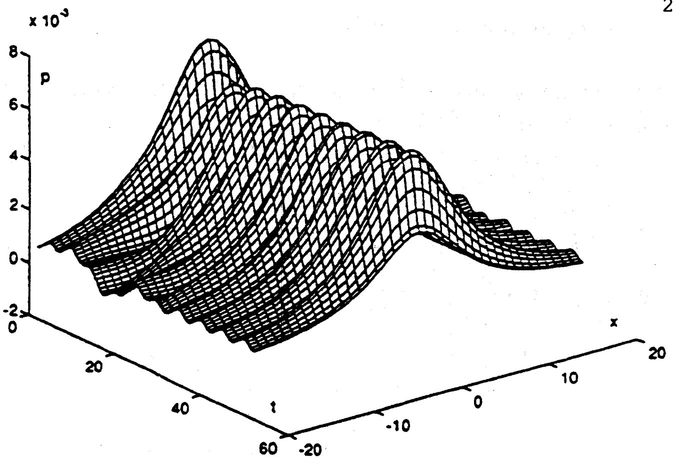
Figure 1 shows the evolution of the free surface profile in the case when

$$p(x, t, \varepsilon) = (\lambda \sin^2 (2\pi t/\varepsilon) - \alpha)(a^2 + x^2)^{-1}$$

where $a = 6.0$, $\alpha = 0.2$, $\lambda = 0.21/\pi^{1/2}$ and $\varepsilon = 10.0$. One sees that even for surprisingly large ε the principal term $\widehat{\eta}$ (see (2.39) and fig. 1*a*) gives a good approximation to the exact free surface elevation η (fig. 1*b*).

For the same surface pressure the evolution of the hydrodynamical pressure at the depth 0.5 is shown in fig. 2. The principal term of the asymptotics (2.38) (fig. 2*a*), combining the averaged value $\widehat{p}$ (see (2.37) and fig. 2*c*) with oscillations, practically coincides with the exact pressure (fig. 2*b*).

2a


x 10-3
8
6
4
2
0
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p
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20
40
60
t
-20
-10
0
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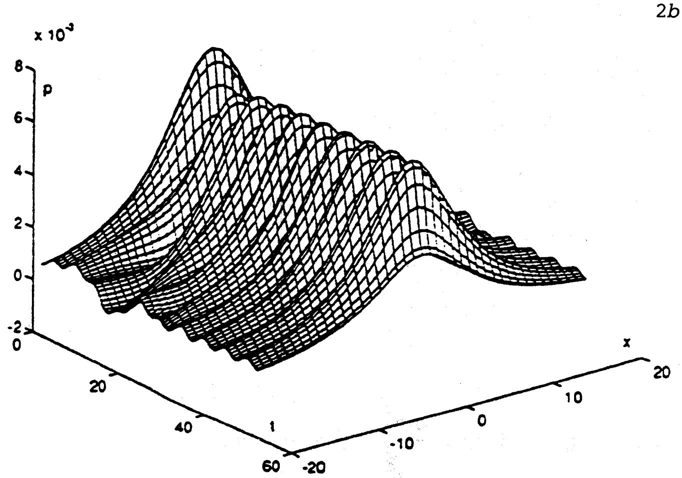
2b


x 10-3
8
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0
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2c

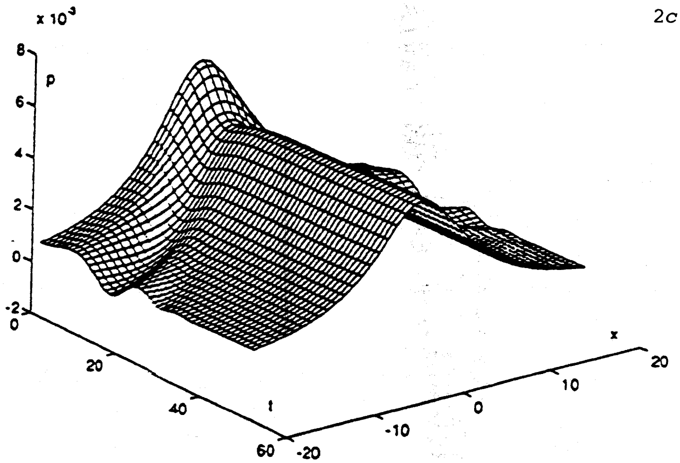
x 10-3
8
6
4
2
0
-2
p
0
20
40
60
t
-20
-10
0
10
20
x

Fig. 2.

2.5. *Justification of the asymptotic expansion.*

In the present section we justify the expansion (1.1) (see also (2.6)). First, we estimate every term in the series (2.6).

In what follows we shall use an operator K. It maps φ defined on F to $\phi_y(x, 0, z)$, where ϕ is the solution of

$$\nabla^2 \phi = 0 \quad \text{in } W, \qquad \partial\phi/\partial n = 0 \quad \text{on } S \cup B, \qquad \phi = \varphi \quad \text{on } F.$$

It is well-known that for any s

$$\|K\varphi\|_{s-1} \leqslant M_s \|\varphi\|_s . \tag{2.40}$$

Here $\|\cdot\|_s$ is the norm in the Sobolev space $H^s(F)$.

Proposition 1. *Let $\kappa \in L^2(0, 1)$ and let $\mathcal{P}$ belong to $H^{m/2}(F)$ if m is odd and to $H^{(m+1)/2}(F)$ if m is even. Then the functions in* (2.6) *satisfy*

$$\|\varphi_{2l-1}\|_{1/2} \leqslant C(2\pi)^{-2l+1} \|\kappa\|_{L^2(0,1)} \|K^{l-1}\mathcal{P}\|_{1/2} , \tag{2.41}$$

$$\|\psi_m\|_{1/2} \leqslant C(2\pi)^{-m} \|\kappa\|_{L^2(0,1)} \|w_m\|_{1/2} \tag{2.42}$$

and

$$\|w_m\|_{1/2} \leqslant \begin{cases} Ct\|K^l\mathcal{P}\|_{1/2} & \textit{when } m = 2l , \\ C\|K^{l-1}\mathcal{P}\|_{1/2} & \textit{when } m = 2l - 1 . \end{cases} \tag{2.43}$$

Proof. Since $\varphi_{2l} = 0$, we have to estimate

$$\varphi_{2l-1}(P, \tau) = \beta_l(\tau) v_l(P) .$$

From (2.23) by using the Cauchy inequality we get

$$|\beta_l| \leqslant C(2\pi)^{1-2l} \|\kappa\|_{L^2(0,1)} . \tag{2.44}$$

Taking into account, that the functions v_m are defined recurrently (see (2.18) and (2.22)), we can write

$$v_l = K^{l-1}\mathcal{P} \quad \text{on } F . \tag{2.45}$$

Thus, (2.44) and (2.45) imply (2.41).

Now, we estimate $\psi_m(P, t) = \alpha_m w_m(P, t)$. By the definition of α_m we obtain (2.42) in the same way as (2.41).

Since w_m is the solution of (2.24)-(2-28), its trace on F (we shall use the same notation w_m for the trace) is given by

$$w_m(\cdot, t) = \begin{cases} K^{-1/2} \sin(K^{1/2}t) v_{l+1} & \text{for } m = 2l , \\ \cos(K^{1/2}t) v_l & \text{for } m = 2l - 1 , \end{cases} \qquad l = 1, 2, \ldots .$$

This and (2.45) imply (2.43). ∎

Now we shall estimate the remainder term

$$(2.46)\qquad R_{2N+1}(P,t,\varepsilon)=\phi(P,t,\varepsilon)-\\ -\sum_{m=0}^{2N+1}\varepsilon^m\alpha_m w_m(P,t)-\sum_{m=1}^{N+1}\varepsilon^{2m-1}\beta_m(t/\varepsilon)v_m(P),$$

what will complete the justification of asymptotics (1.1).

It follows from (2.1), (2.4) and (2.18), (2.22) that

$$(2.47)\qquad \nabla^2 R_{2N+1}=0 \quad \text{in } W,$$

$$(2.48)\qquad \partial R_{2N+1}/\partial n=0 \quad \text{on } S\cup B,$$

for $t\geqslant 0$. By (2.2), (2.25) and (2.46) we have on F

$$\frac{\partial^2 R_{2N+1}}{\partial t^2}+\frac{\partial R_{2N+1}}{\partial y}=-\varepsilon^{-1}\kappa'(\tau)\mathcal{P}(x,z)-\\ -\sum_{m=1}^{N+1}\varepsilon^{2m-1}\left[\varepsilon^{-2}\beta''_m(\tau)v_m(x,0,z)+\beta_m(\tau)\frac{\partial v_m}{\partial y}(x,0,z)\right].$$

According to (2.17) and (2.21) this condition takes the form

$$(2.49)\qquad \frac{\partial^2 R_{2N+1}}{\partial t^2}+\frac{\partial R_{2N+1}}{\partial y}=-\varepsilon^{2N+1}\beta_{N+1}\left(\frac{t}{\varepsilon}\right)\frac{\partial v_{N+1}}{\partial y}.$$

Setting $t=0$ in (2.46) and taking into account (2.4) and (2.27), we find

$$(2.50)\qquad R_{2N+1}=0 \quad \text{on } F \quad \text{when } t=0.$$

Differentiating (2.46) with respect to t, we obtain by (2.5), (2.28) and the definition of α_m

$$\frac{\partial R_{2N+1}}{\partial t}=\sum_{m=0}^{N}\varepsilon^m\beta'_{m+1}(0)v_{m+1}-\sum_{m=1}^{N+1}\varepsilon^{2m-2}\beta'_m(0)v_m \quad \text{on } F \quad \text{when } t=0.$$

Hence,

$$(2.51)\qquad \partial R_{2N+1}/\partial t=0 \quad \text{on } F \quad \text{when } t=0.$$

Thus, the remainder R_{2N+1} satisfies the initial-boundary value problem (2.47)-(2.51).

PROPOSITION 2. *Let* $\mathcal{P}\in H^{(2N+3)/2}(F)$ *and* $\kappa\in L^2(0,1)$. *Then*

$$(2.52)\qquad \|R_{2N+1}\|_{1/2}\leqslant(\varepsilon/2\pi)^{2N+2}Ct\|K^{N+1}\mathcal{P}\|_{1/2},$$

where C *does not depend on* t *and* $\mathcal{P}$.

PROOF. Since R_{2N+1} satisfies (2.47)-(2.51), then we have the explicit expression for its trace on F:

$$(2.53)\qquad R_{2N+1}(\cdot,t,\varepsilon)=-\varepsilon^{2N+1}\left(\int_0^t\beta_{N+1}(\mu/\varepsilon)\sin\left((t-\mu)K^{1/2}\right)d\mu\right)K^{N+1/2}\mathcal{P}.$$

Here (2.45) is also taken into account.

Integrating by parts in the last integral, we get

(2.54) $R_{2N+1}(\cdot, t, \varepsilon) =$

$$= -\varepsilon^{2N+2}\left(\int_0^t\left(\int_0^{\mu/\varepsilon}\beta_{N+1}(\gamma)\,d\gamma\right)\cos\left((t-\mu)K^{1/2}\right)d\mu\right)K^{N+1}\mathcal{P}.$$

From the fact that β_{N+1} is a 1-periodic function with zero mean value, it follows

$$\left|\int_0^{\mu/\varepsilon}\beta_{N+1}(\gamma)\,d\gamma\right| \leqslant \max_{\gamma}|\beta_{N+1}(\gamma)|.$$

Combining this inequality with (2.44) and (2.54), we arrive at (2.52). ∎

In order to formulate a theorem, justifying asymptotic expansion (1.1), we define the remainder

$$R_{2N}(P, t, \varepsilon) = \phi(P, t, \varepsilon) - \sum_{m=0}^{2N}\varepsilon^m\alpha_m w_m(P, t) - \sum_{m=1}^{N}\varepsilon^{2m-1}\beta_m(t/\varepsilon)w_m(P).$$

THEOREM 1. *Let $\mathcal{P} \in H^{(n+2)/2}(F)$ and $\kappa \in L^2(0, 1)$, then*

$$|R_n| + \|R_n\|_{1/2} \leqslant \varepsilon^{n+1}C_n t\|\mathcal{P}\|_{(n+2)/2}. \tag{2.55}$$

Here $|\cdot|$ is the norm in $H^1(W)$. The constant C_n does not depend on t and $\mathcal{P}$.

PROOF. First let $n = 2N+1$. Then (2.55) follows from Proposition 2, (2.40) and from the known inequality

$$|R_n| \leqslant C\|R_n\|_{1/2}. \tag{2.56}$$

When $n = 2N$, we write

$$R_{2N} = R_{2N+1} + \varepsilon^{2N+1}(\varphi_{2N+1} + \psi_{2N+1}).$$

The required estimate for the sum in brackets follows from Proposition 1 and (2.40).

Applying (2.40) and (2.44) to (2.53), we get the inequality

$$\|R_{2N+1}\|_{1/2} \leqslant \varepsilon^{2N+1}tC_N\|\mathcal{P}\|_{N+1},$$

which together with (2.56) completes the proof.

For the case when W is the layer we can obtain the following condition for convergence of the asymptotic series (2.6).

THEOREM 2. *Let W be the layer of the constant depth h or the half-space $\{y < 0\}$. If $\kappa \in L^2(0, 1)$ and if the support of the Fourier transform $\tilde{\mathcal{P}}$ is placed in the disk $\{\sigma: |\sigma| < (2\pi/\varepsilon)^2\}$, then the series (2.6) converges absolutely with respect to the $H^{1/2}(F)$-norm.*

PROOF. It follows from Proposition 1 that if

$$(\varepsilon/(2\pi))^n \|K^{n/2}\mathcal{P}\|_{1/2} \leqslant Cq^n, \tag{2.57}$$

where C does not depend on n and $q < 1$, then series (2.6) converges absolutely with respect to the $H^{1/2}(F)$-norm.

Using the Fourier transform, one obtains that for the layer

$$(K\mathcal{P})(x, z) = (2\pi)^{-2} \int_{R^2} \tilde{\mathcal{P}}(\sigma)\,|\sigma| \tanh(|\sigma| h)\, e^{i(x\xi + z\zeta)}\, d\xi\, d\zeta\,.$$

When $W = \{y < 0\}$, the hyperbolic tangent should be omitted. In both cases we get by the Parseval theorem:

$$\|K^{n/2}\mathcal{P}\|_{1/2}^2 \leqslant C \int_{R^2} (1 + |\sigma|^2)^{1/2} |\sigma|^n\, |\tilde{\mathcal{P}}(\sigma)|^2\, d\xi\, d\zeta\,.$$

By the hypothesis of Theorem 2 there exists $q \in (0, 1)$, such that $\tilde{\mathcal{P}}(\sigma) = 0$ when $|\sigma| \geqslant (q\,2\pi/\varepsilon)^2$. Therefore,

$$\|K^{n/2}\mathcal{P}\|_{1/2}^2 \leqslant C\left(\int_0^{(q2\pi/\varepsilon)^2} |\sigma|^{n+1} d|\sigma| + \int_0^{(q2\pi/\varepsilon)^2} |\sigma|^{n+2} d|\sigma| \right) \leqslant C(q2\pi/\varepsilon)^{2n}\,.$$

Hence (2.57) is proved, what completes the proof of Theorem 2.

3. Waves due to a high-frequency submerged source

3.1. *Statement of the problem.*

In the present section we consider the problem on waves, arising when a high-frequency submerged source starts pulsations in the calm fluid. As above we seek an asymptotic expansion for velocity potential, assuming that $\varepsilon \ll 1$, where ε is the dimensionless period of oscillations.

We suppose that the geometry of the fluid region W is the same as in Section 2. If a source of strength $-\kappa(t/\varepsilon)$, where κ is a 1-periodic function, is placed at $P_0 \in W$, then its velocity potential $G(P, P_0, t, \varepsilon)$ satisfies the initial-boundary value problem:

$$\nabla^2 G = -\kappa(t/\varepsilon)\,\delta(P - P_0) \quad \text{in } W, \tag{3.1}$$

$$G_{tt} + G_y = 0 \quad \text{on } F, \tag{3.2}$$

$$\partial G/\partial n = 0 \quad \text{on } S \cup B, \tag{3.3}$$

for $t \geqslant 0$

$$G = 0, \tag{3.4}$$

$$G_t = 0, \tag{3.5}$$

on F when $t = 0$. In the next section we construct an asymptotic series for G.

3.2. *Asymptotic expansion.*

It is natural to suppose, that the waves due to the rapidly oscillating source are the same (up to a term $O(\varepsilon)$) as the waves from the source of the mean strength. We represent the corresponding velocity potential in the form $\langle\kappa\rangle w_0$, where w_0 satisfies

$$\nabla^2 w_0 = -\delta(P-P_0) \quad \text{in } W,$$

$$\frac{\partial^2 w_0}{\partial t^2} + \frac{\partial w_0}{\partial y} = 0 \quad \text{on } F,$$

$$\partial w_0 / \partial n = 0 \quad \text{on } S \cup B,$$

for $t \geqslant 0$

$$w_0 = \partial w_0 / \partial t = 0 \quad \text{on } F \quad \text{when } t = 0. \tag{3.6}$$

Now, we seek the asymptotic series for $G - \langle\kappa\rangle w_0$ in the form (2.6). Then we arrive at the relations (2.7)-(2.13) for $m = 0, 1, \ldots$ with two exceptions. We have

$$\nabla^2 \varphi_0 = [\langle\kappa\rangle - \kappa(\tau)]\,\delta(P-P_0) \quad \text{in } W \quad \text{for } \tau \geqslant 0, \tag{3.7}$$

instead of the Laplace equation and

$$\partial^2 \varphi_1 / \partial\tau^2 = 0 \quad \text{on } F \quad \text{for } \tau \geqslant 0, \tag{3.8}$$

instead of (2.13).

Since (2.12) implies

$$\varphi_0 = 0 \quad \text{on } F \quad \text{for } \tau \geqslant 0,$$

by putting

$$\varphi_0(P,\tau) = b_0(\tau) v_0(P), \qquad \text{where } b_0(\tau) = \kappa(\tau) - \langle\kappa\rangle,$$

we get the boundary value problem for v_0

$$\begin{cases} \nabla^2 v_0 = -\delta(P-P_0) \quad \text{in } W, \quad v_0 = 0 \quad \text{on } F, \\ \partial v_0 / \partial n = 0 \quad \text{on } S \cup B. \end{cases} \tag{3.9}$$

Here (3.7) and (2.9) are taken into account. Then, by (3.8), (2.9) and (2.7) we find

$$\varphi_1 = 0 \quad \text{in } W \quad \text{for } \tau \geqslant 0,$$

since φ_1 is periodic in τ and decays as $|P| \to \infty$.

Using (2.15) in the same manner as in Section 2.2, we obtain

$$\varphi_{2k+1} = 0 \quad \text{in } W \quad \text{for } \tau \geqslant 0, \quad k = 1, 2, \ldots$$

and

$$\varphi_{2k}(P,\tau) = b_k(\tau) v_k(P), \qquad k = 1, 2, \ldots$$

where v_k satisfies (2.22) and

$$b_k(\tau) = (-1)^k (2\pi)^{-2k} \sum_{n \neq 0} b_n^{(0)} n^{-2k} e^{2\pi i n \tau}. \tag{3.10}$$

By $b_n^{(0)}$ we denote the Fourier coefficients of $b_0 = \kappa - \langle\kappa\rangle$.

Let

$$a_m = -(i/(2\pi))^{m+1+(-1)^{m+1}} \sum_{n \neq 0} (b_n^{(0)} / n^{m+1+(-1)^{m+1}})$$

so that for $k = 1, 2, \ldots$

$$a_m = \begin{cases} -b_k(0) & \text{for } m = 2k , \\ -b'_{k+1}(0) & \text{for } m = 2k-1 . \end{cases}$$

Seeking the functions ψ_m $(m = 1, 2, \ldots)$ in the form $a_m w_m$ we get from (2.7)-(2.11) the Cauchy-Poisson problem (see (2.24)-(2.26)) with the initial conditions on F when $t = 0$ (cf. (2.27) and (2.28)):

$$w_m = \begin{cases} 0 & \text{for } m = 2k-1 , \\ v_k & \text{for } m = 2k \end{cases} \tag{3.11}$$

and

$$\frac{\partial w_m}{\partial t} = \begin{cases} v_{k+1} & \text{for } m = 2k-1 , \\ 0 & \text{for } m = 2k . \end{cases} \tag{3.12}$$

Here $k = 1, 2, \ldots$.

Putting $a_0 = \langle \kappa \rangle$, we arrive at the asymptotic representation for the source potential

$$G(P, P_0, t, \varepsilon) \sim \sum_{m=0}^{\infty} \varepsilon^m a_m w_m(P, t) + \sum_{m=0}^{\infty} \varepsilon^{2m} b_m(t/\varepsilon) v_m(P) , \tag{3.13}$$

where a_m and $b_m(\tau)$ are given above (see, *e.g.*, (3.10)), the functions w_0 and v_0 satisfy (3.6) and (3.9) respectively, and the next functions are to be found recurrently with the help of (2.22) and (2.24)-(2.26), (3.11), (3.12).

To justify the asymptotic expansion (3.13) we consider the initial-boundary value problem for the remainder term

$$R_{2N}(P, t, \varepsilon) = G(P, P_0, t, \varepsilon) - \sum_{m=0}^{2N} \varepsilon^m a_m w_m(P, t) - \sum_{m=0}^{N} \varepsilon^{2m} b_m(t/\varepsilon) v_m(P) .$$

It can be written down in the same way as the corresponding problem in Section 2.5 and has the form

$$\nabla^2 R_{2N} = 0 \quad \text{in } W ,$$

$$\frac{\partial^2 R_{2N}}{\partial t^2} + \frac{\partial R_{2N}}{\partial y} = -\varepsilon^{2N} b_n\left(\frac{t}{\varepsilon}\right) \frac{\partial v_N}{\partial y} \quad \text{on } F ,$$

$$\partial R_{2N} / \partial n = 0 \quad \text{on } S \cup B ,$$

for $t \geqslant 0$

$$R_{2N} = \partial R_{2N} / \partial t = 0 \quad \text{on } F \quad \text{when } t = 0 .$$

We see that this problem is similar to (2.47)-(2.51). Hence, the justification of asymptotic expansion (3.13) can be performed following the scheme in Section 2.5. There is only one difference. We have to use the trace of $\partial v_0 / \partial y$ on F instead of $\mathcal{P}$.

3.3. *Pulsations of a source in the layer.*

The functions in (3.13) can be written down explicitly, when W is the layer of constant depth h, or the half-space $\{y < 0\}$. For this purpose one has to use a well-known identity

$$(a^2 + b^2)^{-1/2} = \int_0^\infty e^{-\mu b} J_0(\mu a)\, d\mu$$

in the same way as in [8, Sect. 6.9]. The corresponding formulae for v_m are:

$$v_0 = \frac{1}{4\pi}\left[\frac{1}{r} - \frac{1}{r_0} + 2\int_0^\infty e^{-\mu h} \frac{\sinh \mu y \sinh \mu y_0}{\cosh \mu h} J_0(\mu \varrho)\, d\mu\right] \tag{3.14}$$

and for $m \geqslant 1$

$$v_m = \frac{(-1)^{m+1}}{2\pi} \int_0^\infty \mu(\mu \tanh \mu h)^{m-1} \frac{\cosh \mu(y+h) \cosh \mu(y_0 + h)}{\cosh^2 \mu h} J_0(\mu\varrho)\, d\mu\,.$$

Here

$$\varrho^2 = (x - x_0)^2 + (z - z_0)^2\,, \qquad r^2 = \varrho^2 + (y - y_0)^2 \text{ and } r_0^2 = \varrho^2 + (y + y_0)^2\,.$$

The functions w_m are given by

$$w_0 = \frac{1}{4\pi}\left\{\frac{1}{r} - \frac{1}{r_0} + 2\int_0^\infty e^{-\mu h} \frac{\sinh \mu y \sinh \mu y_0}{\cosh \mu h} J_0(\mu\varrho)\, d\mu + \right. \tag{3.15}$$

$$\left. +2\int_0^\infty \frac{1 - \cos((\mu \tanh \mu h)^{1/2} t)}{\tanh \mu h} \frac{\cosh \mu(y+h) \cosh \mu(y_0+h)}{\cosh^2 \mu h} J_0(\mu\varrho)\, d\mu\right\}$$

and for $m \geqslant 1$

$$w_m = \frac{1}{2\pi} \int_0^\infty \mu(\mu \tanh \mu h)^{(m-2)/2} \frac{\cosh \mu(y+h) \cosh \mu(y_0+h)}{\cosh^2 \mu h}\,\cdot$$

$$\cdot J_0(\mu\varrho) \cos\left(\frac{m\pi}{2} - t(\mu \tanh \mu h)^{1/2}\right) d\mu\,.$$

The explicit solution of (3.1)-(3.5) for the layer is

$$G(P, P_0, t, \varepsilon) = \frac{\kappa(t/\varepsilon)}{4\pi}\left[\frac{1}{r} - \frac{1}{r_0} + 2\int_0^\infty e^{-\mu h} \frac{\sinh \mu y \sinh \mu y_0}{\cosh \mu h} J_0(\mu\varrho)\, d\mu\right] + \tag{3.16}$$

$$+\frac{1}{2\pi}\int_0^\infty \left(\frac{\mu}{\tanh \mu h}\right)^{1/2} \frac{\cosh \mu(y+h) \cosh \mu(y_0+h)}{\cosh^2 \mu h} J_0(\mu\varrho)\, d\mu\,\cdot$$

$$\cdot \int_0^t \kappa\left(\frac{\gamma}{\varepsilon}\right) \sin((\mu \tanh \mu h)^{1/2}(t - \gamma))\, d\gamma\,.$$

Hence, (3.13) gives the expansion of the unsteady integral in (3.16).

When the depth is infinite, the first integrals in (3.15) and (3.16) should be omitted as well as the integral in (3.14). Also, $\tanh \mu h$ should be replaced by 1 and instead of

$$\cosh \mu(y+h) \cosh \mu(y_0+h)/(\cosh^2 \mu h)$$

one should insert $\exp\{\mu(y+y_0)\}$. In this case the integrals, expressing v_m, can be found explicitly and we get

$$v_m = \frac{-1}{2\pi} \frac{\partial^m r_0^{-1}}{\partial y^m}$$

for $m = 1, 2, \ldots$.

3.4. *Principal terms in the asymptotics of wave characteristics.*

First we consider the principal term in the asymptotics of a source. By (3.13) we have

$$G(P, P_0, t, \varepsilon) = [\kappa(t/\varepsilon) - \langle\kappa\rangle] v_0(P, P_0) + \langle\kappa\rangle w_0(P, P_0, t) + O(\varepsilon).$$

According to (3.9), the first term in the right-hand side describes a stationary ε-periodic waveless source with zero mean strength. It produces no wave pattern because of the homogeneous Dirichlet condition on F in (3.9).

The second term is proportional to $\langle\kappa\rangle$ and w_0 satisfies (3.6). Hence, it describes the source having the mean strength and starting at $t = 0$. The waves are given only by $\langle\kappa\rangle w_0$, since the asymptotics of the free surface elevation has the form

$$\eta(x, z, t, \varepsilon) = -\langle\kappa\rangle(\partial w_0/\partial t)(x, 0, z, t) + O(\varepsilon).$$

The force and the moment, applied to the submerged body D, depend both on v_0 and w_0:

$$\boldsymbol{F}(t, \varepsilon) = \varepsilon^{-1} \kappa'(t/\varepsilon) \int_S v_0 \boldsymbol{n}\, dS + \langle\kappa\rangle \int_S \frac{\partial w_0}{\partial t} \boldsymbol{n}\, dS + O(\varepsilon),$$

$$\boldsymbol{M}(t, \varepsilon) = \varepsilon^{-1} \kappa'(t/\varepsilon) \int_S v_0 \boldsymbol{r} \times \boldsymbol{n}\, dS + \langle\kappa\rangle \int_S \frac{\partial w_0}{\partial t} \boldsymbol{r} \times \boldsymbol{n}\, dS + O(\varepsilon).$$

We see that the principal terms of $\boldsymbol{F}$ and $\boldsymbol{M}$ increase infinitely as $\varepsilon \to 0$, while the contributions due to w_0 remain finite.

The force impulse during the time interval t has the asymptotics:

$$\boldsymbol{S}(t, \varepsilon) = [\kappa(t/\varepsilon) - \langle\kappa\rangle] \int_S v_0 \boldsymbol{n}\, dS + \langle\kappa\rangle \int_S w_0 \boldsymbol{n}\, dS + O(\varepsilon).$$

Thus, the two terms, having different orders in the formula for the force, give contributions of the same order into $\boldsymbol{S}(t, \varepsilon)$, because of oscillations of the term, which has the order ε^{-1}.

4. High-frequency oscillations of underwater surfaces

4.1. *Statement of the problem and description of results.*

Let the surfaces S and B perform small amplitude rapid oscillations. This can be described by the initial-boundary value problem for the velocity potential $\phi(P, t, \varepsilon)$:

$$\nabla^2 \phi = 0 \quad \text{in } W, \tag{4.1}$$

$$\phi_{tt} + \phi_y = 0 \quad \text{on } F, \tag{4.2}$$

$$\partial\phi / \partial n = \kappa(t/\varepsilon) f(P) \quad \text{on } S \cup B, \tag{4.3}$$

for $t \geqslant 0$

$$\phi = 0, \tag{4.4}$$

$$\phi_t = 0 \tag{4.5}$$

on F when $t = 0$. It is assumed that $f(P)$ decays as $|P| \to \infty$ and $P \in B$.

The solution of (4.1)-(4.5) has an asymptotic expansion of the same form (3.13) as in the case of submerged source. Moreover, for $m \geqslant 1$ the functions v_m and w_m satisfy the same problems (2.22) and (2.24)-(2.26), (3.11), (3.12) respectively. The function v_0 is determined by the stationary boundary value problem:

$$\begin{cases} \nabla^2 v_0 = 0 \quad \text{in } W, \quad v_0 = 0 \quad \text{on } F, \\ \partial v_0 / \partial n = f \quad \text{on } S \cup B. \end{cases} \tag{4.6}$$

The function w_0 is obtained from the initial-boundary value problem:

$$\begin{cases} \nabla^2 w_0 = 0 \quad \text{in } W, \\ \partial^2 w_0 / \partial t^2 + \partial w_0 / \partial y = 0 \quad \text{on } F, \\ \partial w_0 / \partial n = f \quad \text{on } S \cup B \end{cases} \tag{4.7}$$

for $t \geqslant 0$

$$w_0 = \partial w_0 / \partial t = 0 \quad \text{on } F \quad \text{when } t = 0.$$

From (3.13) we see that, if $\langle \kappa \rangle = 0$, then the fluid motion is determined by v_0 up to a term $O(\varepsilon)$. Hence, the amplitude of surface waves is of order ε, because v_0 satisfies the homogeneous Dirichlet condition on F.

As above (see Sections 2.4 and 3.3), the explicit expressions for v_m and w_m can be written down when W is the layer of constant depth h. Using the Fourier transform, we get:

$$v_0(P) = \frac{1}{(2\pi)^2} \int_{R^2} \tilde{f}(\sigma) \frac{\sinh|\sigma| y}{|\sigma| \cosh|\sigma| h} e^{i(x\xi + z\zeta)} \, d\sigma$$

and for $m \geqslant 1$

$$v_m(P) = \frac{(-1)^m}{(2\pi)^2} \int_{R^2} \tilde{f}(\sigma) (|\sigma| \tanh|\sigma| h)^{m-1} \frac{\cosh|\sigma|(y+h)}{\cosh^2|\sigma| h} e^{i(x\xi + z\zeta)} \, d\sigma.$$

The functions w_m are given by

$$w_m(P,t) = \frac{1}{(2\pi)^2} \int_{R^2} \tilde{f}(\sigma)\,(|\sigma| \tanh |\sigma| h)^{(m-1)/2} e^{i(x\xi + z\zeta)} \cdot$$
$$\cdot \frac{\cosh |\sigma|(y+h)}{\cosh^2 |\sigma| h} \sin\left(\frac{m\pi}{2} - t(|\sigma| \tanh |\sigma| h)^{1/2}\right) d\sigma$$

for $m = 0, 1, \ldots$.

4.2. *Asymptotics of the energy, the force and the moment.*

We have similar asymptotic expansions, when either a submerged source or the surfaces S and B oscillate rapidly. The series is given by (3.13) in both cases.

According to (2.33) and (2.30), we get by (3.13):

$$E(t,\varepsilon) = (1/2)\Big\{ \int_W |(\kappa(t/\varepsilon) - \langle\kappa\rangle)\nabla v_0 + \langle\kappa\rangle \nabla w_0|^2\, dx\,dy\,dz +$$
$$+ \langle\kappa\rangle^2 \int_F (\partial w_0/\partial t)^2\, dx\,dz \Big\} + O(\varepsilon).$$

Using the Green formula and the boundary value problems (4.6), (4.7), we write

$$2\langle\kappa\rangle(\kappa - \langle\kappa\rangle) \int_W \nabla w_0 \cdot \nabla v_0\, dx\,dy\,dz + (\kappa - \langle\kappa\rangle)^2 \int_W |\nabla v_0|^2\, dx\,dy\,dz =$$
$$= (\kappa^2 - \langle\kappa\rangle^2) \int_W |\nabla w_0|^2\, dx\,dy\,dz.$$

Thus, the asymptotic formula for the energy takes the form

$$E(t,\varepsilon) = (\langle\kappa\rangle^2/2)\left\{ \int_W |\nabla w_0|^2\, dx\,dy\,dz + \int_F (\partial w_0/\partial t)^2\, dx\,dz \right\} + \tag{4.8}$$
$$+ (1/2)(\kappa^2(t/\varepsilon) - \langle\kappa\rangle^2) \int_W |\nabla v_0|^2\, dx\,dy\,dz + O(\varepsilon).$$

By averaging the energy over the period ε, we get

$$\langle E\rangle = (\langle\kappa\rangle^2/2)\left\{ \int_W |\nabla w_0|^2\, dx\,dy\,dz + \int_F (\partial w_0/\partial t)^2\, dx\,dz \right\} + \tag{4.9}$$
$$+ (1/2)(\langle\kappa\rangle^2 - \langle\kappa\rangle^2) \int_W |\nabla v_0|^2\, dx\,dy\,dz + O(\varepsilon).$$

In particular, if the mean value of κ is zero then

$$\langle E\rangle = (\langle\kappa\rangle^2/2) \int_W |\nabla v_0|^2\, dx\,dy\,dz + O(\varepsilon). \tag{4.10}$$

We remark that formulae (4.8)-(4.10) are also valid if the number of dimensions is two. The Dirichlet integral in (4.10) can be evaluated explicitly in the two-dimensional case when

$$W = \mathbf{R}_-^2 \setminus \{ |z+i| \leq a \}, \qquad a < 1, \qquad z = x + iy .$$

In order to solve the problem:

$$\nabla^2 v_0 = 0 \quad \text{in } W, \qquad v_0 = 0 \quad \text{when } y = 0,$$

$$\partial v_0 / \partial n = f(e^{i\varphi}) \qquad \text{when } z + i = ae^{i\varphi}$$

we apply the transform $\varsigma = (z + i(1-a^2)^{1/2})/(z - i(1-a^2)^{1/2})$. It maps W onto an annulus $\{r < |\varsigma| < 1\}$ on the ς-plane, where

$$r = a/(1 + (1-a^2)^{1/2}) . \tag{4.11}$$

Putting $\varsigma = \varrho e^{i\theta}$, we arrive at the problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \{r < \varrho < 1\}, \\ u\,|_{\varrho = 1} = 0, \ \dfrac{\partial u}{\partial \varrho}\bigg|_{\varrho = r} = 2(1-a^2)^{1/2} g(e^{i\theta}) . \end{cases} \tag{4.12}$$

Here $u(\varrho, \theta) = v_0(z(\varsigma))$ and $g(e^{i\theta}) = f(z(re^{i\theta}))/(1 - 2r\cos\theta + r^2)$.

Seeking the solution of (4.12) in the form:

$$u(\varrho, \theta) = 2(1-a^2)^{1/2} \left\{ a_0 \log \varrho + \sum_{n=1}^{\infty} (\varrho^n - \varrho^{-n})(a_n \cos n\theta + b_n \sin n\theta) \right\}$$

we have to determine a_n and b_n in order to satisfy the Neumann condition. Then

$$a_0 = r\alpha_0/2, \qquad a_n + ib_n = r^{n+1}(\alpha_n + i\beta_n)/(n(1 + r^{2n})), \qquad n \geq 1,$$

where α_n, β_n are the Fourier coefficients of g.

We have the obvious equalities

$$\int_W |\nabla v_0|^2 \, dx\, dy = \int_{r < \varrho < 1} |\nabla u|^2 \, d\xi\, d\eta = -r \int_0^{2\pi} \left[u \frac{\partial u}{\partial \varrho} \right]_{\varrho = r} d\theta .$$

With the help of the series, obtained for u, we get

$$\int_W |\nabla v_0|^2 \, dx\, dy = 4\pi r^2 (1-a^2) \left[\frac{\alpha_0^2}{2} \log \frac{1}{r} + \sum_{n=1}^{\infty} n^{-1} \frac{1 - r^{2n}}{1 + r^{2n}} (\alpha_n^2 + \beta_n^2) \right].$$

Thus, the Dirichlet integral in (4.10) is expressed in terms of the Fourier coefficients of g.

Now we calculate principal terms in the asymptotics of $\mathbf{F}(t, \varepsilon)$ and $M(t, \varepsilon)$. We have

$$\int_{|z+i| = a} v_0 \mathbf{n} \, dS = 2(1-a^2)^{1/2} r \int_0^{2\pi} \frac{u(r, \theta)\, \mathbf{v}}{1 - 2r\cos\theta + r^2} \, d\theta$$

where $\boldsymbol{v} = -(\sin\theta, \cos\theta)$. Substituting the series for u and calculating the integrals, we get the explicit asymptotic formula for the force

$$(4.13)\qquad \boldsymbol{F}(t,\varepsilon) = \frac{4\pi(1-a^2)^{1/2} r}{\varepsilon}\, \kappa'(t/\varepsilon) \sum_{n=1}^{\infty} \frac{r^n(1-r^{2n})}{n(1+r^{2n})} \begin{pmatrix} \beta_n \\ \alpha_n \end{pmatrix} + O(1).$$

As before, α_n and β_n are the Fourier coefficients of g in (4.12).

Similarly, the moment with respect to the point $P_0 = (x_0, y_0)$ is expressed by

$$(4.14)\qquad M(t,\varepsilon) = \frac{4\pi(1-a^2)^{1/2} r}{\varepsilon}\, \kappa'(t/\varepsilon) \sum_{n=1}^{\infty} \frac{r^n(1-r^{2n})}{n(1+r^{2n})} (\beta_n(y_0+1) - \alpha_n x_0) + O(1).$$

In the special case when a surface of a circular cylinder rapidly oscillates with zero mean value and $f(e^{i\varphi}) = A = \text{const}$ formulae (4.10), (4.13) and (4.14) can be made quite explicit, since

$$\alpha_n = 2r^n A/(1-r^2), \qquad \beta_n = 0.$$

Hence,

$$\langle E \rangle = 2\langle \kappa^2 \rangle \pi a^2 A^2 \left(\frac{1}{2} \log \frac{1}{r} + \varphi(r) \right) + O(\varepsilon),$$

$$\boldsymbol{F}(t,\varepsilon) = (4\pi A a/\varepsilon)\, \kappa'(t/\varepsilon)\, \varphi(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(\varepsilon),$$

$$M(t,\varepsilon) = -(4\pi A a x_0/\varepsilon)\, \kappa'(t/\varepsilon)\, \varphi(r) + O(\varepsilon),$$

where

$$\varphi(r) = \sum_{n=1}^{\infty} \frac{r^{2n}(1-r^{2n})}{n(1+r^{2n})} = \log\left(\frac{1-r^2}{2r^{1/2}}\, \theta_2(0,r^2)\, \theta_3(0,r^2) \right).$$

Here r is defined by (4.11) and θ_2, θ_3 are the theta-functions (see [10, Section 22.5]).

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References

[1] A. Friedman - M. Shinbrot, *The initial value problem for the linearized equation of water waves, I.* J. Math. Mech., 17, 1967, 107-180.

[2] A. Friedman - M. Shinbrot, *The initial value problem for the linearized equation of water waves, II.* J. Math. Mech., 18, 1969, 1177-1194.

[3] R. M. Garipov, *On the linear theory of gravity waves: the theorem of existence and uniqueness*. Arch. Rat. Mech. Anal., 24, 1967, 352-362.

[4] I. S. GRADSHTEIN - I. M. RYZHIK, *Tables of Integrals, Sums, Series and Products*. Fizmatgiz, Moscow 1962 (in Russian).
[5] K. HAMDACHE, *Forward speed motions of a submerged body. The Cauchy problem*. Math. Meth. Appl. Sci., 6, 1984, 371-392.
[6] J. N. NEWMAN, *The theory of ship motions*. Adv. Appl. Mech., 18, 1978, 221-283.
[7] L. N. SRETENSKIY, *The Theory of Wave Motions of a Fluid*. Nauka, Moscow 1977 (in Russian).
[8] J. J. STOKER, *Water Waves*. Interscience, New York 1957.
[9] J. V. WEHAUSEN - E. V. LAITONE, *Surface waves*. Handbuch der Physik, Springer-Verlag, Berlin, 9, 1960, 446-778.
[10] E. T. WHITTAKER - G. N. WATSON, *A Course of Modern Analysis*. University Press, Cambridge 1927.

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