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A special version of the Schwarz lemma on an infinite dimensional domain

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Funzioni di variabili complesse. — *A special version of the Schwarz lemma on an infinite dimensional domain.* Nota di TATSUHIRO HONDA, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Let B be the open unit ball of a Banach space E, and let f: $B \to B$ be a holomorphic map with $f(0) = 0$. In this paper, we discuss a condition whereby f is a linear isometry on E.

KEY WORDS: Banach space; Schwarz lemma; Complex geodesic; Projective space.

RIASSUNTO. — *Una versione speciale del lemma di Schwarz su un dominio di dimensione infinita.* Sia B il disco unità aperto di uno spazio di Banach complesso. Si determina una condizione perché un'applicazione olomorfa f: $B \to B$, con $f(0) = 0$, sia un'isometria lineare.

1. INTRODUCTION

Let $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$ denote the unit disc in the complex plane $\mathbb{C}$. The classical Schwarz lemma is as follows:

THEOREM (The classical Schwarz lemma).

(1) *Let f: $\Delta \to \Delta$ be a holomorphic map such that $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in \Delta$.*

(2) *Moreover, if there exists $z_0 \in \Delta \backslash \{0\}$ such that $|f(z_0)| = |z_0|$ then there exists a complex number λ with $|\lambda| = 1$ such that $f(x) = \lambda x$ and f is an automorphism of Δ.*

Let E be a Banach space, and let D be a domain in E. The Carathéodory and Kobayashi invariant pseudo-distances have been introduced in D, together with the corresponding infinitesimal pseudo-metrics. A holomorphic map from Δ into D which is an isometry for the Poincaré distance of Δ and the Carathéodory or Kobayashi pseudo-distance of D is called a complex geodesic. It is known that complex geodesics do not always exist on D. However, their existence is a useful tool in the study of the group of all holomorphic isometry on D.

J. P. Vigué [13] generalized the above classical Schwarz lemma to the unit ball B in $\mathbb{C}^n$, for some norm such that every boundary point of B is a complex extreme point of $\overline{B}$ and to a holomorphic map f: $B \to B$ for which $C_B(f(0), f(w)) = C_B(0, w)$ on an open subset U of B. H. Hamada [8] extended Vigué's results for some local complex submanifold of codimension 1 instead of an open subset U.

The notion of a complex geodesic on infinite dimensional spaces was first introduced by E. Vesentini [10]. He showed that if every boundary point of B_2 is a complex extreme point of $\overline{B_2}$ and if $C_{B_2}(f(0), f(w)) = C_{B_1}(0, w)$ for every $w \in B_1$, then f is a linear $\|\cdot\|$-isometry, where B_1 and B_2 are the open unit balls for normed spaces E_1 and E_2 over $\mathbb{C}$.

(*) Nella seduta del 7 febbraio 1997.

In this paper, we give an infinite dimensional version of the above classical Schwarz lemma as follows:

MAIN THEOREM. *Let E be a complex Banach space, let B be the unit ball of E, and let $f\colon B \to B$ be a holomorphic map such that $f(0) = 0$. We assume that every boundary point of B in E is a complex extreme point of the closure $\overline{B}$ of B. Let X be a non-empty subset of B such that X is mapped homeomorphically onto an open subset Ω of the complex projective space $\mathbb{P}(E)$ by the quotient map from E onto $\mathbb{P}(E)$. If $C_B(f(0), f(w)) = C_B(0, w)$ for every $w \in X$, then f is a linear isometry on E.*

If $E = \mathbb{C}$, $\mathbb{P}(E)$ is the set of only one element. This main theorem contains the part (2) of the classical Schwarz lemma.

2. NOTATIONS AND PRELIMINARIES

Let $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. The Poincaré distance ϱ on Δ is defined as follows:

$$\varrho(z, w) = (1/2) \log (1 + |(z - w)/(1 - z\overline{w})|)/(1 - |(z - w)/(1 - z\overline{w})|) \quad (z, w \in \Delta).$$

Let D_1 and D_2 be domains in complex Banach spaces. We denote by $H(D_1, D_2)$ the set of all holomorphic mappings on D_1 into D_2. Let E be a complex Banach space, and let D be a domain in E. The Carathéodory distance C_D on D is defined as follows:

$$C_D(p, q) = \sup\{\varrho(f(p), f(q)); f \in H(D, \Delta)\} \qquad (p, q \in D).$$

A mapping $g \in H(\Delta, D)$ is called a complex geodesic if $C_D(g(z), g(w)) = \varrho(z, w)$ (for all $z, w \in \Delta$).

THEOREM 1 [11, 4]. *Let E be a complex Banach space, and let D be a convex domain in E. A mapping $g \in H(\Delta, D)$ is a complex geodesic if and only if there exist distinct points $z, w \in \Delta$ such that $C_D(g(z), g(w)) = \varrho(z, w)$.*

THEOREM 2 [11]. *Let E_1 and E_2 be two locally convex, locally bounded, complex vector spaces. Let D_1 and D_2 be two bounded, convex, balanced open neighborhoods of 0 in E_1 and E_2, and let $f\colon D_1 \to D_2$ be a holomorphic map such that $f(0) = 0$. We assume that every boundary point of D_2 is a complex extreme point of the closure $\overline{D_2}$ of D_2. If $C_{D_2}(f(0), f(w)) = C_{D_1}(0, w)$ holds for all $w \in D_1$, then f is a linear map of E_1 into E_2.*

3. MAIN RESULTS

PROPOSITION 3 [6]. *Let E be a complex Banach space with the norm $\|\cdot\|$, let B be the open unit ball of E for the norm $\|\cdot\|$. Then $C_B(0, x) = C_\Delta(0, \|x\|)$ for every $x \in B$.*

Let f be a holomorphic map from B to B such that $f(0) = 0$. By Proposition 3 and the distance decreasing property of the Carathéodory distances, we have $C_\Delta(0, \|z\|) = C_B(0, z) \geqslant C_B(0, f(z)) = C_\Delta(0, \|f(z)\|)$ for all $z \in B$. Since $C_\Delta(0, r)$ is strictly increasing for $0 \leq r < 1$, we obtain that $\|f(z)\| \leq \|z\|$ for $z \in B$. This is a generalization of part (1) of the classical Schwarz lemma.

PROPOSITION 4. *Let E be a complex Banach space with the norm $\|\cdot\|$, let B be the open unit ball of E for the norm $\|\cdot\|$, and let $f\colon B \to B$ be a holomorphic map such that $f(0) = 0$. We assume that every boundary point of B is a complex extreme point of the closure $\overline{B}$ of B. Let U be a non-empty open subset of B. If $C_B(f(0), f(w)) = C_B(0, w)$ for every $w \in U$, then f is a linear isometry on E.*

PROOF. By Theorem 2, f is linear on E. So we show that f is injective.

By Proposition 3, the conditions $\|f(w)\| = \|w\|$ and $C_B(f(0), f(w)) = C_B(0, w)$ are equivalent.

Let z be a point of E with $f(z) = 0$ and let $w \neq 0$ be a point of U. Since U is open, there exists a positive number $r > 0$ such that $w + \zeta z \in U$ for $\zeta \in \mathbb{C}$, $|\zeta| < r$. Then

$$\|f(w + \zeta z)\| = \|w + \zeta z\|. \tag{4.1}$$

On the other hand,

$$\|f(w + \zeta z)\| = \|f(w) + \zeta f(z)\| = \|f(w)\| = \|w\|. \tag{4.2}$$

By (4.1) and (4.2), we have $\|w + \zeta z\| = \|w\|$. So

$$\left\| \frac{w}{\|w\|} + \frac{\zeta}{\|w\|} z \right\| = 1 \text{ for } |\zeta| < r.$$

Since $w/\|w\|$ is a complex extreme point of $\overline{B}$, we have $z = 0$. Therefore f is injective. ■

Now we introduce the projective space $\mathbb{P}(E)$. Let E be a Banach space. Let z and z' be points in $E \backslash \{0\}$. z and z' are said to be *equivalent* if there exists $\lambda \in \mathbb{C}^*$ such that $z = \lambda z'$. We denote by $\mathbb{P}(E)$ the quotient space of $E \backslash \{0\}$ by this equivalence relation. Then $\mathbb{P}(E)$ is a Hausdorff space. The Hausdorff space $\mathbb{P}(E)$ is called the *complex projective space induced by E*. We denote by Q the quotient map from $E \backslash \{0\}$ to $\mathbb{P}(E)$.

THEOREM 5 (Main theorem). *Let E be a complex Banach space with the norm $\|\cdot\|$, let B be the open unit ball of E for the norm $\|\cdot\|$, and $f\colon B \to B$ a holomorphic map such that $f(0) = 0$. We assume that every boundary point of B is a complex extreme point of the closure $\overline{B}$ of B. Let X be a non-empty subset of B such that X is mapped homeomorphically onto an open subset Ω of the projective space $\mathbb{P}(E)$ by the quotient map Q. If $C_B(f(0), f(w)) = C_B(0, w)$ holds for every $w \in X$, then f is a linear isometry on E.*

PROOF. We take a point $w \neq 0$, $w \in X$. We set $\varphi(\zeta) = \zeta(w/\|w\|)$ for $\zeta \in \Delta$. Then φ is a complex geodesic of B. We have $C_B(f \circ \varphi(0), f \circ \varphi(\|w\|)) = C_\Delta(0, \|w\|)$. By Theorem 1, $f \circ \varphi$ is a complex geodesic of B. So there exists a point $y \in B \backslash \{0\}$ such that

$$f \circ \varphi(\zeta) = \zeta(y/\|y\|). \tag{5.1}$$

(see *e.g.* [10-12]). On the other hand, let $f(x) = \sum_{n=1}^{\infty} P_n(x)$ be the development of f by n-homogeneous continuous polynomials P_n in a neighborhood V of 0 in E. Then we have

$$f \circ \varphi(\zeta) = \sum_{n=1}^{\infty} P_n\left(\zeta \frac{w}{\|w\|}\right) = \sum_{n=1}^{\infty}\left(\frac{\zeta}{\|w\|}\right)^n P_n(w) \tag{5.2}$$

in a neighborhood of 0 in Δ. By (7.1) and (7.2), we obtain $P_n(w) = 0$ for $w \in X$, $n \geqslant 2$.

We take any point $y \in \mathbb{C}^* X = \{tx;\ t \in \mathbb{C}^*,\ x \in X\}$. Then there exist $t \in \mathbb{C}^*$ and $x \in X$ such that $y = tx$. So

$$P_n(y) = P_n(tx) = t^n P_n(x) = 0\,.$$

Thus $P_n \equiv 0$ on $\mathbb{C}^* X \subset E$ for every $n \geqslant 2$. Since Q is continuous, the set $\mathbb{C}^* X = Q^{-1}(\Omega)$ is an open subset of E. By the identity theorem, $P_n \equiv 0$ on E for every $n \geqslant 2$. Therefore $f = P_1$ on B. So we have $\|f(tx)\| = \|tf(x)\| = |t|\,\|f(x)\| = |t|\,\|x\| = \|tx\|$ for every $t \in \mathbb{C}^*$, $x \in X$. Then $\|f(y)\| = \|y\|$ for all $y \in \mathbb{C}^* X$. By Proposition 4, f is a linear isometry on E. ■

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