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Generalization of Fueter's result to $\mathbb{R}^{n+1}$

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Funzioni di variabile complessa. — *Generalization of Fueter's result to* $\boldsymbol{R}^{n+1}$. Nota di TAO QIAN, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Fueter's result (see [6, 8]) on inducing quaternionic regular functions from holomorphic functions of a complex variable is extended to Euclidean spaces $\boldsymbol{R}^{n+1}$. It is then proved to be consistent with M. Sce's generalization for n being odd integers [6].

KEY WORDS: Clifford analysis; Harmonic analysis; Complex analysis; Singular integrals; Fourier multiplier.

RIASSUNTO. — *Generalizzazione del risultato di Fueter allo spazio* $\boldsymbol{R}^{n+1}$. Il risultato di Fueter [6, 8] sulle funzioni regolari determinate in funzioni olomorfe di una variabile complessa viene esteso allo spazio euclideo $\boldsymbol{R}^{n+1}$. Viene poi dimostrata, per n intero dispari, la compatibilità con la generalizzazione di M. Sce [6].

We will be working in $\boldsymbol{R}^{n+1}$, the real-linear span of $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, where $\boldsymbol{e}_0$ is identical with 1 and $\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = -2\delta_{ij}$. $\boldsymbol{R}^{n+1}$ is embedded into the Clifford algebra $\boldsymbol{R}^{(n)}$ generated by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. A typical element in $\boldsymbol{R}^{n+1}$ is denoted $x = x_0 + \underline{x}$, where $x_0 \in \boldsymbol{R}$ and $\underline{x} = x_1\boldsymbol{e}_1 + \ldots + x_n\boldsymbol{e}_n$, $x_j \in \boldsymbol{R}$. If $x \neq 0$, then its inverse x^{-1} exists: $x^{-1} = \bar{x}|x|^{-2}$, where $\bar{x} = x_0 - \underline{x}$. We will study $\boldsymbol{R}^{n+1}$-variable and Clifford-valued functions and the concepts of left- and right-monogeneity are introduced via the Dirac operator $D = \dfrac{\partial}{\partial_0} + \boldsymbol{e}_1 \dfrac{\partial}{\partial x_1} + \ldots + \boldsymbol{e}_n \dfrac{\partial}{\partial x_n}$ in the usual way. In this *Note*, a function is said to be monogenic if it is both left- and right-monogenic. The Cauchy kernel stands for $E(x) = \bar{x}/|x|^{n+1}$ and the Kelvin inversion of a function f is $I(f)(x) = E(x)f(x^{-1})$. The symbol $\boldsymbol{Z}$ and $\boldsymbol{Z}^+$ denote the sets of all integers and positive integers, respectivley.

We will use Fourier transform of functions f on $\boldsymbol{R}^{n+1}$ defined by

$$\mathcal{F}(f)(\xi) = \int_{\boldsymbol{R}^{n+1}} e^{2\pi i\langle x, \xi\rangle} f(x)\, dx\,,$$

and the result (see [7])

$$\mathcal{F}\left(\frac{P_k(\cdot)}{|\cdot|^{k+n+1-\alpha}}\right)(\xi) = \gamma_{k,\alpha}\,\frac{P_k(\xi)}{|\xi|^{k+\alpha}}\,, \tag{1}$$

(*) Nella seduta del 7 marzo 1997.

where $0 < \alpha < n+1$, $k \in \boldsymbol{Z}^+$, P_k is a homogeneous harmonic polynomial of degree k, and

$$\gamma_{k,\alpha} = i^k \pi^{(n+1)/2-\alpha}\,\Gamma(k/2+\alpha/2)/\Gamma(k/2+(n+1)/2-\alpha/2),$$

(Γ denotes the ordinary gamma function).

The inverse Fourier transform of a function g, denoted by $\mathcal{R}(g)$, is defined by

$$\mathcal{R}(g) = \int_{\boldsymbol{R}^{n+1}} e^{-2\pi i\langle x,\xi\rangle} g(\xi)\,d\xi\,.$$

Functions in the Schwarz class have Fourier transforms which are still in the class, and in this case the Fourier inversion formula holds: $\mathcal{R}\mathcal{F}(f) = f$. In the sequel, Fourier and inverse Fourier transforms will be used in the distribution sense.

For a function g defined in $\boldsymbol{R}^{n+1}$ one can introduce a Fourier multiplier transform M_g by $M_g f = \mathcal{R}(g\mathcal{F}f)$. It is easy to verify that the Laplace differential operator is identical with the Fourier multiplier transform induced by $-4\pi^2|\xi|^2$.

Let f^0 be a complex-valued function defined in an open set O in the upper half complex plane. Write $f^0 = u + iv$, where u and v are real-valued. Denote, for $x \in \vec{O}$,

$$\vec{f}^{\,0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|),$$

where

$$\vec{O} = \{x \in \boldsymbol{R}^{n+1} : (x_0\, |\underline{x}|) \in O\}.$$

$\vec{f}^{\,0}$ is said to be the *induced function* from f^0, and $\vec{O}$ the *induced set* from O.

We will be working with functions of the form

$$g(x) = p(x_0, |\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|} q(x_0, |\underline{x}|),$$

where p and q are real-valued. We will call p and q the real and the imaginary parts of g, respectively.

The concepts of intrinsic sets and functions naturally fit into our theory. In the complex plane $\boldsymbol{C}$ a set is said to be *intrinsic*, if it is open and symmetric with respect to the real axis; and a function f^0 an *intrinsic function* if it is defined in an intrinsic set and satisfies $\overline{f^0(z)} = f^0(\bar{z})$ within its domain (see [5]). In the notation $f^0 = u + iv$, the above condition is equivalent to requiring that u is even and v is odd in their second argument. In particular, $v(x_0, 0) = 0$, *i.e.* f^0 is real-valued if restricted to the real line in its domain.

Denote by τ the mapping

$$\tau(f^0) = \Delta^{(n-1)/2}\vec{f}^{\,0},$$

where f^0 is any holomorphic intrinsic function and the differential operation in the distribution sense. Here we adopt the convention that $\vec{f}^{\,0} = 0$ outside the induced set $\vec{O}$.

Note that for $n \in \boldsymbol{Z}^+$ being odd, the operator $\Delta^{(n-1)/2}$ is a pointwise differential operator, while for $n \in \boldsymbol{Z}^+$ being even, it is the Fourier multiplier operator induced by $(2\pi i|\xi|)^{n-1}$ mapping some functions to merely distributions.

OBSERVATION. *If h is a complex function defined in an intrinsic set, then $g^0(z) = (h(z) + \bar{h}(\bar{z}))/2$, and $h^0(z) = (h(z) - \bar{h}(\bar{z}))/2i$ both are intrinsic, defined in the same set, and $h = g^0 + ih^0$.*

The observation enables us to extend the domain of τ to complex functions h defined in intrinsic sets but not necessarily intrinsic themselves. We define, for such a function h: $\tau(h) = \tau(g^0) + i\tau(h^0)$. The mapping τ extended in such a way is linear under addition and real-scalar multiplication. By virtue of the observation and the extended definition of τ we may concentrante only on holomorphic intrinsic functions. A characterization of such functions is that coefficients of their Laurent series expansions in annuli centered at real points in their domains are all real. Our study (see [2, 4]) shows that it is essential to concentrate in the functions $\tau((\cdot)^{-k})$, $k \in \boldsymbol{Z}$.

Define, for $k \in \boldsymbol{Z}^+$,

$$P^{(-k)} = \tau((\cdot)^{-k}), \qquad P^{(k-1)} = I(P^{(-k)}).$$

We have

THEOREM 1. *Let $k \in \boldsymbol{Z}^+$. Then* (i) *$P^{(-k)}$ and $P^{(k-1)}$ both are monogenic;* (ii) *$P^{(-k)}$ is homogeneous of degree $-n+1-k$ and $P^{(k-1)}$ is homogeneous of degree $(k-1)$;* (iii) *if n is odd, then $P^{(k-1)} = \tau((\cdot)^{n+k-2})$.*

The underlying idea of Theorem 1 is to explore the similarity between Clifford analysis and complex analysis of one variable. Some close similarity between the quaternionic and the complex analysis of one variable has been established in [4] via the corresponding relation $z^k \to P^{(k)}$. An analogous theory for general Clifford algebras is studied in [2].

In the quaternionic space, which is not quite identical with our case $n = 3$ for the former is a full algebra but the latter is not, Fueter's result states that τ maps holomorphic functions of a complex variable to quaternionic regular functions (see *e.g.* [8]). M. Sce proved that, if n is odd, then τ maps holomorphic functions defined in open sets of the upper half complex plane to Clifford monogenic functions [6] which generalizes Fueter's result. The assertion (*iii*) shows that, for n being odd, our generalization through the Kelvin inversion is consistent with Sce's on the functions $f^0(z) = z^k$, $k \in \boldsymbol{Z}$. For n being even, however, Fueter or Sce's device in terms of the differential operator $\Delta^{(n-1)/2}$ can only be generalized, using Fourier multiplier transform, to the power functions of negative powers, *i.e.* to $f^0(z) = z^k$, $-k \in \boldsymbol{Z}^+$; while the other half corresponding to nonnegative powers cannot be generalized using the differential operator. This can be seen, *e.g.* from the following example: for $f^0(z) = z$, $\tau(z)$ is a distribution but not a function. The setting of using the Kelvin inversion to define $P^{(k)}$, $k \geqslant 0$, is suggested by the author's earlier work [4] where the setting is used for the convenience of proving estimates. Having proved the assertions (*i*) and (*ii*) the author got to know through J. Ryan about Sce's result and the reference [6]. The proof of (*iii*) is similar to the proof of the analogous relation in the quaternions case (see [4]), but uses, in a slightly developed form, some techniques of [6]. It would be interesting to see alternative proofs of (*iii*) simpler than what we give in below, *e.g.* using conformal covariance of intertwining differential operators like those used in [3].

PROOF. (*i*) Using the Fourier transform result cited above and the relation

$$\overrightarrow{(\cdot)^{-k}}(x) = \left(\frac{\bar{x}}{|x|^2}\right)^k = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\left(\frac{\bar{x}}{|x|^2}\right),$$

we have

$$P^{(-k)}(x) = \tau((\cdot)^{-k})(x) = \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{R}\mathcal{F}\left(\Delta^{(n-1)/2}\frac{\bar{x}}{|x|^2}\right) =$$

$$= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1} \mathcal{R}\left(\gamma_{1,n}(2\pi i|\xi|)^{n-1}\frac{\bar{\xi}}{|\xi|^{1+n}}\right) =$$

$$= \frac{(-1)^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial x_0}\right)^{k-1}\gamma_{1,n}^2(2\pi i)^{n-1}\frac{\bar{x}}{|x|^{1+n}} = \frac{(-1)^{k-1}}{(k-1)!}\kappa_n\left(\frac{\partial}{\partial x_0}\right)^{k-1}E(x),$$

where we let $\kappa_n = (2\pi i)^{n-1}\gamma_{1,n}^2 = (2i)^{n-1}\Gamma^2((n+1)/2)$. This implies that $P^{(-k)}$ are monogenic for $k \in \mathbf{Z}^+$. The monogeneity of $P^{(k)}$ is from the property of the Kelvin inversion (see [1]) or a special case of Bojarski's result on intertwining Dirac operators, for which we refer the reader to [3].

(*ii*) It is a consequence of the expression of $P^{(-k)}$ obtained above and the property of the Kelvin inversion.

(*iii*) Let $n = 2m+1$. We have $\kappa_n = (-1)^m 2^{2m}(m!)^2 = (-1)^m((2m)!!)^2$. We will use the mathematical induction. The case $k=1$ reduces to verifying $\Delta^m(x^{2m}) = = (-1)^m(2m)!!$. The following lemma will be used.

LEMMA 1. *Let $f^0(z) = u(x_0, y) + iv(x_0, y)$ be a function holomorphically defined in an open set U of the upper half complex plane. Denote $u_0 = u$, $v_0 = v$, and, for $s \in \mathbf{Z}^+$,*

$$u_s = 2s\frac{\partial u_{s-1}}{\partial y}\frac{1}{y}, \qquad v_s = 2s\left(\frac{\partial v_{s-1}}{\partial y}\frac{1}{y} - \frac{v_{s-1}}{y^2}\right) = 2s\frac{\partial}{\partial y}\frac{v_{s-1}}{y}.$$

Then

$$\Delta^s\overrightarrow{f}^0(x) = u_s(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v_s(x_0, |\underline{x}|), \qquad x_0 + i|\underline{x}| \in U.$$

The lemma may be proved using the mathematical induction via a computation of $\Delta(u_{s-1} + iv_{s-1})$ invoking the following relation proved in [6]:

$$\frac{\partial u_{s-1}}{\partial x_0} = \frac{\partial v_{s-1}}{\partial y} + 2(s-1)\frac{v_{s-1}}{y}, \qquad \frac{\partial u_{s-1}}{\partial y} = -\frac{\partial v_{s-1}}{\partial x_0}.$$

We will frequently use the following formula given in [6]: For any function

$f^0 = u + iv$ and $r \in \boldsymbol{Z}^+$,

$$(2) \qquad (\vec{f}^{\,0})^r(x) = \sum_{l=0}^{[r/2]} (-1)^l \binom{r}{2l} u^{r-2l} v^{2l} + \frac{\underline{x}}{|\underline{x}|} \sum_{l=0}^{[r/2]} (-1)^l \binom{r}{2l+1} u^{r-2l-1} v^{2l+1},$$

where $\binom{r}{l}$ are binomial coefficients with the convention that $\binom{r}{l} = 0$ for $l > r$, and $[s]$ the largest integer that does not exceed s.

Using formula (2) for $f^0(z) = z$, $r = 2m$ and Lemma 1, one immediately deduces $\Delta^m(x^{2m}) = (-1)^m((2m)!!)^2$ which proves the case $k = 1$. Now assuming $P^{(k)} = \tau((\cdot)^{n+k-1})$, we need to show $P^{(k+1)} = \tau((\cdot)^{n+k})$. The verification is a bit crumblesome. First, this reduces to proving

$$(3) \qquad \frac{-1}{k+1} \frac{\partial}{\partial x_0}(I(\Delta^m((\cdot)^{2m+k}))) = I(\Delta^m((\cdot)^{2m+k+1})),$$

where $k \in \boldsymbol{Z}^+$ or $k = 0$.

Using formula (2) and Lemma 1, we have

$$\Delta((\cdot)^{2m+k})(x) =$$

$$= (2m)!! \left[\sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2)\ldots(2l-2m+2) x_0^{2m+k-2l} y^{2l-2m} + \right.$$

$$\left. + \frac{\underline{x}}{y} \sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l+1} (2l)(2l-2)\ldots(2l-2m+2) x_0^{2m+k-2l-1} y^{2l+1-2m} \right],$$

where we have put $y = |\underline{x}|$.

By applying the Kelvin inversion, *i.e.* replacing x_0, y and $\underline{x}/y$ by $x_0|x|^{-2}$, $y|x|^{-2}$ and $-\underline{x}/y$, respectively, the above becomes

$$(4) \qquad (2m)!! \frac{\bar{x}}{|x|^{n+2k+1}} \cdot$$

$$\cdot \left[\sum_{l=0}^{m+[k/2]} (-1)^l \binom{2m+k}{2l} (2l)(2l-2)\ldots(2l-2m+2) x_0^{2m+k-2l} y^{2l-2m} + \right.$$

$$\left. + \frac{\underline{x}}{y} \sum_{l=0}^{m+[k/2]} (-1)^{l+1} \binom{2m+k}{2l+1} (2l)(2l-2)\ldots(2l-2m+2) x_0^{2m+k-2l-1} y^{2l+1-2m} \right].$$

Applying the differential operator $[-1/(k+1)]\,\partial/\partial x_0$ to the expression (4), we obtain

$$(5) \qquad \frac{-(2m)!!}{k+1} E(x) \frac{1}{|x|^{2k+2}} \left\{ \left(-(n+2k)x_0 + \frac{\underline{x}}{y} y \right) [\ldots] + (x_0^2 + y^2) \frac{\partial}{\partial x_0} [\ldots] \right\},$$

where $[\ldots]$ is as $[\ldots]$ in (4).

Now

$$\left(-(n+2k)x_0+\frac{x}{y}y\right)[\ldots]=$$

$$=\left\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}\binom{2m+k}{2l}(n+2k)(2l)(2l-2)\ldots(2l-2m+2)x_0^{2m+k-2l+1}y^{2l-2m}+\right.$$

$$\left.+\sum_{l=0}^{m+[k/2]}(-1)^{l}\binom{2m+k}{2l+1}(2l)(2l-2)\ldots(2l-2m+2)x_0^{2m+k-2l-1}y^{2l+1-2m+1}\right\}+$$

$$+\frac{x}{y}\left\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}\binom{2m+k}{2l+1}(n+2k)(2l)(2l-2)\ldots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}+\right.$$

$$\left.+\sum_{l=0}^{m+[k/2]}(-1)^{l}\binom{2m+k}{2l}(2l)(2l-2)\ldots(2l-2m+2)x_0^{2m+k-2l}y^{2l+1-2m}\right\},$$

and,

$$(x_0^2+y^2)\frac{\partial}{\partial x_0}[\ldots]=$$

$$=\left\{\sum_{l=0}^{m+[k/2]}(-1)^{l}\binom{2m+k}{2l}(2l)(2l-2)\ldots(2l-2m+2)(2m+k-2l)\cdot\right.$$

$$\left.\cdot(x_0^{2m+k-2l+1}y^{2l-2m}+x_0^{2m+k-2l-1}y^{2l-2m+2})\right\}+$$

$$+\frac{x}{y}\left\{\sum_{l=0}^{m+[k/2]}(-1)^{l+1}\binom{2m+k}{2l+1}(2l)(2l-2)\ldots(2l-2m+2)(2m+k-2l-1)\cdot\right.$$

$$\left.\cdot(x_0^{2m+k-2l}y^{2l+1-2m}+x_0^{2m+k-2l-2}y^{2l+1-2m+2})\right\}.$$

By comparing the coefficients of a general nomial $x_0^{2m+k+1-2l}y^{2l-2m}$ in the real part of (5) with those in the real part of $I(\Delta^m((\cdot)^{2m+k+1}))(x)=E(x)(\Delta^m((\cdot)^{2m+k+1}))(x^{-1})$, the later being of the expression (4) but with $k+1$ in place of k, we are reduced to verifying

$$(6)\qquad -2l(n+2k)\binom{2m+k}{2l}+(2m-2l)\binom{2m+k}{2l-1}+2l(2m+k-2l)\binom{2m+k}{2l}+$$
$$+(2m-2l)(2m+k-2l+2)\binom{2m+k}{2l-2}=-(k+1)2l\binom{2m+k+1}{2l}.$$

Using the relation

$$(s-l)\binom{s}{l}=(l+1)\binom{s}{l+1},$$

the second and fourth entries on the left hand side of (6) add up to

$$2l(2m-2l)\binom{2m+k}{2l-1}, \tag{7}$$

while the first and third to

$$-2l(2l+k+1)\binom{2m+k}{2l} = [-4l^2-2l(k+1)]\binom{2m+k}{2l} = \tag{8}$$
$$= -2l(2m+k-2l+1)\binom{2m+k}{2l-1} - 2l(k+1)\binom{2m+k}{2l}.$$

Combining (7) with the right hand side of (8) and using the relation

$$\binom{s}{l} + \binom{s}{l-1} = \binom{s+1}{l},$$

we obtain (6). The verification of the equality between the imaginary part of (5) and that of $I(\Delta^m((\cdot)^{2m+k+1}))$ is similar, and the proof of *(iii)* is complete.

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