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Differentiability of the Feynman-Kac semigroup and a control application

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Analisi matematica. — *Differentiability of the Feynman-Kac semigroup and a control application.* Nota di GIUSEPPE DA PRATO e JERZY ZABCZYK, presentata (*) dal Corrisp. G. Da Prato.

ABSTRACT. — The Hamilton-Jacobi-Bellman equation corresponding to a large class of distributed control problems is reduced to a linear parabolic equation having a regular solution. A formula for the first derivative is obtained.

KEY WORDS: Stochastic control problem; Feynman-Kac formula; Hamilton-Jacobi equations.

RIASSUNTO. — *Differenziabilità del semigruppo di Feynman-Kac e applicazioni*. L'equazione di Hamilton-Jacobi-Bellman corrispondente a un'ampia classe di problemi di controllo distribuiti viene ridotta a una equazione parabolica lineare avente una soluzione regolare. Viene inoltre ottenuta una formula per la derivata prima della soluzione.

1. INTRODUCTION

The *Note* is concerned with a distributed control system in which the *control action is perturbed by noise*. To write down the system assume that H is a separable Hilbert space and $A: D(A) \subset H \to H$, $F: H \to H$, $B: H \to L(H, H)$ are mappings such that:

$A1)$ Operator A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$,

$A2)$ Mappings F and B are Lipschitz and

$$\sup_{x \in H} (\|B(x)\| + \|B^{-1}(x)\|) < +\infty .$$

Let, in addition, $W(t)$, $t \geq 0$, be a cylindrical Wiener process on H, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}\}_t$. The control system under consideration is described by the equations:

$$dX = (AX + F(X))\,dt + B(X)(zdt + \sigma dW(t)), \qquad X(0) = x, \tag{1.1}$$

where $z(t)$, $t \geq 0$, is a control process and σ is a constant. Note that if $\sigma = 0$, the equation (1.1) is deterministic and the action of the controller is not affected by random perturbations. The effect of the noise is present if $\sigma \neq 0$.

Assume that the cost functional is of the form

$$J_t(x, z) = \mathbb{E}\left(\int_0^t [g(X(s,x)) + |z(s)|^2]\,ds + \psi(X(t,x)) \right),$$

where $X(\cdot, x)$ denotes the solution to (1.1) and g and ψ are given functions. The corre-

(*) Nella seduta del 19 giugno 1997.

sponding Hamilton-Jacobi-Bellman equation for the value function

$$V(t,x) = \inf_{z(\cdot)} J_t(x,z),$$

is then

$$(1.2) \qquad \begin{cases} V_t(t,x) = (\sigma^2/2)\operatorname{Tr}[B(x)B^*(x)V_{xx}(t,x)] + \langle Ax + F(x), V_x(t,x)\rangle - \\ \qquad\qquad - |B^*(x)V_x(t,x)|^2/4 + g(x), \\ V(0,\cdot) = \psi . \end{cases}$$

It is not difficult to show that if equation (1.2) has a classical solution, then it can be identified with the value function and the control

$$(1.3) \qquad z(s) = -\sigma^2 B^*(X(s,x))\,V_x(t-s, X(s,x)), \qquad s \in [0,t[,$$

is an optimal one. However the classical solution does not exist in general. Under some condition one can prove existence of a viscosity solution, see [9-11, 13], which is however only continuous and the formula (1.3) loses its meaning. Another way of solving the problem was proposed for systems with diffusion B independent of x. This approach was introduced in [1, 2] and it was developped on in [6-8, 3]. One writes (1.2) in the so-called mild form:

$$(1.4) \qquad V(t,\cdot) = P_t\psi + \int_0^t P_{t-s}(\langle F, V_x(s,\cdot)\rangle - |B^* V_x(s,\cdot)|^2/4 + g)\,ds,$$

where P_t, $t \geq 0$ is the transition semigroup corresponding to (1.1) with $F = 0$, $\sigma = 0$. Under assumptions implying regularizing properties of P_t, $t \geq 0$ one shows existence of a solution to (1.4) for which V_x does exist. There are essential difficulties to extend this method to the case of B dependent of x. The corresponding semigroup P_t, $t \geq 0$ has regularizing properties under very strong assumptions. In the present paper we show that the so-called logarithmic transform can be used to obtain solution of class C^1.

Note that setting

$$(1.5) \qquad V(t,x) = -2\sigma^2 \ln u(t,x),$$

one arrives, after straightforward calculations, to a linear equation on u:

$$(1.6) \qquad \begin{cases} u_t(t,x) = (\sigma^2/2)\operatorname{Tr}[B(x)B^*(u)u_{xx}(t,x)] + \\ \qquad\qquad + \langle Ax + F(x), u_x(t,x)\rangle - g(x)u(t,x)/(2\sigma^2), \\ V(0,\cdot) = e^{-\psi/(2\sigma^2)} = \varphi, \end{cases}$$

which is of the Feynman-Kac type.

In the next section we will derive a formula for u_x giving a meaning to the feedback law (1.5). Some of the assumptions of our theorem could be removed as we intend to show in a future paper. We *conjecture* that if the functions g, F and B are Gateaux differentiable with bounded and weakly continuous derivatives, and φ is bounded continuous, then equation (1.6) has a solution and the optimal control is of the form (1.5).

2. The Feynman-Kac semigroup

Let g be a bounded and continuous function from H into $\mathbb{R}$. We shall denote by P_t^g, $t \geqslant 0$ the *Feynman-Kac semigroup*

$$(2.1)\qquad P_t^g \varphi(x) = E[\varphi(X(t,x)) e^{-\int_0^t g(X(s,x))\,ds}], \qquad t \geqslant 0,$$

for φ on the space $B_b(H)$ of bounded Borel functions. The function

$$u(t,x) = P_t^g \varphi(x), \qquad t \geqslant 0, \qquad x \in H,$$

is a candidate for a solution to equation (1.2) in which for simplicity, we set $\sigma = 1$ and replace $g/2$ by g.

We show that under conditions $A1$), $A2$) the semigroup P_t^g has regularizing properties similarly as in the finite dimensional case. The formula (2.2) below is new. It is well known in the special case $g = 0$, see [5, 12].

Theorem 2.1. *Assume that conditions* $A1$), $A2$) *hold. Assume moreover that* F, B, g *are twice differentiable functions with bounded and continuous derivatives up to the second order. If* $\varphi \in C_b(H)$ *then* $P_t^g \varphi$ *is differentiable in any direction* $h \in H$ *and*

$$(2.2)\qquad D_x^h P_t^g \varphi(x) = \mathbb{E}\left[\varphi(X(t,x)) e^{-\int_0^t g(X(\sigma,x))\,d\sigma} \left(t^{-1} \int_0^t \langle B^{-1}(X(s,x) X_x^h(s,x), dW(s)\rangle - \right.\right.$$

$$\left.\left. - \int_0^t (1 - s/t) \langle X_x^h(s,x), D_x g(X(s,x))\rangle\, ds \right)\right],$$

where D_x^h *denotes the derivative in the direction* h.

Proof. Let $\{e_n\}$ be a complete orthonormal system on H. For each $n \in \mathbb{N}$ let $X_n(\cdot, x)$ be the solution of the problem

$$(2.3)\qquad dX_n = (A_n X_n + F(X_n))\,dt + B(X_n) Q_n\, dW(t) \qquad X_n(0) = x,$$

where $A_n = nA(n - A)^{-1}$ is the Yosida approximation of A and Q_n is the orthogonal projection of H onto $\text{lin}\{e_1, \ldots, e_n\}$. Then the function

$$u_n(t,x) = \mathbb{E}[\varphi(X_n(t,x)) e^{-\int_0^t g(X_n(s,x))\,ds}],$$

is a strict solution of parabolic equation

$$D_t u_n(t,x) = \text{Tr}\,[B(x) Q_n B^*(x) D^2 u_n(t,x)]/2 +$$

$$+ \langle A_x + F(x), D^2 u_n(t,x)\rangle - g(x) u_n(t,x) = \mathcal{L}_n u_n(t,x)$$

$$u_n(0,x) = \varphi(x).$$

Fix $t > 0$. Applying Itô's formula to the process

$$e^{-\int_0^s g(X_n(\sigma,x))\,d\sigma} u_n(t - s, X_n(s,x)), \qquad s \in [0,t]$$

we have

$$de^{-\int_0^s g(X_n(\sigma,x))\,d\sigma} u_n(t-s, X_n(s,x)) =$$

$$= (-D_t u_n(t-s, X_n(s,x)) + \mathcal{L}_n u_n(t-s, X_n(s,x)))\, e^{-\int_0^s g(X_n(\sigma,x))\,d\sigma} ds +$$

$$+ \langle D_x u_n(t-s, X_n(s,x)), B(X_n(s,x))\, Q_n\, dW(s)\rangle\, e^{-\int_0^s g(X_n(\sigma,x))\,d\sigma}.$$

Therefore

$$e^{-\int_0^t g(X_n(\sigma,x))\,d\sigma} \varphi(X_n(t,x)) = u_n(t,x) +$$

$$+ \int_0^t e^{-\int_0^s g(X_n(\sigma,x))\,d\sigma} \langle D_x u_n(t-s, X_n(s,x)), B(X_n(s,x))\, Q_n\, dW(s)\rangle.$$

Letting n tend to infinity we obtain that

$$e^{-\int_0^t g(X(\sigma,x))\,d\sigma} \varphi(X(t,x)) = u(t,x) +$$

$$+ \int_0^t e^{-\int_0^s g(X(\sigma,x))\,d\sigma} \langle D_x u(t-s, X(s,x)), B(X(s,x))\, Q_n\, dW(s)\rangle.$$

Multiplying this identity by

$$\int_0^t \langle B^{-1}(X(s,x))\, X_x^h(s,x), dW(s)\rangle,$$

and taking expectation, we arrive at

$$K := \mathbb{E}\left(e^{-\int_0^t g(X(\sigma,x))\,d\sigma} \varphi(X(t,x)) \int_0^t \langle B^{-1}(X(s,x))\, X_x^h(s,x), dW(s)\rangle \right) =$$

$$= \mathbb{E}(e^{-\int_0^s g(X(\sigma,x))\,d\sigma} \langle D_x u(t-s, X(s,x)), X_x^h(s,x)\rangle).$$

On the other hand

$$K = D_x^h [e^{-\int_0^s g(X(\sigma,x))\,d\sigma} u(t-s, X(s,x))] =$$

$$= -e^{-\int_0^s g(X(\sigma,x))\,d\sigma} \int_0^s \langle D_x g(X(\sigma,x)), X_x^h(\sigma,x)\rangle\, d\sigma\, u(t-s, X(s,x)) +$$

$$+ \langle D_x u(t-s, X(s,x)), X_x^h(s,x)\rangle.$$

Therefore

$$K = \mathbb{E}\left[\left(\int_0^t D_x^h \left(e^{-\int_0^s g(X(\sigma,x))\,d\sigma} u(t-s, X(s,x))\right) ds\right)\right] +$$

$$+\mathbb{E}\left[\int_0^t e^{-\int_0^s g(X(\sigma,x))\,d\sigma} \left(\langle D_x g(X(\sigma,x), X_x^h(\sigma,x)\rangle d\sigma\right) u(t-s, X(s,x))\right].$$

Applying to both the expressions the Markov property we obtain that

$$K = D_x^h \int_0^t P_s^g (P_{t-s}^g \varphi)(x)\,ds +$$

$$+\mathbb{E}\left(\varphi(X(t,x))\, e^{-\int_0^t g(X(\sigma,x))\,d\sigma} \int_0^t \int_0^s \langle D_x g(X(\sigma,x)), X_x^h(\sigma,x)\rangle\, d\sigma\, ds\right) =$$

$$= tD_x^h P_t^g \varphi(x) + \mathbb{E}\left(\varphi(X(t,x))\, e^{\int_0^t g(X(\sigma,x))\,d\sigma} \int_0^t (t-\sigma)\langle D_x g(X(\sigma,x)), X_x^h(\sigma,x)\rangle\, d\sigma\right),$$

and the result follows for $\varphi \in C_b^2(H)$.

Let now φ be an arbitrary function from $C_b(H)$. Then there exists a sequence $\{\varphi_n\} \in C_b^2(H)$ uniformly bounded and pointwise convergent to φ. Let us fix $t > 0$, $x \in H$ and $h \in H$, and consider functions

$$v_n(\sigma) = P_t^g \varphi_n(x + \sigma h), \qquad 0 \leqslant \sigma \leqslant 1 .$$

Then

$$v_n'(\sigma) = D_x^h P_t^g \varphi_n(x + \sigma h), \qquad 0 \leqslant \sigma \leqslant 1 .$$

From the definition of v_n and the formula for $D_x^h P_t^g \varphi_n$ valid for $\{\varphi_n\} \in C_b^2(H)$ one easily gets that functions v_n, v_n' are convergent, in a bounded way, to $P_t^g \varphi(x + \sigma h)$ and to

$$\mathbb{E}\left[\varphi(X(t, x+\sigma h))\, e^{-\int_0^t g(X(\sigma, x+\sigma h))\,d\sigma} \left(t^{-1}\langle B^{-1}(X(s, x+\sigma h) X_x^h(s, x+\sigma h), dW(s)\rangle -\right.\right.$$

$$\left.\left. - \int_0^t (1 - s/t)\langle X_x^h(s, x+\sigma h), D_x g(X(s, x+\sigma h))\rangle\, ds\right)\right],$$

for $\sigma \in]0, 1]$. Taking $\sigma = 0$ one gets that the formula for the directional derivative is true in general. ■

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