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Fisica matematica. — *On the temperature distribution in cold ice.* Nota (*) di JAMES N. FLAVIN e SALVATORE RIONERO, presentata dal Corrisp. S. Rionero.

ABSTRACT. — The linear heat equation predicts that the variations of temperature along a cold ice sheet (*i.e.* at a temperature less than is freezing point) due to a sudden increase in air temperature, are very very slow. Based on this we represent the nonlinear evolution of an ice sheet as a sequence of steady states. As a first fundamental indication that this model is correct well posedness with respect to the variations of initial and boundary data is proved. Further an estimate of the error made in evaluating the thickness is given.

KEY WORDS: Cold ice; Nonlinear heat equation; Stability.

RIASSUNTO. — *Sul campo di temperatura nel ghiaccio «cold».* La teoria lineare prevede che in una lastra di ghiaccio «cold» (cioè a temperatura inferiore a quella di fusione) le variazioni di temperatura dovute ad improvvise variazioni di quella dell'aria siano molto lente. Per tale motivo si propone qui che l'evoluzione non lineare di una lastra di ghiaccio possa rappresentarsi con una successione di stati stazionari. Come prima fondamentale indicazione che tale modello sia corretto, si prova la dipendenza continua rispetto alle perturbazioni dei dati iniziali ed al contorno. Inoltre viene fornita una stima dell'errore che si commette nella valutazione dello spessore della lastra.

1. INTRODUCTION

As is well known, ice at a temperature less than its freezing point is called «cold», while ice at freezing point (which is essentially a two phase mixture of ice and water) is called «temperate» [1]. Cold ice occurs in many situations: in glaciers, on frozen lakes and seas, on mountains tops .... In the range $[-40\,°C, 0\,°C]$ for the temperature T, the thermal conductivity k of cold ice depends on the temperature. Specifically the empirical relation of Dilland and Timmerhans's [1, p. 151; 2, p. 360] is

$$k(T) = (2.1725 - 3.403 \times 10^{-3} T + 9.085 \times 10^{-5} T^2)\,\mathrm{Kg\,ms^{-3}(deg)^{-1}}. \tag{1}$$

Therefore, the temperature in such cold ice is governed by the equation

$$\varrho c_p \frac{\partial T}{\partial t} = \nabla \cdot [k(T)\,\nabla T] \tag{2}$$

where the density ϱ and the specific heat at a constant pressure c_p are given by [1, 2]

$$\varrho = 900\,\mathrm{Kgm}, \qquad c_p = 2 \times 10^3\,\mathrm{JKg^{-1}(K)^{-1}}. \tag{3}$$

Many relevant problems depend on the solvability of (2). We quote here the following:

i) In a cold ice glacier occupying a domain Ω, only a part Σ of $\partial\Omega$ is accessible (the surface of glacier). Therefore measurements of the temperature T and heat flux $\boldsymbol{n}\cdot\nabla T$, $\boldsymbol{n}$ being the unit upward normal, are possible only on Σ. Consequently the prob-

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lem of solving (2) in a bounded region Ω knowing T and $\boldsymbol{n}\cdot\nabla T$ on only part of $\partial\Omega$, arises.

ii) On frozen lakes and seas the problem of determining the temperature distribution in a cold ice sheet and the thickness of the sheet arises. Because the bottom (*i.e.* the ice-water interface) is at freezing temperature and the temperature and the heat flux on the top can be measured, the problem of solving (2) and determining the sheet thickness under the aforesaid boundary conditions arises.

The question *i*), (in one spatial dimension) has been considered recently in [3]. The authors – among other things – obtain an L^2-Holder continuous dependence result and, in a sense, pointwise continuous dependence. Remaining in one spatial dimension, our aim is to contribute to the solution of question *ii*). Our starting point is the fact that the variations of temperature along a cold ice sheet due to a sudden increase in air temperature, are very very slow. In fact, for an ice sheet of thickness $l = 0.5$ m the linear version of equation (2) predicts [2, p. 390] that about two days are needed in order that 75 per cent of the final effect of any change in surface temperature penetrate three-quarters of the thickness. Based on this, it seems realistic to represent the evolution of an ice sheet as a sequence of steady states U_n requiring – as a first fundamental indication that this model is correct – well posedness with respect to the variations of the initial and boundary data. The present paper is devoted to this problem and its plan is as follows. First of all, we briefly recall an appropriate nondimensionalization (Section 2). In Section 3, we obtain the general form of a steady state U – as done in [4] for a similar equation – and further we obtain the associated thickness determined by the appropriate boundary data. Section 4 is dedicated to obtaining some preliminary lemmas used later in the analysis of well posedness. In Section 5 we perform an L^2-continuous dependence analysis, while in the Section 6 pointwise continuous dependence is treated. In the last Section an estimate of the error made in evaluating the thickness is given.

2. An appropriate nondimensionalization

Equation (2) is written in nondimensional form by introducing the scalings

$$\begin{cases} x_i = l_0 x_i' \,, \\ t = t_0 t' \,, \\ T = T_0 T' \,. \end{cases} \tag{4}$$

This leads to

$$\frac{\varrho c_p}{t_0}\frac{\partial T'}{\partial t'} = \frac{2.1725}{l_0^2}\nabla'\left\{\left[1 - \frac{3.403\,T_0}{2172.5}T' + \frac{9.085\,T_0^2}{217250}(T')^2\right]\nabla' T'\right\}, \tag{5}$$

where

$$\nabla' = \left(\frac{\partial}{\partial x_1'}, \frac{\partial}{\partial x_2'}, \frac{\partial}{\partial x_3'}\right).$$

On choosing

$$\begin{cases} \dfrac{l_0^2 \varrho c_p}{2.1725} = t_0 , \\ T_0 = 166.8\ °\mathrm{C} , \end{cases} \tag{6}$$

setting

$$\begin{cases} u = -T' , \\ \varepsilon_1 = 0.261 , \\ \varepsilon_2 = 1.163 \end{cases} \tag{7}$$

and omitting the primes, it follows that

$$u_t = \nabla \cdot [(1 + \varepsilon_1 u + \varepsilon_2 u^2) \nabla u] . \tag{8}$$

We notice that (8) is the nondimensional version used in [3].

3. The steady state

We determine here the steady state temperature distribution and the thickness of the ice sheet, corresponding to known fixed values for the temperature and its gradient on its top.

On setting

$$F = u + \varepsilon_1 u^2/2 + \varepsilon_2 u^3/3 \tag{9}$$

equation (8) becomes

$$u_t = \Delta F \tag{10}$$

and, in one spatial dimension, admits the steady solution

$$u + \varepsilon_1 u^2/2 + \varepsilon_2 u^3/3 = ax + b \tag{11}$$

where a and b are constants to be determined by the boundary conditions. We suppose that $x = 0$ denotes the top of the piece of cold ice, while $x = l > 0$ denotes the other end. On putting

$$\mu = 6(ax + b) \tag{12}$$

it turns out that

$$6u + 3\varepsilon_1 u^2 + 2\varepsilon_2 u^3 = \mu \tag{13}$$

which, by the substitution

$$u = w - \varepsilon_1/(2\varepsilon_2) , \tag{14}$$

is reduced to

$$\alpha w^3 + \beta w + \gamma = 0 \tag{15}$$

where

$$(16) \qquad \begin{cases} \alpha = 4\varepsilon_2^3 \,, \\ \beta = 3\varepsilon_2(4\varepsilon_2 - \varepsilon_1^2) \,, \\ \gamma = \varepsilon_1^3 - 6\varepsilon_1\varepsilon_2 - 2\varepsilon_1^2\mu \,. \end{cases}$$

The unique real solution of (15) is given by

$$(17) \qquad w = \sqrt[3]{\sqrt{f_1^2 + h_1} + f_1} - \sqrt[3]{\sqrt{f_1^2 + h_1} - f_1}$$

where

$$(18) \qquad f_1 = -\gamma/(2\alpha) \,, \qquad h_1 = [\varepsilon_2(4\varepsilon_2 - \varepsilon_1)]^3/\alpha^3 \,.$$

On taking into account (12)-(18), we obtain the steady state solution

$$(19) \qquad U = \sqrt[3]{\sqrt{f_1^2 + h_1} + f_1} - \sqrt[3]{\sqrt{f_1^2 + h_1} - f_1} - \varepsilon_1/(2\varepsilon_2)$$

where

$$(20) \qquad \begin{cases} h_1 = ((4\varepsilon_2 - \varepsilon_1^2)/(4\varepsilon_2^2))^3 \,, \\ f_1 = 3(ax + b)/(2\varepsilon_2) + (6\varepsilon_2 - \varepsilon_1^2)\,\varepsilon_1/(8\varepsilon_2^3) \,. \end{cases}$$

Let us evaluate now the values of a and b and the thickness l assuming that temperature and its gradient on the top ($x = 0$) are known. Denoting – in line with (7) – by p, q respectively the values of $-T$ and $-\partial T/\partial x$ on $x = 0$, from (11) it follows that

$$(21) \qquad \begin{cases} a = q(1 + \varepsilon_1 p + \varepsilon_2 p^2) \,, \\ b = p(1 + \varepsilon_1 p/2 + \varepsilon_2 p^2/3) \,. \end{cases}$$

Concerning the thickness l, from $u(l) = 0$ and (11), (21), it follows that

$$(22) \qquad l = -b/a = p(1 + 0.13p + 0.387p)/(q(1 + 0.261p + 1.163p)) \,.$$

We notice that $4\varepsilon_2 > \varepsilon_1^2$ implies that

$$(23) \qquad P(\xi) = 1 + \varepsilon_1\xi/2 + \varepsilon_2\xi^2/3 > 0 \,, \qquad \forall \xi \in R \,,$$

$$(24) \qquad Q(\xi) = 1 + \varepsilon_1\xi + \varepsilon_2\xi^2 > 0 \,, \qquad \forall \xi \in R \,.$$

Therefore $\{p > 0, q < 0\}$ give $l > 0$. Concerning $q < 0$, we observe that (11) gives

$$(25) \qquad \frac{\partial U}{\partial x}(1 + \varepsilon_1 U + \varepsilon_2 U^2) = a \,, \qquad \forall x \,.$$

On taking into account (24), it follows that $\partial U/\partial x$ does not change sign. But U decreases from p °C to 0 °C hence $\partial U/\partial x < 0$.

4. Preliminary lemmas to the stability

Let $t = 0$ be the time at which the ice began to form and $[0, \mathcal{T}]$, $\mathcal{T} = \text{const} > 0$, be an interval of time on which the temperature of upper surface of the piece of ice is non constant and less or equal to zero °C. Let $[t_0, t_1] \subset]0, \mathscr{T}]$ and denote by p and q the values

$$p = -T \quad (x = 0, t = t_0), \tag{26}$$

$$q = -\partial T/\partial x \quad (x = 0, t = t_0) \tag{27}$$

(evaluated experimentally) and by U the steady state corresponding to the data (26)-(27) and by $l = -pP(p)/qQ(p)$ the corresponding thickness. The following question arises:

representing the temperature distribution by U and the thickness by l in the interval $[t_0, t_1]$ what are the consequential errors? Concerning the temperature distribution, in Sections 5-6 an estimate is made of the error both in the L^2 and pointwise norms. Next, Section 7 estimates the error made in evaluating the thickness. In the present section we prove some preliminary lemmas to the stability.

On setting

$$F(u) = u + \varepsilon_1 u^2/2 + \varepsilon_2 u^3/3 \tag{28}$$

equation (8), in one spatial dimension, becomes

$$u_t = \frac{\partial^2}{\partial x^2} F(u). \tag{29}$$

Let us consider the problem

$$\begin{cases} u_t = \dfrac{\partial^2}{\partial x^2} F(u), & (x,t) \in [0,l] \times [t_0, t_1], \\ u(0,t) = h(t), & t \in [t_0, t_1], \\ u_x(0,t) = g(t), & t \in [t_0, t_1], \\ u[l,t] = 0, & t \in [t_0, t_1], \\ u(x,0) = u_0(x), & x \in [0,l], \end{cases} \tag{30}$$

$h(t)$ and $g(t)$ being prescribed functions on $[t_0, t_1]$. Our aim is to estimate the error made when the problem (30) is replaced by

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} F(U) = 0, & x \in [0,l], \\ U = p, & x = 0, \\ U_x = q, & x = 0, \\ U(x) = 0, & x = l. \end{cases} \tag{31}$$

On setting

$$v = u - U \tag{32}$$

from (30)-(31) it follows that

$$
(33) \quad \begin{cases} v_t = \dfrac{\partial^2}{\partial x^2}[F(v+U)-F(U)], & (x,t) \in [0,l] \times [t_0, t_1], \\ v(0,t) = h(t) - p, & t \in [t_0, t_1], \\ v_x(0,t) = g(t) - q, & t \in [t_0, t_1], \\ v[l,t] = 0, & t \in [t_0, t_1], \\ v(x,0) = u_0 - U, & x \in [0,l]. \end{cases}
$$

Therefore putting

$$
(34) \quad L(U,v) = F(v+U) - F(U) = F'(U)v + F''(U)v^2/2 + F'''(U)v^3/6
$$

it follows that

$$
(35) \quad L(U,v) = (1 + \varepsilon_1 U + \varepsilon_2 U^2)v + (\varepsilon_1 + 2\varepsilon_2 U)v^2/2 + \varepsilon_2 v^3/3
$$

and (33) becomes

$$
(36) \quad \begin{cases} v_t = \dfrac{\partial^2}{\partial x^2} L[U(x), v(x,t)], & (x,t) \in [0,l] \times [t_0, t_1], \\ v(0,t) = h(t) - p, & t \in [t_0, t_1], \\ v_x(0,t) = g(t) - q, & t \in [t_0, t_1], \\ v(l,t) = 0, & t \in [t_0, t_1], \\ v(x,0) = u_0 - U, & x \in [0,l]. \end{cases}
$$

In order to estimate v in the L^2 and in the pointwise topology, we use the following theorem.

THEOREM 1. *Let $v \in R$ and $n \in N$, then*

$$
(37) \quad \begin{cases} L^{2n+2}(v) \geq A \displaystyle\int_0^v L^{2n+1}(\bar{v})\, d\bar{v}, \\ A = 1 - 3\varepsilon_1^2/(16\varepsilon_2). \end{cases}
$$

For any easy proof of (37) and for other purposes we need the following Lemmas:

LEMMA 1. *$L^{2n+1}(v)$, $(n \in N)$, is an increasing function on R.*

PROOF. It turns out that

$$
\frac{d}{dv} L^{2n+1} = (2n+1) L^{2n} L' .
$$

But

$$
\begin{aligned}
L'(v) &= 1 + \varepsilon_1 U + \varepsilon_2 U^2 + (\varepsilon_1 + 2\varepsilon_2 U)v + \varepsilon_2 v^2 = \\
&= 1 + \varepsilon_1 U + \varepsilon_2 U^2 + \varepsilon_2 [v + (\varepsilon_1 \varepsilon_2^{-1} + 2U)/2]^2 - \varepsilon_2(\varepsilon_1 \varepsilon_2^{-1} + 2U)^2/4 \geq \\
&\geq 1 + \varepsilon_1 U + \varepsilon_2 U^2 - (\varepsilon_1^2 \varepsilon_2^{-1} + 4\varepsilon_2 U^2 + 4\varepsilon_1 U)/4 = 1 - \varepsilon_1^2/(4\varepsilon_2) > 0 .
\end{aligned}
$$

LEMMA 2. *One has*

$$(38) \qquad \begin{cases} L(v) \geqslant Av\,, & v \geqslant 0\,, \\ L(v) \leqslant Av\,, & v \leqslant 0\,. \end{cases}$$

PROOF. For $v \neq 0$, it follows that

$$L(v)/v = [6(1+\varepsilon_1 U + \varepsilon_2 U^2) + 3(\varepsilon_1 + 2\varepsilon_2 U)v + 2\varepsilon_2 v^2]/6\,.$$

But $U \geqslant 0$ and $4\varepsilon_2 > \varepsilon_1^2$ imply

$$9(\varepsilon_1 + 2\varepsilon_2 U)^2 - 48\varepsilon_2(1+\varepsilon_1 U + \varepsilon_2 U^2) =$$
$$= 9(\varepsilon_1 + 2\varepsilon_2 U)^2 - 12(4\varepsilon_2 + 4\varepsilon_1\varepsilon_2 U + 4\varepsilon_2^2 U^2) \leqslant -3(\varepsilon_1 + 2\varepsilon_2 U)^2 < 0\,.$$

Therefore

$$L(v)/v > 0\,, \qquad \forall v \in R - \{0\}\,,$$

and attains its l.b. at $v = -3(\varepsilon_1 + 2\varepsilon_2 U)/(4\varepsilon_2)$. Hence it follows that

$$6L(v)/v > 6(1+\varepsilon_1 U + \varepsilon_2 U^2) - 9(\varepsilon_1 + 2\varepsilon_2 U)^2/(4\varepsilon_2) + 9(\varepsilon_1 + 2\varepsilon_2 U)^2/(8\varepsilon_2) =$$
$$= 6(1+\varepsilon_1 U + \varepsilon_2 U^2) - 9(\varepsilon_1 + 2\varepsilon_2 U)^2/(8\varepsilon_2) \geqslant$$
$$\geqslant (6 - 9\varepsilon_1^2/(8\varepsilon_2)) + (6 - 9/2)\varepsilon_1 U + (6 - 9/2)\varepsilon_2 U^2 \geqslant 6 - 9\varepsilon_1^2/(8\varepsilon_2)$$

recalling that $\varepsilon_1 > 0$. Therefore it immediately follows that

$$L(v)/v > 1 - 3\varepsilon_1^2/(16\varepsilon_2) = A\,.$$

LEMMA 3. *For any $v \in R$ one has*

$$(39) \qquad 0 \leqslant \int_0^v L^{2n+1}(\bar v)\,d\bar v < vL^{2n+1}(v)\,.$$

PROOF. Because $L(0) = 0$, Lemma 1 gives

$$0 < \bar v < v \Rightarrow \int_0^v L^{2n+1}(\bar v)\,d\bar v < L^{2n+1}(v)\int_0^v dv = vL^{2n+1}(v)\,,$$

$$v < \bar v < 0 \Rightarrow \int_v^0 L^{2n+1}(\bar v)\,d\bar v > L^{2n+1}(v)\int_v^0 dv = -vL^{2n+1}(v)\,.$$

The proof of (37) is now trivial. In fact for $v \neq 0$, (39) implies

$$L^{2n+2}(v)\Big/\int_0^v L^{2n+1}(\bar v)\,d\bar v > L^{2n+2}(v)/(L^{2n+1}(v)\cdot v) = L/v\,.$$

Therefore on taking (38) into account, (37) immediately follows.

5. L^2-ERROR ESTIMATE FOR THE TEMPERATURE

We set

$$(40) \quad G[U, v(x,t)] = \int_0^v L(U,v)\,dv = \frac{v^2}{12}[6(1+\varepsilon_1 U + \varepsilon_2 U^2) + 2(\varepsilon_1 + 2\varepsilon_2 U)v + \varepsilon_2 v^2]$$

and notice that the quantity in the square brackets is always positive (see the Proof of Lemma 1). Hence G is a positive definite function of v:

$$\begin{cases} G > 0\,, & v \neq 0\,, \\ G = 0\,, & v = 0\,. \end{cases} \tag{41}$$

We use now the Liapunov function introduced by Rionero in [7]

$$V = \int_0^l G[U(x), v(x,t)]\,dx\,. \tag{42}$$

Along the solution to (36), it turns out that

$$\dot{V} = \int_0^l v_t L\,dx\,. \tag{43}$$

But

$$v = 0 \Rightarrow \{G = 0, L = 0\} \tag{44}$$

hence from $(36)_4$, (43)-(44) it turns out that

$$\dot{V} = -[LL_x]_{x=0}^{x=l} - \int_0^l \left(\frac{\partial L}{\partial x}\right)^2 dx\,, \tag{45}$$

and (44) and the Poincaré inequality [5, p. 338] imply

$$\dot{V} = -[LL_x]_{x=0} - \frac{\pi^2}{4l^2}\int_0^l L^2\,dx\,. \tag{46}$$

But (37), for $n = 0$, gives

$$L^2 > AG\,, \tag{47}$$

hence (46) gives

$$\begin{cases} \dot{V} \leqslant -\left[L\,\dfrac{\partial L}{\partial x}\right]_{x=0} - \sigma V\,, \\ \sigma = A\pi^2/(4l^2)\,, \end{cases} \tag{48}$$

and therefore it follows that

$$V(t) < V_0 e^{-\sigma t} - e^{-\sigma t}\int_{t_0}^{t}\left\{\left[L\,\frac{\partial L}{\partial x}\right]_{x=0} e^{\sigma\tau}\right\}d\tau \tag{49}$$

where $V_0 = V(U, v_0)$.

Now it turns out that

$$\begin{cases} |L|_{x=0} \leqslant |h(t) - p|\,[6Q(p) + 3(\varepsilon_1 + 2\varepsilon_2 p)|h(t) - p| + 2\varepsilon_2\,|h(t) - p|^2]/6\,, \\ |\partial L/\partial x|_{x=0} \leqslant |q|\,[(\varepsilon_1 + 2\varepsilon_2\,|p|) + \varepsilon_2\,|q|\,\,|h(t) - p|]\,|h(t) - p| + \\ \qquad + |g(t) - q|\,[Q(p) + (\varepsilon_1 + 2\varepsilon_2 p)|h(t) - p| + \varepsilon_2\,|h(t) - p|^2]\,. \end{cases} \tag{50}$$

Suppose

$$\begin{cases} |h(t) - p| \leqslant \delta, & t \in [t_0, t_1], \\ |g(t) - q| \leqslant \varepsilon, & t \in [t_0, t_1], \end{cases} \tag{51}$$

and set

$$\begin{cases} m_1 = [6Q(p) + 3(\varepsilon_1 + 2\varepsilon_2 p)\delta + 2\varepsilon_2 \delta^2]/6, \\ m_2 = |q|[(\varepsilon_1 + 2\varepsilon_2 p) + \varepsilon_2 |q| \delta], \\ m_3 = Q(p) + (\varepsilon_1 + 2\varepsilon_2 p)\delta + \varepsilon_2 \delta^2. \end{cases} \tag{52}$$

From (49) it turns out that $\forall t \in [t_0, t_1]$

$$V(t) \leqslant V_0 e^{-\sigma t} + m_1 \delta (m_2 \delta + m_3 \varepsilon)(1 - e^{-\sigma t})/\sigma, \tag{53}$$

i.e. L^2 continuous dependence with respect to perturbations in the initial data and in the boundary data.

In fact from (38) it follows that

$$v > 0 \Rightarrow L(v) > Av \Rightarrow \int_0^v L(v) > Av^2/2,$$

$$v < 0 \Rightarrow L(v) < Av \Rightarrow \int_v^0 L(v)\, dv < A \int_v^0 v\, dv = -Av^2/2,$$

i.e. $\forall v \in R$

$$v^2 \leqslant \frac{2}{A} \int_0^v L(v)\, dv = G/A$$

and hence

$$\int_0^l v^2\, dv \leqslant V/\alpha. \tag{54}$$

6. Pointwise error estimate for temperature

Set

$$E = \int_0^l \widetilde{F}\, dx, \tag{55}$$

$$\widetilde{F} = \int_0^v L^{2n+1}\, dx, \qquad n \in N. \tag{56}$$

It follows that

$$\dot{E} = \int_0^l L^{2n+1} L_{xx}\, dx = [L^{2n+1} L_x]_{x=0}^{x=l} - (2n+1) \int_0^l L^{2n} L_x^2\, dx. \tag{57}$$

But $v(l,t)=0 \Rightarrow \{L=\widetilde{F}=0,\ \text{at}\ x=l\}$, hence

$$(58)\qquad \dot{E} = -[L^{2n+1}L_x]_{x=0} - \left(\frac{2n+1}{(n+1)^2}\right)\int_0^l \left(\frac{\partial}{\partial x}L^{n+1}\right)^2 dx \leqslant$$

$$\leqslant -[L^{2n+1}L_x]_{x=0} - \frac{(2n+1)\pi^2}{4(n+1)^2 l^2}\int_0^l L^{2n+2}\,dx\,.$$

On integrating (37), it follows that

$$(59)\qquad \int_0^l L^{2n+2}\,dx \geqslant AE$$

hence from $(48)_2$ and (58) we obtain

$$(60)\qquad \dot{E} \leqslant -[L^{2n+1}L_x]_{x=0} - \sigma(2n+1)(n+1)^{-2}E$$

i.e.

$$(61)\qquad E \leqslant E_0 e^{-\sigma_n t} - e^{-\sigma_n t}\int_{t_0}^{t}\left[L^{2n+1}\frac{\partial L}{\partial x}\right]_{x=0} e^{\sigma_n \tau}\,d\tau$$

where

$$(62)\qquad \begin{cases} E_0 = E(t_0)\,, \\ \sigma_n = [(2n+1)(n+1)^{-2}]\,\sigma\,. \end{cases}$$

On taking into account (50) and (51), it follows that

$$(63)\qquad E \leqslant E_0 e^{-\sigma_n t} + (m_1\delta)^{2n+1}(m_2\delta + m_3\varepsilon)(1-e^{-\sigma_n t})/\sigma_n\,.$$

From (38) it turns out that

$$L^{2n+1}/v^{2n+1} \geqslant A^{2n+1}$$

hence

$$v>0 \Rightarrow L^{2n+1}(v) \geqslant A^{2n+1}v^{2n+1} \Rightarrow \int_0^v L^{2n+1}\,dv \geqslant A^{2n+1}v^{2n+2}/(2n+2)\,,$$

$$v<0 \Rightarrow L^{2n+1}(v) \leqslant A^{2n+1}v^{2n+1} \Rightarrow \int_v^0 L^{2n+1}\,dv \leqslant -A^{2n+1}v^{2n+2}/(2n+2)\,,$$

i.e.

$$\int_0^v L^{2n+1}\,dv \geqslant \frac{A^{2n+1}}{2n+2}v^{2n+2}\,,\qquad \forall v$$

and integrating

$$(64)\qquad E \geqslant \frac{A^{2n+1}}{2n+2}\int_0^l v^{2n+2}\,dx\,.$$

Finally we obtain

$$\left(\int_0^l v^{2n+2}\,dx\right)^{1/(2n+2)} \leqslant \left\{\left(\frac{2n+2}{A^{2n+1}}\right)E\right\}^{1/(2n+2)}. \tag{65}$$

On taking into account that [1]

$$\sqrt[n]{\lambda_1+\lambda_2} \leqslant \sqrt[n]{\lambda_1} + \sqrt[n]{\lambda_2}, \qquad n, \lambda_1, \lambda_2 > 0$$

from (63) and (65) it follows that

$$\left(\int_0^l v^{2n+2}\,dx\right)^{1/(2n+2)} \leqslant \frac{\sqrt[2n+2]{2n+2}}{A^{(2n+1)/(2n+2)}}\left[E_0^{1/(2n+2)}\, e^{-\sigma_n t/(2n+2)} + m_1\delta\left(\frac{m_2\delta + m_3\varepsilon}{m_1\delta\sigma}\right)^{1/(2n+2)}\right]$$

and hence

$$\lim_{n\to\infty}\left(\int_0^l v^{2n+2}\,dx\right)^{1/(2n+2)} \leqslant \frac{1}{A}\left[\lim_{n\to\infty} E_0^{1/(2n+2)} + m_1\delta\right]. \tag{66}$$

But, from (37), it turns out that

$$E_0 = \int_0^l dx \int_0^{v_0} L^{2n+1}\,dv \leqslant \frac{1}{A}\int_0^l L^{2n+2}(v_0)\,dx. \tag{67}$$

Therefore in view of (70), we obtain

$$\lim_{n\to\infty}\left[\int_0^l v^{2n+2}dx\right]^{1/(2n+2)} \leqslant \frac{1}{A}\left[\lim_{n\to\infty}\left(\int_0^l L^{2n+2}(v_0)\,dx\right)^{1/(2n+2)} + m_1\delta\right] \tag{68}$$

and hence pointwise continuous dependence follows according to

$$|v(x,t)| \leqslant \frac{1}{A}\left[\sup_{[0,\,l(0)]} |L(v_0)| + m_1\delta\right]. \tag{69}$$

REMARK 1. We conclude with a simple analysis which provides the basis for an alternative pointwise continuous dependence estimate. However, the estimate requires a knowledge of certain time rates of change, which are not required in the previous analysis.

Referring to the problem defined by (36), define

$$E(t) = \int_0^l \left(\frac{\partial L}{\partial x}\right)^2 dx. \tag{70}$$

(1) From $\xi_1^n + \xi_2^n \leqslant (\xi_1+\xi_2)^n$, $\xi_1, \xi_2, n > 0$, it follows that $\sqrt[n]{\xi_1^n+\xi_2^n} \leqslant \xi_1+\xi_2$. Hence, setting $\xi_i^n = \lambda_i$, $i = 1, 2$, it turns out that $\sqrt[n]{\lambda_1+\lambda_2} \leqslant \sqrt[n]{\lambda_1} + \sqrt[n]{\lambda_2}$, $\lambda_1, \lambda_2, n > 0$.

Differentiating gives

$$(71)\qquad \frac{dE}{dt} = 2\int_0^l \frac{\partial L}{\partial x}\frac{\partial^2 L}{\partial t\,\partial x}\,dx = -2\int_0^l \frac{\partial^2 L}{\partial x^2}\frac{\partial L}{\partial t}\,dx + \left[2\,\frac{\partial L}{\partial x}\frac{\partial L}{\partial t}\right]_{x=0}^{x=l}$$

where integration by parts has been used in the last step.

Using the definition of L and (36) it follows that

$$(72)\qquad -2\,\frac{\partial^2 L}{\partial x^2}\frac{\partial L}{\partial t} = -2[(1+\varepsilon_1 U+\varepsilon_2 U^2)+(\varepsilon_1+2\varepsilon_2 U)v+\varepsilon_2 v^2]\left(\frac{\partial^2 L}{\partial x^2}\right)^2 \leqslant$$

$$\leqslant -2\left(1-\frac{1}{4}\varepsilon_1^2\varepsilon_2^{-1}\right)\left(\frac{\partial^2 L}{\partial x^2}\right)^2 \leqslant 0\,.$$

Since $v=0$ on $x=l$, it follows that $L=0$ thereon. Thus

$$(73)\qquad \partial L/\partial t = 0$$

thereon. It follows from (71)-(73) that,

$$\frac{dE}{dt} \leqslant -2\,\frac{\partial L}{\partial x}(0,t)\,\frac{\partial L}{\partial t}(0,t)$$

(where $L=L(x,t)$). Integration gives

$$(74)\qquad E(t) \leqslant E(t_0) - 2\int_{t_0}^{t} \frac{\partial L}{\partial x}(0,t')\,\frac{\partial L}{\partial t}(0,t')\,dt'\,.$$

A pointwise estimate is deducible from this in view of

$$(75)\qquad L^2(x,t) \leqslant (l-x)\,E(t)$$

a consequence of Schwarz's inequality and

$$(76)\qquad L^2(x,t) \geqslant v^2(x,t)[1+\varepsilon_1 U+\varepsilon U^2+(\varepsilon_1+2\varepsilon_2 U)v/2+\varepsilon_2 v^2/3] \geqslant$$

$$\geqslant v^2(x,t)[1-3\varepsilon_1^2\varepsilon_2^{-1}/16]\,.$$

In the above analysis it is, of course, assumed that the requisite smoothness obtains.

7. On an estimate of thickness

Let us recall the equation commonly used to predict the thickness of ice on lakes from forecasts. For the sake of simplicity, we return to using dimensional quantities. Let $\delta(t)$ be the thickness at time t and T_Σ (<0) the temperature in °C of the upper surface Σ such that $\{T_\Sigma(0)=0,\ T_\Sigma(0)<0 \text{ for } t>0\}$. The equation that has been found to describe the growth of the ice sheets remarkably well is [2, pp. 389-390]

$$(77)\qquad \delta^2(t) = \frac{2}{\varrho\lambda}\int_0^t T_\Sigma K(T_\Sigma)\,d\xi$$

where λ is the latent heat of melting per unit mass and $t = 0$ is the time at which the ice began to form. The equation (77) is a consequence of the following two assumptions

i) the quantity of heat dQ conducted through unit area of the sheet in time dt is

(78) $$dQ = -(k(T_\Sigma)/\delta)\, T_\Sigma\, dt$$

ii) it is allowed to neglect any change in the temperature of the ice which has formed and therefore

(79) $$dQ = -\varrho\lambda\, d\delta\,.$$

We recall that (78) is suggested by experiments when the steady state of temperature has been reached [6, p. 2]. It is quite natural then to compare, at any fixed time $\bar{t}$, the value $\delta(\bar{t})$ given by (77) to the sheet thickness δ^* evaluated through a steady state $\bar{T}(x)$

(80) $$a_1 \bar{T} + a_2 \bar{T}^2/2 + a_3 \bar{T}^3/3 = \bar{a}x + \bar{b}$$

where

(81) $$a_1 = 2.1725\,, \quad a_2 = -3.403 \times 10^{-3}\,, \quad a_3 = 9.085 \times 10^{-5}$$

and $\bar{a}$, $\bar{b}$ being constants to be determined by the boundary data

(82) $$\bar{T}(0) = T_\Sigma(\bar{t})\,,$$

(83) $$[d\bar{T}/dx]_{x=0} = S(\bar{t})$$

where S: $t \in [0, \mathcal{T}] \Rightarrow S(t) \in R$ denotes the heat flux on the top. From (80), (82) and (83) it turns out that

(84) $$\begin{cases} \bar{a} = S(\bar{t})\, K[T_\Sigma(\bar{t})]\,, \\ \bar{b} = T_\Sigma(\bar{t})[a_1 + (a_2/2)\, T_\Sigma(\bar{t}) + (a_3/3)\, T_\Sigma^2(\bar{t})]\,, \\ \delta^* = -\bar{b}/\bar{a}\,. \end{cases}$$

A comparison between $\delta(\bar{t})$ and δ^* can be found noticing that (78) assumes that

(85) $$S(\bar{t}) = (T[\delta(\bar{t}), \bar{t}] - T_\Sigma(\bar{t}))/\delta(\bar{t}) = -T_\Sigma(\bar{t})/\delta(\bar{t})\,.$$

Consequently (83) and (84) imply

(86) $$\delta^*/\delta = (a_1 + a_2 T_\Sigma/2 + a_3 T_\Sigma^2/3)/(K(T_\Sigma))$$

i.e.

(87) $$\delta^*/\delta = (1 + \varepsilon_1 p/2 + \varepsilon_2 p^2/3)/(1 + \varepsilon_1 p + \varepsilon_2 p^2) = \\ = (2 + (4 + \varepsilon_1 p)/(1 + \varepsilon_1 p + \varepsilon_2 p^2))/6$$

where

(88) $$p = -T_\Sigma/166.8\,.$$

It is easily seen that δ^*/δ is a decreasing function of $p \geq 0$. In particular, in the range $T \in [-40\,°\mathrm{C}, 0\,°\mathrm{C}]$ easily follows.

TABLE I.

T_Σ	p	δ^*/δ
0 °C	0	1
− 10 °C	0.059	0.9896
− 20 °C	0.118	0.9738
− 30 °C	0.179	0.9555
− 40 °C	0.23	0.932

TABLE II.

T_Σ	δ	δ^*
0 °C	1000 mm	1000 mm
− 10 °C	1000 mm	989 mm
− 20 °C	1000 mm	973 mm
− 30 °C	1000 mm	955 mm
− 40 °C	1000 mm	932 mm

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