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Straightening cell decompositions of cusped hyperbolic 3-manifolds

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Geometria. — *Straightening cell decompositions of cusped hyperbolic 3-manifolds.* Nota di Marina Pescini, presentata (*) dal Socio E. Vesentini.

Abstract. — Let M be an oriented cusped hyperbolic 3-manifold and let τ be a topological ideal triangulation of M. We give a characterization for τ to be isotopic to an ideal geodesic triangulation; moreover we give a characterization for τ to flatten into a partially flat triangulation. Finally we prove that straightening combinatorially equivalent topological ideal cell decompositions gives the same geodesic decomposition, up to isometry.

Key words: Hyperbolic 3-manifolds; Flat triangulations; Ideal cell decompositions.

Riassunto. — *Raddrizzamento di decomposizione di 3-varietà iperboliche con cuspidi.* In questo articolo studiamo le condizioni necessarie e sufficienti affinché una triangolazione topologica di una 3-varietà orientata iperbolica con cuspidi possa essere raddrizzata in modo da fornire una triangolazione geodetica ideale con tetraedri eventualmente piatti. Inoltre proviamo che raddrizzando decomposizioni topologiche ideali combinatoriamente equivalenti si ottiene la stessa cellularizzazione geodetica a meno di isometria.

0. Introduction

Let M be an oriented cusped hyperbolic 3-manifold, *i.e.* an oriented, non-compact, hyperbolic, complete 3-manifold of finite volume. It is well known that M is isometric to $\mathbb{H}^3/\Gamma$ for a suitable group Γ of orientation preserving isometries of $\mathbb{H}^3$. One of the unsolved problems in hyperbolic geometry is the existence of straight ideal triangulations of such a manifold. D. B. A. Epstein and R. C. Penner have proved the existence of straight ideal cell decompositions [2]; if we retriangulate such a decomposition, we do not necessarily obtain an ideal straight triangulation, but only a geodesic decomposition into tetrahedra of non-negative volume; this means that some tetrahedra may be flat. On the other hand, we know from standard spine theory that there exist topological ideal triangulations of M [1]. In this work we study equivalent conditions for such a triangulation to be isotopic to a straight ideal one. Moreover, we observe that under weaker hypotheses some 3-simplices may flatten when straightened, so that the resulting triangulation is made up of tetrahedra of possibly null volume.

The starting point of our arguments is the following result: let $\Delta \subset \mathbb{R}^3$ be the abstract tetrahedron and let Δ^* be Δ with vertices v_0, v_1, v_2, v_3 removed. Let f be an immersion of Δ^* into M such that $f(\Delta^*)$ is a tetrahedron of a fixed topological ideal triangulation τ of M; finally, let $\widetilde{f} : \Delta^* \to \mathbb{H}^3$ be a lifting of f. Then it is proved that $\widetilde{f}$ extends to a continuous function from Δ to $\overline{\mathbb{H}^3}$ such that $\{\widetilde{f}(v_i)\}_{i=0,\dots,3}$ is made up of points of $\partial\mathbb{H}^3$.

(*) Nella seduta del 9 gennaio 1998.

If we repeat this process for every lifting of f and for every tetrahedron in M, we obtain a topological ideal Γ-invariant triangulation $\widetilde{\tau}$ of $\mathbb{H}^3$. We prove that it is possible to straighten the triangulation τ of M if and only if for each tetrahedron $\widetilde{f}(\Delta^*)$ of $\widetilde{\tau}$ one of the following occurs:

1. no hyperbolic plane contains the vertices of $\widetilde{f}(\Delta)$ and the orientation on $\widetilde{f}(\Delta^*)$ induced by $\mathbb{H}^3$ coincides with the one induced by $\mathbb{R}^3$ through the map $\widetilde{f}$;
2. the vertices of $\widetilde{f}(\Delta)$ are distinct and contained in one hyperbolic plane.

If for all the tetrahedra 1) occurs, we obtain an ideal geodesic triangulation; otherwise we obtain a partially flat ideal geodesic triangulation of M.

Finally we show that if $\mathcal{R}$ and $\mathcal{R}'$ are two combinatorially equivalent geodesic ideal decompositions of M with polyhedra of non-negative volume, then $\mathcal{R}$ and $\mathcal{R}'$ are isometric. As a consequence of this theorem, we have that straightening equivalent ideal topological triangulations gives isometric ideal geodesic triangulation.

1. Straightening of tetrahedra

Throughout all this paper, M will indicate an oriented cusped hyperbolic 3-manifold, and $\Gamma < \mathrm{Isom}^+(\mathbb{H}^3)$ will be the group of orientation preserving isometries of $\mathbb{H}^3$ such that M is isometric to $\mathbb{H}^3/\Gamma$.

We give now the definition of *topological ideal decomposition* of M in a constructive way: let $\Delta_1, \ldots, \Delta_n$ be a finite number of abstract polyhedra; if σ_1, σ_2 are two distinct faces of a polyhedron Δ_i or of two different polyhedra Δ_i, Δ_j, let $f_{(\sigma_1,\sigma_2)}$ be either the empty set or a simplicial isomorphism between σ_1 and σ_2 such that, if $f_{(\sigma_1,\sigma_2)} \neq \emptyset$, then $f_{(\sigma_2,\sigma_1)} \neq \emptyset$ and $f_{(\sigma_2,\sigma_1)} = f^{-1}_{(\sigma_1,\sigma_2)}$; let $\sim$ be the relation on $\bigsqcup_{i=1}^n \Delta_i$ given by: $x \sim y$ if there exist σ_1, σ_2 such that $x \in \sigma_1, y \in \sigma_2, f_{(\sigma_1,\sigma_2)} \neq \emptyset$ and $f_{(\sigma_1,\sigma_2)}(x) = y$; let $\widetilde{Q}$ be the topological space defined as

$$\Big(\bigsqcup_{i=1}^{n} \Delta_i\Big) \Big/ \sim;$$

let us suppose that the link of every vertex in $\widetilde{Q}$ is homeomorphic to a torus; then we define the topological space Q as $\widetilde{Q}$ with vertices removed. Now, if we suppose that M and Q are homeomorphic, we have automatically defined a topological decomposition without vertices on M, that is a *topological ideal decomposition* of M.

Definition. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two decompositions of M with the same abstract polyhedra $\Delta_1, \ldots, \Delta_n$ and let $f_i, f_i' : \Delta_i^* \to M$ be the associated maps. We say that $\mathcal{P}$ and $\mathcal{P}'$ are* isotopic *if there exists a homeomorphism*

$$\Phi : M \times [0,1] \to M \times [0,1]$$

of the form

$$\Phi(x,t) = \big(\phi_t(x), t\big)$$

such that $\phi_0 = \mathrm{id}$ and $f_i' = \phi_1 \circ f_i$.

Definition. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two decompositions of M with the same abstract polyhedra $\Delta_1, \ldots, \Delta_n$ and let $f_i, f_i' : \Delta_i^* \to M$ be the associated maps. We say that $\mathcal{P}$* flattens *into $\mathcal{P}'$ if there exists a homeomorphism*

$$\Phi : M \times [0,1) \to M \times [0,1)$$

of the form

$$\Phi(x, t) = \big(\phi_t(x), t\big)$$

such that $\phi_0 = \mathrm{id}$ and such that Φ extends continuously from $M \times [0,1]$ to $M \times [0,1]$ so that $f_i' = \phi_1 \circ f_i$.

In particular, the case we are interested in is the following: let $\mathcal{P}$ be any topological ideal triangulation and $\mathcal{P}'$ a geodesic ideal one. Let us suppose either that $\mathcal{P}$ and $\mathcal{P}'$ are isotopic or that $\mathcal{P}$ flattens into $\mathcal{P}'$; let Φ be the function defining the isotopy or the flattening. Then in both cases $\Phi|_{\Delta^* \times [0,1]}$ parametrizes a deformation of $f(\Delta^*)$ into the geodesic ideal tetrahedron with the same vertices. If $f(\Delta^*)$ is straightenable non-flat, then we obtain a geodesic tetrahedron of positive volume; while if $f(\Delta^*)$ is straightenable flat, we obtain a geodesic flat tetrahedron.

Now consider two decompositions τ_1 and τ_2 of M with the same set of abstract polyhedra $\Delta_1, \ldots, \Delta_n$ and let $f_i, f_i' : \Delta_i^* \to M$ be the corresponding maps. Let us suppose τ_1 and τ_2 isotopic and let $\Phi : M \times [0,1] \to M \times [0,1]$ be the relative homeomorphism. Let us define a map $F_i : \Delta_i^* \times [0,1] \to M$ for each $i = 1, \ldots, m$ by $F_i(x, t) = \phi_t\big(f_i(x)\big)$. We fix i and omit all subscripts. Let us fix a lifting $\widetilde{F} : \Delta^* \times [0,1] \to \mathbb{H}^3$ of the function F.

Theorem 1 [3]. *For every vertex v of Δ there exists a point w on $\partial\mathbb{H}^3$ such that the following holds: if B is a horoball centred at w there exists a neighbourhood U of v in Δ such that*

$$\widetilde{F}\big((U \cap \Delta^*) \times [0,1]\big) \subset B;$$

moreover w is the fixed point at infinity of some parabolic element of Γ. In particular setting $\widetilde{F}(v, t) = w$ one gets a continuous extension of $\widetilde{F}$ from $\Delta \times [0,1]$ to $\overline{\mathbb{H}^3}$.

To prove Theorem 1 we will use the following trivial result:

Lemma 2. *Let M be topologically decomposed. Let Δ be an abstract polyhedron which appears in the glueing, let Δ^* be the version of Δ without vertices and let $f : \Delta^* \to M$ be the natural induced map. Then f is a proper function.*

Proof of Theorem 1. Let $\epsilon > 0$ be small enough that the ϵ-ends of M are disjoint. Denote by $M_{[\epsilon,\infty)}$ the ϵ-thick part of M, and recall that by finiteness of the volume this thick part is compact. The function F is proper, being composition of the following proper functions:

$$\Delta^* \times [0,1] \ni (x, t) \mapsto \big(f(x), t\big) \in M \times [0,1],$$

$$\Phi : M \times [0,1] \ni (x, t) \mapsto \big(\varphi_t(x), t\big) \in M \times [0,1],$$

$$M \times [0\,,1] \ni (x\,,t) \mapsto x \in M;$$

so $F^{-1}(M_{[\epsilon\,,\infty)})$ is compact in $\Delta^* \times [0\,,1]$, and its complement is an open subset of $\Delta \times [0\,,1]$. Therefore $F^{-1}(M_{[\epsilon\,,\infty)})$ contains some set of the form $U \times [0\,,1]$ with U neighbourhood of v. We can assume that U is connected, so that $F\big(U \times [0\,,1]\big)$ is contained in one of the ϵ-ends. Then

$$\widetilde{F}\left((U \cap \Delta^*) \times [0\,,1]\right)$$

is contained in some horoball B of $\mathbb{H}^3$. Call w the center of B. Let us show that for every horoball centred at w the first assertion of the proposition holds: let B' be another horoball centred at w, we can suppose $B' \subset B$; then it is sufficient to choose a neighbourhood $U' \subset U$ of v in Δ such that $U' \times [0\,,1]$ is contained in the complement of $F^{-1}(M_{[\epsilon\,,\infty)})$ in $\Delta \times [0\,,1]$, and such that

$$\widetilde{F}\left((U' \cap \Delta^*) \times [0\,,1]\right)$$

is contained in B'. Hence the first assertion is established.

By construction w is the fixed point at infinity of some parabolic element of Γ, and the last assertion of the proposition follows at once. □

Corollary 3. *Let τ be an ideal topological triangulation of M, let $f : \Delta^* \to M$ be an immersion of Δ^* into M such that $f(\Delta^*)$ is an element of τ, and let $\widetilde{f} : \Delta^* \to \mathbb{H}^3$ be a lifting of f. Then $\widetilde{f}$ extends to a continuous function $\widetilde{f}$ from Δ to $\mathbb{H}^3$ such that $\widetilde{f}(v_i) \in \partial\mathbb{H}^3$, where v_i, $i = 0\,, \ldots\,, 3$, are the vertices of Δ.*

Proof. It is enough to apply Theorem 1 in the case $\tau_1 = \tau_2 = \tau$, $\Phi = \mathrm{id}$. □

Let us note that if we take every lifting of f for every tetrahedron in M, we obtain a topological ideal Γ-invariant triangulation of $\mathbb{H}^3$.

Corollary 4. *If two ideal polyhedra decompositions of M are isotopic or one flattens into the other one, then they coincide.*

Proof. If the two decompositions are isotopic, it follows from Theorem 1 that their liftings have the same vertices at infinity; as an ideal geodesic polyhedron in $\mathbb{H}^3$ is uniquely determined by its points at infinity, the conclusion follows at once. Suppose now that one decomposition flattens into the other one; consider the continuous function $\widetilde{F}|_{\{v\}\times[0\,,1)}$; we know from the previous theorem that it is constant, so extending it by continuity on $\{v\} \times [0\,,1]$ we have a constant function and the two decompositions have the same vertices at infinity and the conclusion follows. □

Now let τ be an ideal topological triangulation of M, let $f : \Delta^* \to M$ be an immersion of Δ^* into M such that $f(\Delta^*)$ is a cell of τ, and let $\widetilde{f} : \Delta \to \overline{\mathbb{H}^3}$ be the extension of a lifting of f. Under these hypotheses we give the following definitions:

Definition. *A tetrahedron $\widetilde{f}(\Delta^*)$ in $\mathbb{H}^3$ is* straightenable flat *if all the vertices of $\widetilde{f}(\Delta)$ are contained in a hyperbolic plane, but distinct.*

Definition. *A tetrahedron $\widetilde{f}(\Delta^*)$ in $\mathbb{H}^3$ is* straightenable non-flat *if*

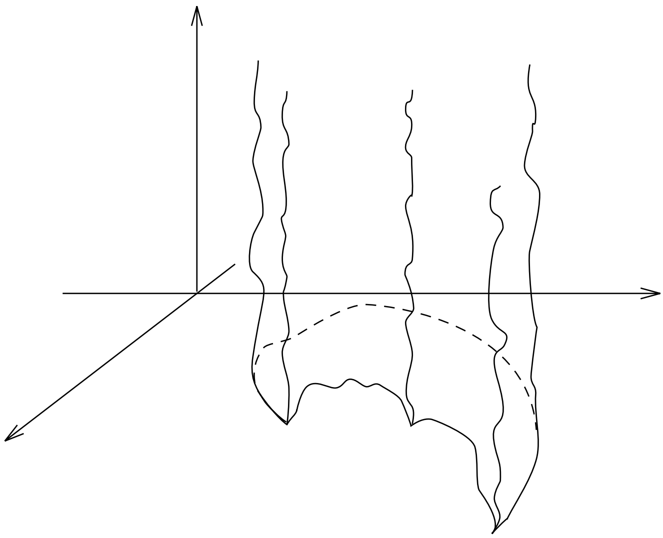

Fig. 1. – A 3-simplex which cannot be straightened.

1. *no hyperbolic plane contains the vertices of* $\widetilde{f}(\Delta)$, *and in particular such points are distinct;*
2. *the orientation on* $\widetilde{f}(\Delta^*)$ *induced by* $\mathbb{H}^3$ *coincides with the one induced by* $\mathbb{R}^3$ *through the map* $\widetilde{f}$.

It is easily checked that the two definitions are independent of the choice of the lifting, since all the relevant properties are preserved by orientation preserving isometries of $\mathbb{H}^3$.

Definition. *A tetrahedron* $f(\Delta^*)$ *in* τ *is* straightenable flat *(respectively* non-flat*) if* $\widetilde{f}(\Delta^*)$ *is straightenable flat (respectively* non-flat*).*

Proposition 5 [3]. *A topological triangulation* τ *is isotopic to a geodesic ideal decomposition with tetrahedra of positive volume if and only if every* 3*-simplex of* τ *is straightenable non-flat.*

Proof. The arrow ($\Rightarrow$) is obvious. Let us prove the necessity of the assertion: let $\Delta_1, \ldots, \Delta_n$ be the abstract polyhedra involved in the decomposition τ; let $f_i : \Delta_i^* \to M$ be the associated maps and let $\widetilde{f}_i : \Delta \to \overline{\mathbb{H}^3}$ be the extension of a lifting of f_i for every i; the hypotheses imply that there exists a homeomorphism

$$\Psi : \overline{\mathbb{H}^3} \times [0, 1] \to \overline{\mathbb{H}^3} \times [0, 1]$$

of the form

$$\Psi(x, t) = \big(\psi_t(x), t\big)$$

such that:

1. $\psi_t|_{\partial\mathbb{H}^3} = \mathrm{id}$ for every t,

2. $\psi_0 = \mathrm{id}$,
3. for every i, $(\psi_t \circ \widetilde{f}_i)(\Delta)$ parametrizes a deformation of a topological ideal simplex with fixed vertices $w_{i,0}, \dots, w_{i,3}$,
4. for every i, $(\psi_1 \circ \widetilde{f}_i)(\Delta)$ is the geodesic ideal tetrahedron of positive volume with vertices $w_{i,0}, \dots, w_{i,3}$.

So the conclusion follows. □

Proposition 6. *A topological decomposition into n ideal tetrahedra of M flattens into a geodesic ideal decomposition with tetrahedra of non-negative volume if and only if every 3-simplex is straightenable, flat or not.*

Proof. The arrow $(\Rightarrow)$ is obvious. The proof runs similarly to the previous one, with the only difference that in this case Ψ is a homeomorphism from $\mathbb{H}^3 \times [0, 1)$ in $\mathbb{H}^3 \times [0, 1)$ and it is continuous from $\mathbb{H}^3 \times [0, 1]$ in $\mathbb{H}^3 \times [0, 1]$, and, for every i, $(\psi_1 \circ \widetilde{f}_i)(\Delta)$ is the geodesic ideal tetrahedron with non-negative volume with vertices in $w_{i,0}, \dots, w_{i,3}$. □

2. Combinatorially equivalent decompositions

In this section we prove the following:

Theorem 7. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two combinatorially equivalent geodesic ideal decompositions of M with polyhedra of non-negative volume. Then $\mathcal{R}$ and $\mathcal{R}'$ are isometric.*

Let us recall that, if (X, d) and (X', d') are two metric spaces then $g : X \to X'$ is a *pseudo-isometry* if it induces an isomorphism between the respective fundamental groups and there exist two constants k and l such that $k^{-1}d(x, y) - l \leq d'(g(x), g(y)) \leq kd(x, y)$ for every x, y in X.

To prove Theorem 7 we use the following result:

Lemma 8. *Let $\varphi : M \to M$ be a homeomorphism mapping the decomposition $\mathcal{R}$ onto $\mathcal{R}'$. Then there exists a map isotopic to φ which maps $\mathcal{R}$ onto $\mathcal{R}'$ and such that its lifting to $\mathbb{H}^3$ is a pseudo-isometry.*

Proof. We can suppose $\varphi \in C^1$ without loss of generality. Let ϵ be small enough for every component of $M_{(0,\epsilon]}$ to be a topological end of M; then it is easy to show that there exists $\delta(\epsilon)$ such that every component F of $M_{(0,\delta(\epsilon)]}$ is a topological end of M, and $\varphi(F)$ is contained in a component of $M_{(0,\epsilon]}$; since φ is a homeomorphism, it is obvious that for every end F, $\varphi(F)$ is contained in a different component of $M_{(0,\epsilon]}$.

Let F be a component of $M_{(0,\delta(\epsilon)]}$, and let $F' = \varphi(F)$. We know from Margulis' Lemma (see [1]) that F is isometric to $T \times [0, \infty)$, where T is the torus, with a metric of the form

$$ds^2_{(x,t)} = e^{-2t} d\sigma^2_x + dt^2 ,$$

where $d\sigma^2_x$ represents the Euclidean metric on T.

Let us fix geometric coordinates $T\times[0\,,\infty)$ on F; then we can redefine φ on F such that it is still a C^1-homeomorphism respecting the combinatorics of the correspondence between $\mathcal{R}$ and $\mathcal{R}'$, and such that, if we choose suitable geometric coordinates $T'\times[0\,,\infty)$ on F', we have

$$\varphi|_F(x\,,t)=\big(g(x)\,,t\big)\,,$$

where $g: T\to T'$ is a C^1-homeomorphism.

Repeating the process for every component of $M_{(0\,,\delta(\epsilon)]}$, we obtain a C^1-homeomorphism from M to M (which we will indicate with the symbol φ again) isotopic to the starting function by construction.

We show now that the lifting of the new map φ is a pseudo-isometry: let φ_0 be $\varphi|_{M_{[\delta(\epsilon)\,,\infty)}}$, let K be a compact set in $\mathbb{H}^3$ such that $\pi(K)\subset M$ coincides with $M_{[\delta(\epsilon)\,,\infty)}$, and let $\widetilde{\varphi}_0$ be $\widetilde{\varphi}|_K$; since φ_0 is a C^1-map and $M_{[\delta(\epsilon)\,,\infty)}$ and $\varphi(M_{[\delta(\epsilon)\,,\infty)})$ are compact, we have that there exists a constant k such that

$$\big|D\varphi_0\big|\,,\ \big|D\varphi_0^{-1}\big|\leq k\,, \tag{1}$$

so

$$\big|D\widetilde{\varphi}_0\big|\,,\ \big|D\widetilde{\varphi}_0^{-1}\big|\leq k. \tag{2}$$

Let φ_F be the restriction of φ to F. Since $g\in C^1$, there exists a constant c such that

$$\big|Dg\big|\,,\ \big|Dg^{-1}\big|\leq c\,,$$

so we have that

$$\big|D\varphi_F\big(x\,,t\big)\big|^2=\big|\big(Dg(x)\,,1\big)\big|^2=\max\big\{\big|Dg(x)\big|^2\,,1\big\}\leq c^2+1 \qquad \forall\ \big(x\,,t\big)\in T\times[0\,,\infty)\,, \tag{3}$$

$$\big|D\varphi_F^{-1}\big(y\,,s\big)\big|^2=\big|\big(Dg^{-1}(y)\,,1\big)\big|^2=\max\big\{\big|Dg(y)\big|^2\,,1\big\}\leq c^2+1 \qquad \forall\ \big(y\,,s\big)\in T'\times[0\,,\infty). \tag{4}$$

We can suppose $c^2+1<k$, so that from (1)-(4) we get

$$|D\widetilde{\varphi}|\,,\ |D\widetilde{\varphi}^{-1}|\leq k\,, \tag{5}$$

and

$$d\big(\widetilde{\varphi}(x_1)\,,\widetilde{\varphi}(x_2)\big)\leq k\ d\big(x_1\,,x_2\big) \qquad \forall\ x_1\,,x_2\in\mathbb{H}^3\,, \tag{6}$$

$$d\big(\widetilde{\varphi}^{-1}(y_1)\,,\widetilde{\varphi}^{-1}(y_2)\big)\leq k\ d\big(y_1\,,y_2\big) \qquad \forall\ y_1\,,y_2\in\mathbb{H}^3. \tag{7}$$

Applying (6) first and then (7), we get

$$d\big(\widetilde{\varphi}(x_1)\,,\widetilde{\varphi}(x_2)\big)\geq k^{-1}d\big(\widetilde{\varphi}^{-1}\circ\widetilde{\varphi}(x_1)\,,\widetilde{\varphi}^{-1}\circ\widetilde{\varphi}(x_2)\big)=k^{-1}d\big(x_1\,,x_2\big) \qquad \forall\ x_1\,,x_2\in\mathbb{H}^3\,, \tag{8}$$

and the lemma is proved. $\square$

Proof of Theorem 7. Let $\varphi : M \to M$ be a homeomorphism such that $\varphi(\mathcal{R}) = \mathcal{R}'$. By the previous lemma we can suppose, without loss of generality, that, if $\widetilde{\varphi}$ is a lifting of φ, then $\widetilde{\varphi}$ is a pseudo-isometry. Under this hypothesis, as Mostow has showed in the proof of the rigidity theorem [4], we have the following facts:

1. $\widetilde{\varphi}|_{\partial\mathbb{H}^3}$ is a Möbius transformation of S^2;
2. there exists a homotopy $F : \mathbb{H}^3 \to \mathbb{H}^3$ between $\widetilde{\varphi}$ and the Möbius transformation $\widetilde{f} : \mathbb{H}^3 \to \mathbb{H}^3$ extending $\widetilde{\varphi}|_{\partial\mathbb{H}^3}$, and F induces a homotopy between φ and an isometry $f : M \to M$; *i.e.*:

$$\exists\ \Phi : M \times [0\,,1] \to M \text{ continuous such that}$$

$$\Phi(x, 0) = \varphi(x)\,,\ \ \Phi(x, 1) = f(x) \quad \forall\, x \in M;$$

moreover, using the upper hyperboloid model of the hyperbolic space, F is given by the following formula:

$$F(x, t) = \frac{(1-t)\widetilde{\varphi}(x) + t\widetilde{f}(x)}{|||(1-t)\widetilde{\varphi}(x) + t\widetilde{f}(x)|||}.$$

We prove now that f maps $\mathcal{R}$ onto $\mathcal{R}'$. For this purpose we need to show that F is a proper function: we notice that, given $x \in \mathbb{H}^3$, the set $F\big(x, [0\,,1]\big)$ is the geodesic arc joining the points $\widetilde{\varphi}(x)$ and $\widetilde{f}(x)$ in $\mathbb{H}^3$. Being $\widetilde{f}^{-1} \circ \widetilde{\varphi} = \mathrm{id}$ on $\partial\mathbb{H}^3$ and being $\widetilde{f}$ an isometry, we can suppose $\widetilde{f} = \mathrm{id}$. Therefore it is sufficient to show that, if $\widetilde{\varphi}|_{\partial\mathbb{H}^3} = \mathrm{id}$, there is a compact set K' in $\mathbb{H}^3$ such that, for every $x \in K$, we have $\widetilde{\varphi}(x)$, x and the geodesic arc joining them contained in K'. But this comes easily from the uniform continuity of $\widetilde{\varphi}$ on $\overline{\mathbb{H}^3}$.

Now let Δ be an abstract polyhedron and $i : \Delta^* \to M$ an immersion such that $i(\Delta^*)$ is a cell in the decomposition $\mathcal{R}$ of M. Let us consider a lifting $\widetilde{i} : \Delta^* \to \mathbb{H}^3$ of i and define the map

$$\widetilde{L} : \Delta^* \times [0\,,1] \to \mathbb{H}^3$$

as $\widetilde{L} := F \circ (\widetilde{i} \times \mathrm{id})$.

To prove the thesis we show that, for every vertex v of Δ, $\widetilde{L}$ extends to $\{v\} \times [0\,,1]$ and is continuous on this set; in fact this would imply that the isometry $f \equiv \Phi(\,\cdot\,,1)$ maps the cell $i(\Delta^*)$ of $\mathcal{R}$ on the correspondent cell of $\mathcal{R}'$. The assertion follows immediately from the proposition we are going to prove. □

Proposition 9. *For every vertex v of Δ there exists a point w in $\partial\mathbb{H}^3$ such that the following holds: if B is a horoball centred at w there exists a neighbourhood U of v in Δ such that*

$$\widetilde{L}\big((U \cap \Delta^*) \times [0\,,1]\big) \subset B;$$

moreover, w is the fixed point at infinity of some parabolic element of Γ. In particular, setting $\widetilde{L}(v, t) = w$ and repeating the same process for every vertex of Δ, one gets a continuous extension of $\widetilde{L}$ from $\Delta \times [0\,,1]$ to $\overline{\mathbb{H}^3}$.

We omit the proof of Proposition 9 since it is analogous to the proof of Theorem 1.

COROLLARY 10. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two topological ideal triangulations of M. Let us suppose $\mathcal{P}$ and $\mathcal{P}'$ combinatorially equivalent.*

- *If $\mathcal{P}$ is isotopic to a geodesic ideal triangulation $\widetilde{\mathcal{P}}$ of M with tetrahedra of positive volume, then also $\mathcal{P}'$ is isotopic to a geodesic ideal triangulation $\widetilde{\mathcal{P}'}$, and $\widetilde{\mathcal{P}'}$ is isometric to $\widetilde{\mathcal{P}}$;*
- *if $\mathcal{P}$ flattens into a geodesic ideal decomposition with tetrahedra of non-negative volume, then $\mathcal{P}'$ flattens into a geodesic ideal decomposition $\widetilde{\mathcal{P}'}$, and $\widetilde{\mathcal{P}'}$ is isometric to $\widetilde{\mathcal{P}}$.*

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