Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Russell Johnson, Mikhail Kamenskii, Paolo Nistri

On the existence of periodic solutions of an hyperbolic equation in a thin domain

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **8** (1997), n.3, p. 189–195.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1997_9_8_3_189_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1997.

Rend. Mat. Acc. Lincei
s. 9, v. 8:189-195 (1997)

Equazioni a derivate parziali. — *On the existence of periodic solutions of an hyperbolic equation in a thin domain.* Nota di RUSSELL JOHNSON, MIKHAIL KAMENSKII e PAOLO NISTRI, presentata (*) dal Socio E. Magenes.

ABSTRACT. — For a nonlinear hyperbolic equation defined in a thin domain we prove the existence of a periodic solution with respect to time both in the non-autonomous and autonomous cases. The methods employed are a combination of those developed by J. K. Hale and G. Raugel and the theory of the topological degree.

KEY WORDS: Hyperbolic nonlinear equations; Periodic solutions; Topological degree.

RIASSUNTO. — *Esistenza di soluzioni periodiche di una equazione iperbolica in un dominio sottile.* Si prova l'esistenza di soluzioni periodiche di un'equazione iperbolica smorzata definita in un dominio sottile sia nel caso autonomo che in quello non autonomo. I metodi impiegati sono una combinazione di quelli sviluppati da J. K. Hale e G. Raugel e la teoria del grado topologico.

1. INTRODUCTION

The aim of this *Note* is to present an existence result for T-periodic solutions with respect to t of a damped wave equation in a thin domain both in the non-autonomous and autonomous cases.

The considered equation in the non-autonomous case is of the form:

$$\frac{\partial^2 u}{\partial t^2} = \Delta_X u + \frac{\partial^2 u}{\partial Y^2} - \beta \frac{\partial u}{\partial t} - \alpha u + g(t, X, Y, u), \tag{1}$$

where α and β are positive constants and g is a suitable smooth function, which we assume T-periodic in time. (X, Y) is a generic point of the thin domain $Q_\varepsilon = \Omega \times (0, \varepsilon) \subset \boldsymbol{R}^{N+1}$, where Ω is a C^2-smooth bounded domain in $\boldsymbol{R}^N$ and $\varepsilon \in (0, \varepsilon_0)$ is a small parameter.

Associated to equation (1) we consider the Neumann boundary condition

$$\frac{\partial u}{\partial \nu_\varepsilon} = 0 \quad \text{on } \partial Q_\varepsilon . \tag{2}$$

By using topological degree arguments for compact operators we can show the existence of a T-periodic solution to (1)-(2). The main assumption is that the reduced problem at $\varepsilon = 0$ has an isolated T-periodic solution whose topological index is different from zero. Then suitable admissible homotopies allow us to derive the existence of a T-periodic solution of (1)-(2) for sufficiently small $\varepsilon > 0$. It must be noted that in the autonomous case, that is when g is independent of t, the period of the resulting periodic solution of (1)-(2) will be in general different from that of the periodic solution of the reduced system. This solution is not isolated in the autonomous case. Furthermore, in this case additional assumptions are required on the linearized reduced equation. These assumptions

(*) Nella seduta del 18 aprile 1997.

permit to apply the method of functionalization of a parameter (see [3, and references therein]).

The behavior of the period and of the periodic solution when $\varepsilon \to 0$ are also described. The use of topological degree methods is nontrivial in these cases because of the presence of the small parameter $\varepsilon > 0$ tending to zero. In this *Note* we combine these methods with techniques of Hale and Raugel developed to study the properties of the attractor A_ε defined by problem (1), when g does not depend on t, under various boundary conditions (see [4] and also [1, 2, 5, 6, 11]).

This *Note* is organized as follows. In Section 2 we give our existence result for the non-autonomous case. In Section 3 we give the existence result when the nonlinear term g does not depend on t. In both the theorems we only sketch the proof, which will be presented with all the preliminaries and details in two forthcoming papers [7, 8].

2. The non-autonomous case

The basic idea is to convert problem (1)-(2) to a fixed point problem in a suitable Banach space, and study it using the theory of the topological degree. First, following [4], we consider, for fixed $\varepsilon > 0$, the change of variables $X = x$, $Y = \varepsilon y$. The equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta\frac{\partial u}{\partial t} - \alpha u + g(t, x, \varepsilon y, u)\,, \tag{3}$$

with boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial Q\,. \tag{4}$$

Here $Q = \Omega \times (0, 1)$ and ν denotes the outward unit normal vector to ∂Q. We assume that Ω is a C^2-smooth domain.

Following [4], we introduce the following Banach spaces when $\varepsilon > 0$. Let X^1_ε be the space $H^1(Q)$ with the norm

$$\left(\|u\|^2_{1Q} + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{0Q}\right)^{1/2}.$$

Here and below, $\|\cdot\|_{0Q}$ denotes the norm in $L^2(Q)$, and $\|\cdot\|_{1Q}$ that in $H^1(Q)$. Let $U_\varepsilon(t)$ be the semigroup generated by the system of linear equations

$$\frac{\partial u}{\partial t} = v\,, \qquad \frac{\partial v}{\partial t} = \Delta_x u + \frac{1}{\varepsilon^2}\frac{\partial^2 u}{\partial y^2} - \beta v - \alpha u \tag{5}$$

with the boundary condition (4). It is known (see [4, 6]) that $U_\varepsilon(t)$ is a C_0-semigroup in the space $Y^1_\varepsilon \triangleq X^1_\varepsilon \times L^2(Q) \ni (u, v)$.

In the somewhat more general problem considered in [4], this space is defined in another way which yields, however, the space Y^1_ε in the case we are considering. One has the exponential estimate:

$$\|U_\varepsilon(t)\|_{Y_\varepsilon^1 \to Y_\varepsilon^1} \leqslant c e^{-\gamma t} \qquad (t \geqslant 0)$$

where c and γ are positive constants.

If $u \in L^2(Q)$, define its projection by

$$(Pu)(x) = \int_0^1 u(x, y)\, dy \qquad \text{then define } \boldsymbol{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}.$$

Then P maps $L^2(Q)$ to $L^2(\Omega)$, $\boldsymbol{P}$ maps Y_ε^1 to $H^1(\Omega) \times L^2(\Omega)$ and

$$\|Pu\|_{j,\Omega} \leqslant \|u\|_{j,Q}, \qquad u \in H^j(Q),\ j = 0, 1. \tag{6}$$

This projection is important in relating problem (1)-(2) to the limiting problem in Ω obtained by letting $\varepsilon \to 0$ (see lines (14) and (15) below).

Next we discuss the function g. We require that g be T-periodic with respect to t. We further assume that $g: \boldsymbol{R} \times \Omega \times [0, \varepsilon_0) \times \boldsymbol{R} \to \boldsymbol{R}$ is of class C^1 jointly in the variables t, x, Y and u, and that its derivatives satisfy the following estimates:

$$|g_x(t, x, Y, u)| \leqslant a(1 + |u|^{\theta+1}), \tag{7}$$

$$|g_Y(t, x, Y, u)| \leqslant a(1 + |u|^{\theta+1}), \tag{8}$$

$$|g_u(t, x, Y, u)| \leqslant a(1 + |u|^{\theta}). \tag{9}$$

Here a is a positive constant, and θ is determined as follows: $\theta \in [0, \infty)$ if $N = 1$, and $\theta \in [0, 2/(N-1))$ for values $N \geqslant 2$ (recall that $\dim Q = N + 1$).

Now let $C_T(Y_\varepsilon^1)$ be the space of all continuous, T-periodic functions $w = \begin{pmatrix} u \\ v \end{pmatrix}$ from $\boldsymbol{R}$ into Y_ε^1 with the usual norm:

$$\|w\| = \sup_{0 \leqslant t \leqslant T} \|w(t)\|_{Y_\varepsilon^1}.$$

Define the following maps on $C_T(Y_\varepsilon^1)$:

$$f_\varepsilon(w)(t)(x, y) = \begin{pmatrix} 0 \\ g(t, x, \varepsilon y, u(t, x, y)) \end{pmatrix},$$

and

$$J_\varepsilon w(t) = U_\varepsilon(t)(I - U_\varepsilon(T))^{-1} \int_0^T U_\varepsilon(T - s) w(s)\, ds + \int_0^t U_\varepsilon(t - s) w(s)\, ds.$$

Then define:

$$F_\varepsilon(w) = J_\varepsilon f_\varepsilon(w). \tag{10}$$

Using the estimate (6) with $j = 1$, one sees that the right-hand side of (10) is well-defined. Using the Sobolev embedding theory together with the theory of nonlinear Nemytskii operators [9], one can prove that F_ε maps $C_T(Y_\varepsilon^1)$ into itself and is completely continuous, *i.e.* it is continuous and it maps bounded sets into relatively compacts sets.

We identify the set of fixed points of F_ε as the class of T-periodic solutions which we will study. The question of the exact relation between the set of fixed points of F_ε and the set of T-periodic distributional solutions of (3)-(4) has been studied in [9, 10]. It is known that a fixed point of F_ε is always a T-periodic distributional solution of (3)-(4).

Next we pose the limit problem at $\varepsilon = 0$. Let $U_0(t)$ $(t \geqslant 0)$ be the semigroup generated by the equations

$$\frac{\partial u}{\partial t} = v\,, \qquad \frac{\partial v}{\partial t} = \Delta_x u - \beta v - \alpha u \tag{11}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega\,. \tag{12}$$

Let $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ be an element of $H^1(\Omega) \times L^2(\Omega)$. Then $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is in $H^1(\Omega) \times L^2(\Omega)$, and one has the estimate

$$\|U_0(t)\|_{H^1(\Omega)\times L^2(\Omega)\to H^1(\Omega)\times L^2(\Omega)} \leqslant ce^{-\gamma t}\,, \tag{13}$$

where c and γ are positive constants. Writing $i\colon \Omega \to Q$ defined by $i(x) = (x, 0)$, we obtain the inclusion $\mathfrak{J}\colon H^1(\Omega) \times L^2(\Omega) \to Y^1_\varepsilon$ with $\mathfrak{J}(u, v)(x, y) = (u(x), v(x))$. The map $\mathfrak{J}$ is an isometry for all $0 < \varepsilon < \varepsilon_0$, and we identify $U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ with the element $\mathfrak{J}U_0(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ of Y^1_ε.

Define an operator F_0 on $C_T(H^1(\Omega) \times L^2(\Omega))$ as follows: $F_0(w) = J_0 f_0(w)$, where J_0 has the same form as J_ε with $U_\varepsilon(t)$ replaced by $U_0(t)$ and

$$f_0(w)(t)(x) = \begin{pmatrix} 0 \\ g(t, x, 0, u(t, x)) \end{pmatrix}.$$

Then F_0 maps $C_T(H^1(\Omega) \times L^2(\Omega))$ into itself and is completely continuous. We identify periodic solutions of the problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta \frac{\partial u}{\partial t} - \alpha u + g(t, x, 0, u)\,, \tag{14}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{15}$$

with the fixed points of the operator F_0.

We now state our main result, here $\operatorname{ind}(\cdot, \cdot)$ indicates the topological index.

THEOREM 1. *If the problem* (14)-(15) *admits an isolated T-periodic solution* u^0 *defining an element* $\begin{pmatrix} u^0 \\ u^0_t \end{pmatrix} = w_0 \in C_T(H^1(\Omega) \times L^2(\Omega))$ *with* $\operatorname{ind}(F_0, w_0) \neq 0$, *then for sufficiently small* $\varepsilon > 0$ *the problem* (3)-(4) *admits a T-periodic solution* u^ε *and*

$$\left\| \begin{pmatrix} u^\varepsilon \\ u^\varepsilon_t \end{pmatrix} - \mathfrak{J}\begin{pmatrix} u^0 \\ u^0_t \end{pmatrix} \right\|_{C_T(Y^1_\varepsilon)} \to 0 \quad \textit{as } \varepsilon \to 0\,.$$

PROOF. The proof is organized in several steps.

I STEP. By assumption u^0 is an isolated T-periodic solution of (14)-(15), hence there exists a bounded neighborhood $V \subset C_T(H^1(\Omega) \times L^2(\Omega))$ of w_0 such that the operator F_0 has no fixed points on the boundary of V. Then it is not hard to prove that the operators $F_\varepsilon^0 = \mathfrak{J}_0 \boldsymbol{P} f_\varepsilon$ and $\mathfrak{J} F_0 \boldsymbol{P}$ are homotopic on $\overline{\mathfrak{J} V}$.

II STEP. For $r > 0$, define $B_r(\mathfrak{J}V) = \{z \in Y_\varepsilon^1 \colon \operatorname{dist}(z, \mathfrak{J}V) < r\}$ for sufficiently small r, say $0 < r \leq r_0$. From the previous step we obtain that

$$\operatorname{ind}_{\mathfrak{J}C_T(H^1(\Omega) \times L^2(\Omega))}(F_\varepsilon^0, B_r(\mathfrak{J}V) \cap \mathfrak{J}C_T(H^1(\Omega) \times L^2(\Omega))) = \\ = \operatorname{ind}_{\mathfrak{J}C_T(H^1(\Omega) \times L^2(\Omega))}(\mathfrak{J}F_0 \boldsymbol{P}, \mathfrak{J}V).$$

III STEP. This is the most important and difficult step of the proof. It consists in proving the existence of $r_0 > 0$ such that, to any fixed $r \in (0, r_0]$ there corresponds $\varepsilon_r > 0$ with the property that if $0 < \varepsilon \leq \varepsilon_r$ then the operators F_ε and F_ε^0 are linearly homotopic on $B_r(\mathfrak{J}V)$.

We only sketch the proof of this part without entering into details. First of all we fix $r_0 > 0$ such that the operator F_0 has no fixed points on the set: $\boldsymbol{P}[B_r(\mathfrak{J}\partial V) \cap \cap \mathfrak{J}C_T(H^1(\Omega) \times L^2(\Omega))]$ for all $0 \leq r \leq r_0$. Such an r_0 exists by our assumption. Then we argue by contradiction, that is, for fixed $r \in (0, r_0]$ we suppose that there exist sequences $\{\lambda_n\} \subset [0, 1]$, $\{w_n\} \subset \partial B_r(\mathfrak{J}V) \subset Y_\varepsilon^1$ and $\{\varepsilon_n\} \subset \boldsymbol{R}_+$ such that $\lambda_n \to \lambda_0$, $\varepsilon_n \to 0$ and

$$w_n = (1 - \lambda_n) F_{\varepsilon_n}(w_n) + \lambda_n F_{\varepsilon_n}^0(w_n), \tag{16}$$

where $w_n(t) = \begin{pmatrix} u_n(t) \\ u_{tn}(t) \end{pmatrix}$.

Now we observe that the sequence $\{u_n\}$ is uniformly bounded in $C_T(X_{\varepsilon_n}^1)$, and so the set $\{u_n(t) \colon n \in \boldsymbol{N},\ t \in [0, T]\}$ lies in a fixed compact subset of $L^p(Q)$, with $p \geq 2(\theta + 1)$ if $N = 1$ or $p \in [2(\theta + 1), (2N + 2)/(N - 1))$ if $N \geq 2$. By using this fact it is possible to prove (see [8]) that, after rewriting (16) is a suitable way, in passing to the limit we obtain $w^* = F_0(w^*)$ for some $w^* \in \partial B_r(V)$, contradicting our hypothesis.

IV STEP. Collecting all the previous steps and using both the homotopy invariance property of topological degree and its solution property we then conclude the proof of Theorem 1. ■

3. The autonomous case

We consider now the autonomous case for equation (1), that is the case when the nonlinearity g does not depend on t. After the change of variables: $X = x$, $Y = \varepsilon y$ and introducing the same spaces and operators of Section 2, we consider the equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u + \frac{1}{\varepsilon^2} \frac{\partial^2 u}{\partial y^2} - \beta \frac{\partial u}{\partial t} - \alpha u + g(x, \varepsilon y, u), \tag{17}$$

with Neumann boundary condition

$$(18) \qquad \partial u / \partial \nu = 0 \quad \text{on } \partial Q .$$

We can formulate the following result.

THEOREM 2. *Suppose that the equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_x u - \beta \frac{\partial u}{\partial t} - \alpha u + g(x, 0, u)$$

together with

$$\partial u / \partial \nu = 0 \quad on \ \partial \Omega ,$$

has a T_0-periodic solution z in the classical sense such that the linearized equation

$$(19) \qquad \frac{\partial^2 v}{\partial t^2} = \Delta_x v - \beta \frac{\partial v}{\partial t} - \alpha v + g_u(x, 0, z(t, x)) v$$

has a T_0-periodic solutions which are linearly independent of z_t. Furthermore, we suppose that (19) *does not possess any solution of the form:*

$$y(t, x) + (t / T_0) z_t(t, x)$$

where y is T_0-periodic with respect to t.

Then there exists $\varepsilon^0 > 0$ such that for all $\varepsilon \in (0, \varepsilon^0)$ problem (17)-(18) *has a T_ε-periodic solution u^ε with $T_\varepsilon \to T_0$ and*

$$\left\| \begin{pmatrix} \hat{u}^\varepsilon \\ \hat{u}^\varepsilon_t \end{pmatrix} - \mathfrak{J} \begin{pmatrix} z \\ z_t \end{pmatrix} \right\|_{C_{T_0}(Y^1_\varepsilon)} \to 0 \quad as \ \varepsilon \to 0 ,$$

where $\hat{u}^\varepsilon(t) = u^\varepsilon((T_\varepsilon / T_0) t)$.

PROOF. First, we make the change of variable $\tau = (T / T_0) t$ and, for any $\varepsilon > 0$, we rewrite problem (17)-(18) as a fixed point problem depending on the parameter T: $w = F_\varepsilon(T, w)$, where $F_\varepsilon(T, \cdot)$: $C_{T_0}(Y^1_\varepsilon) \to C_{T_0}(Y^1_\varepsilon)$ is a continuous compact operator. As in the non-autonomous case it can be expressed in the form: $F_\varepsilon(T, w) = J_\varepsilon(T) f_\varepsilon(T, w)$. Obviously, in the present case the operators J_ε and f_ε depend also on T.

Under our assumptions, by using methods similar to those of [3], it is possible to prove that for the operator $F_0(T, w) = J_0(T) f_0(T, w)$ corresponding to $\varepsilon = 0$ there exists a continuous functional $T = T(w)$ such that $T(w_0) = T_0$, where $w_0 = \begin{pmatrix} z \\ z_t \end{pmatrix}$; furthermore, if we define the operator $\Gamma_0(w) = F_0(T(w), w)$, then w_0 is an isolated fixed point of Γ_0 and $|\mathrm{ind}(w_0, \Gamma_0)| = 1$.

To finish the proof we proceed as in the non-autonomous case, that is, by means of similar admissible homotopies we prove that the operator $F_\varepsilon(T(Pw), w)$ has topological degree different from zero in a bounded open subset of the space $C_{T_0}(Y^1_\varepsilon)$ for ε sufficiently small. ■

Research partially supported by the MURST, the CNR and the RFFI grant 96-01-00360.

References

[1] I. Ciuperca, *Lower semicontinuity of attractors for a reaction-diffusion equation on thin domains with varying order of thinness*. Preprint, Université de Paris-Sud, 1996.

[2] I. Ciuperca, *Reaction-diffusion equations on thin domains with varying order of thinness*. Jour. Diff. Eqns., 126, 1996, 244-291.

[3] I. N. Gourova - M. I. Kamenskii, *On the method of semidiscretization in periodic solutions problems for quasilinear autonomous parabolic equations*. Differential Equations, 32, 1996, 101-106 (in Russian).

[4] J. Hale - G. Raugel, *A damped hyperbolic equation on thin domains*. Trans. Am. Math. Soc., 329, 1992, 185-219.

[5] J. Hale - G. Raugel, *Reaction-diffusion equations in thin domains*. Jour. Math. Pures et Appl., 71, 1992, 33-95.

[6] J. Hale - G. Raugel, *Limits of semigroups depending on parameters*. Resenhas IME-USP, 1, 1993, 1-45.

[7] R. Johnson - M. I. Kamenskii - P. Nistri, *Existence of periodic solutions for an autonomous damped wave equation in a thin domain*. Submitted.

[8] R. Johnson - M. I. Kamenskii - P. Nistri, *On periodic solutions of a damped wave equation in a thin domain using degree theoretic methods*. Jour. Diff. Eqns., to appear.

[9] M. Krasnoselskii - P. Zabreiko - E. Pustyl'nik - P. Sobolevski, *Integral Operators in Spaces of Summable Functions*. Noordhoff International Publishing, Leyden 1976.

[10] S. Krein, *Linear Differential Equations in Banach Spaces*. Nauka, Moscow 1967.

[11] G. Raugel, *Dynamics of partial differential equations in thin domains*. Lecture Notes in Mathematics, Springer-Verlag, Berlin 1995, 1609, 208-315.

R. Johnson, P. Nistri:
Dipartimento di Matematica
Facoltà di Ingegneria
Università degli Studi di Firenze
Via S. Marta, 3 - 50139 Firenze

M. Kamenskii:
Voronezh State University
Department of Mathematics
Universitetskaja p. 1
Voronezh (Russia)