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Silvana Franciosi, Francesco de Giovanni

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Teoria dei gruppi. — *On groups with many nearly maximal subgroups.* Nota (*) di

Silvana Franciosi e Francesco de Giovanni, presentata dal Socio G. Zappa.

Abstract. — A subgroup M of a group G is nearly maximal if the index $|G : M|$ is infinite but every subgroup of G properly containing M has finite index, and the group G is called nearly IM if all its subgroups of infinite index are intersections of nearly maximal subgroups. It is proved that an infinite (generalized) soluble group is nearly IM if and only if it is either cyclic or dihedral.

Key words: Nearly maximal subgroup; Nearly IM-group; Soluble group.

Riassunto. — *Sui gruppi con molti sottogruppi massimali generalizzati.* Un sottogruppo M di un gruppo G si dice «nearly maximal» se l'indice $|G : M|$ è infinito mentre ogni sottogruppo di G che contenga propriamente M ha indice finito in G, ed il gruppo G si dice «nearly IM» se ogni suo sottogruppo di indice infinito è intersezione di sottogruppi «nearly maximal». Si prova che un gruppo risolubile (generalizzato) infinito è «nearly IM» se e solo se è ciclico oppure diedrale.

1. Introduction

A subgroup M of an infinite group G is said to be *nearly maximal* if it is a maximal element of the set of all subgroups of G having infinite index, *i.e.* if the index $|G : M|$ is infinite but every subgroup of G properly containing M has finite index in G. The *near Frattini subgroup* of a group G is defined to be the intersection of all nearly maximal subgroups of G, with the stipulation that it shall equal G if G has no nearly maximal subgroups. These concepts were introduced by Riles [4] in 1969, and more recently Lennox and Robinson [1] have shown that certain group theoretical properties of finitely generated soluble groups can be detected from the behaviour of nearly maximal subgroups.

A group G is called an *IM-group* if all its subgroups are intersections of maximal subgroups. Soluble IM-groups have been completely characterized by Menegazzo [2]. Here we will consider the corresponding concept obtained replacing maximal by nearly maximal subgroups. A group G is said to be *nearly IM* if every subgroup of infinite index of G is intersection of nearly maximal subgroups. In particular, if G is a nearly IM-group, all infinite factor groups of G have trivial near Frattini subgroup.

The aim of this article is to prove the following result, which shows that (generalized) soluble nearly IM-groups have a very restricted structure.

Theorem. *Let G be an infinite hyper-*(*abelian or finite*) *group. Then G is nearly IM if and only if it is either cyclic or dihedral.*

(*) Pervenuta in forma definitiva all'Accademia l'8 ottobre 1997.

The consideration of Tarski groups shows that in the above theorem the hypothesis that the group G is hyper-(abelian or finite) cannot be omitted.

Most of our notation is standard and can be found in [5].

2. Proof of the Theorem

We shall prove that the Fitting subgroup of an arbitrary infinite nearly IM-group is abelian. This will be done using the following two lemmas.

Lemma 1. *Let G be an infinite nearly IM-group. Then G does not contain finite non-trivial normal subgroups.*

Proof. Let E be a finite normal subgroup of G. If M is any nearly maximal subgroup of G, the index $|EM : M|$ is obviously finite, so that $EM = M$ and E is contained in M. On the other hand, the near Frattini subgroup of G is trivial, and hence $E = 1$. Therefore the group G does not contain finite non-trivial normal subgroups. $\square$

Lemma 2. *Let G be a nearly IM-group, and let N be a normal subgroup of infinite index of G. If M is a nearly maximal subgroup of G which does not contain N, then M is maximal in MN.*

Proof. Let H be a subgroup of MN properly containing M. Then H has finite index in G, and so also the index $|N : H \cap N|$ is finite. Moreover, $H \cap N$ is a normal subgroup of H, and so it has only finitely many conjugates in G. It follows that the core K of $H \cap N$ in G has finite index in N, and N/K is a finite normal subgroup of the infinite group G/K. On the other hand, G/K is nearly IM, and hence $N = K$ by Lemma 1. Thus N is contained in H, and so $H = MN$. Therefore M is a maximal subgroup of MN. $\square$

Proposition 3. *Let G be an infinite nearly IM-group. Then the Fitting subgroup of G is abelian.*

Proof. Let N be any nilpotent normal subgroup of G, and suppose first that N has finite index in G. If a is a non-trivial element of $Z(N)$, the normal closure $A = \langle a\rangle^G$ is a finitely generated abelian group, and hence it contains a proper characteristic subgroup B such that A/B is finite. Since the factor group G/B is nearly IM, it follows from Lemma 1 that G/B is finite. Then also the group $N/Z(N)$ is finite, and so the commutator subgroup N' of N is finite, and another application of Lemma 1 yields that $N' = 1$ and N is abelian. Suppose now that the index $|G : N|$ is infinite, and assume that $N' \neq 1$, so that there exists a nearly maximal subgroup M of G which does not contain N'. Then

$$M < MN' \leq MN,$$

and M is a maximal subgroup of MN by Lemma 2, so that $MN' = MN$. Thus $N = N'(M \cap N)$, and hence $N = M \cap N$ is contained in M. This contradiction shows that also in this case N is abelian. Therefore all nilpotent normal subgroups of G are abelian, and so also the Fitting subgroup of G is abelian. □

Lemma 4. *Let G be a group and let M be a nearly maximal subgroup of G. If A is a finitely generated abelian normal subgroup of G such that $A/(A \cap M)$ is infinite, then $A/(A \cap M)$ is torsion-free.*

Proof. Let $T/(A \cap M)$ be the subgroup consisting of all elements of finite order of the group $A/(A \cap M)$. Then $T/(A \cap M)$ is finite and $T^M = T$, so that M has finite index in TM and hence $TM = M$. Thus T is contained in M, and so $A \cap M = T$. Therefore $A/(A \cap M)$ is a torsion-free group. □

Our last lemma deals with finitely generated soluble-by-finite nearly IM-groups.

Lemma 5. *Let G be a finitely generated soluble-by-finite group. If G is nearly IM, then it is abelian-by-finite.*

Proof. Let S be the soluble radical of G, and let K be the smallest non-trivial term of the derived series of S. By induction on the derived length of S the factor group $Q = G/K$ is abelian-by-finite, and hence also polycyclic-by-finite. In particular, the finitely generated group G is abelian-by-polycyclic-by-finite, and so it satisfies the maximal condition on normal subgroups (see [5, Part 1, Theorem 5.34]). Thus K contains a maximal proper G-invariant subgroup L, and K/L is a simple Q-module. It follows that K/L is finite (see [6]), and hence the factor group G/L must be finite by Lemma 1. Therefore G is abelian-by-finite. □

Proof of the Theorem. Clearly both the infinite cyclic group and the infinite dihedral group are nearly IM.

Conversely, suppose that G is nearly IM, and let

$$1 = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending normal series of G whose factors either are finite or abelian. Consider any ordinal $\alpha < \tau$ such that the subgroup G_α has infinite index in G. Then the factor group G/G_α does not contain finite non-trivial normal subgroups by Lemma 1, and hence $G_{\alpha+1}/G_\alpha$ is abelian. It follows that the group G is hyperabelian-by-finite. Let N be a hyperabelian normal subgroup of finite index of G, and assume that N is not soluble. Then there exists an ascending chain

$$K_1 < K_2 < \ldots < K_n < K_{n+1} < \ldots$$

of soluble normal subgroups of N such that the subgroup

$$K = \bigcup_{n \in \mathbb{N}} K_n$$

is not soluble (see [3, Lemma 5]). Moreover, since N has finite index in G, the subgroups K_n can be chosen to be normal in G. The Fitting subgroup F of G is abelian by Proposition 3, and hence we may also suppose that K_1 contains F. As N is not soluble, the index $|G : F|$ is infinite, and the subgroup F is not cyclic. Let a be any non-trivial element of F, and let $\mathfrak{L}$ be the set of all nearly maximal subgroups of G containing a but not F. If M is any element of $\mathfrak{L}$, we have

$$M < MF \leq MK_1 \leq MK_2 \leq \ldots \leq MK_n \leq MK_{n+1} \leq \ldots ,$$

and M is a maximal subgroup of each MK_n by Lemma 2. It follows that $MK_n = MF$ for all n, and hence $MK = MF$, so that in particular $M \cap F$ is a normal subgroup of MK. Moreover,

$$\langle a \rangle = F \cap \Big(\bigcap_{M \in \mathfrak{L}} M \Big) = \bigcap_{M \in \mathfrak{L}} (M \cap F) ,$$

and so $\langle a \rangle$ is a normal subgroup of K. Therefore K acts on F as a group of power automorphisms, and hence $K' \leq C_N(F) = F$, so that K' is abelian and K is soluble. This contradiction proves that N is soluble, and so G is soluble-by-finite.

Assume now that the theorem is false, and choose a counterexample G whose soluble radical S has minimal derived length. If K is the smallest non-trivial term of the derived series of S, it follows that the factor group G/K is finitely generated. Suppose first that G/K is finite, and let a be a non-trivial element of K. Then the normal closure $\langle a \rangle^G$ is a finitely generated torsion-free abelian group, so that $G/(\langle a \rangle^G)^2$ contains a finite non-trivial normal subgroup, and hence it is finite by Lemma 1. It follows that G is finitely generated. Assume now that G is not finitely generated, so that the factor group G/K is infinite, and let E be a finitely generated subgroup of G such that $G = EK$. Then the index $|G : E|$ is infinite, and so E is contained in a nearly maximal subgroup M of G. Clearly $G = MK$ and $M \cap K$ is a normal subgroup of G. Since $M/(M \cap K)$ is a finitely generated nearly maximal subgroup of $G/(M \cap K)$, the group $G/(M \cap K)$ is also finitely generated. On the other hand, $G/(M \cap K)$ is the semidirect product of its infinite subgroups $M/(M \cap K)$ and $K/(M \cap K)$, and so it is a counterexample to the theorem. This argument shows that without loss of generality it can be assumed that the group G is finitely generated. Then G is abelian-by-finite by Lemma 5. In particular, the Fitting subgroup A of G is a finitely generated torsion-free abelian group. Suppose that A is not cyclic, so that it contains a subgroup B such that A/B is infinite and the subgroup T/B of all elements of finite order of A/B is not trivial. Let X be any nearly maximal subgroup of G containing B. Then the index $|A : A \cap X|$ is infinite, and so $A/(A \cap X)$ is torsion-free by Lemma 4. It follows that T is contained in X, a contradiction, since B is intersection of nearly maximal subgroups of G. Therefore A is infinite cyclic. Clearly A is contained in the centre of the normal subgroup $C = C_G(A)$ of G, and so $C/Z(C)$ is finite. Then also C' is finite, and hence $C' = 1$ and $A = C$, so that G/A has order at most 2. If G is not cyclic, it contains a non-trivial subgroup H of infinite index (see [5, Part 1, Theorem 4.33]). Then $H \cap A = 1$ and H has order 2, so that G is infinite dihedral. The theorem is proved. □

Note finally that the class of nearly IM-groups is not subgroup closed. To see this, let T_p and T_q be a Tarski p-group and a Tarski q-group, respectively (where p and q are distinct primes), and let P be a subgroup of order p of T_p. Then the direct product $G = T_p \times T_q$ is nearly IM, but its subgroup $H = PT_q$ does not have the same property.

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Dipartimento di Matematica e Applicazioni
Università degli Studi di Napoli «Federico II»
Complesso Monte S. Angelo
Via Cintia - 80126 Napoli
degiova@matna2.dma.unina.it
francios@matna2.dma.unina.it