Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Castrenze Polizzotto

A maximum reduced dissipation principle for nonassociative plasticity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.2, p. 115–129.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_2_115_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:115-129 (1998)

Meccanica dei solidi. — *A maximum reduced dissipation principle for nonassociative plasticity.* Nota (*) di Castrenze Polizzotto, presentata dal Socio G. Maier.

Abstract. — The concept of reduced plastic dissipation is introduced for a perfectly plastic rate-independent material not obeyng the associated normality rule and characterized by a strictly convex plastic potential function. A maximum principle is provided and shown to play the role of variational statement for the nonassociative constitutive equations. The Kuhn-Tucker conditions of this principle describe the actual material behaviour as that of a (fictitious) composite material with two plastic constituents, each of which is associative in some suitably enlarged stress and strain spaces. The proposed principle is shown to identify with the classical one in case of associative plasticity. A simple illustrative example is reported.

Key words: Plasticity; Nonassociative yielding laws; Variational principles.

Riassunto. — *Un principio della massima dissipazione plastica ridotta nell'ambito della plasticità non associativa.* Per un materiale perfettamente plastico (privo di effetti viscosi) di tipo non associativo e caratterizzato da una funzione potenziale plastico strettamente convessa, si introduce il concetto di dissipazione plastica ridotta. Si propone un principio di massimo e si mostra che esso rappresenta una formulazione variazionale delle equazioni costitutive della plasticità non associativa. Le condizioni di Kuhn-Tucker relative al suddetto principio descrivono il comportamento costitutivo del materiale reale come quello di un materiale (fittizio) composito con due costituenti ciascuno dei quali è di tipo plastico associativo in taluni spazi di tensioni e di deformazioni opportunamente ampliati. Il principio proposto si identifica con quello classico nel caso di plasticità associativa. Si riporta una semplice applicazione illustrativa.

1. Introduction

The constitutive behaviour of elastic-plastic rate-independent material models can be described through two convex stress functions, *i.e.* the yield function, $f(\boldsymbol{\sigma}) \leq 0$, and the plastic potential function, $g(\boldsymbol{\sigma})$. The material behaves elastically for stress states such that $f(\boldsymbol{\sigma}) < 0$, whereas, for stress states on the yield surface, $f(\boldsymbol{\sigma}) = 0$, it may undergo a plastic strain increment the direction of which coincides with that of the gradient of the equipotential surface passing through $\boldsymbol{\sigma}$. Many material models, especially metals and alloys, obey the so-called *associative* normality rule; that is, they deform exhibiting plastic strain increments normal to the yield surface, such that $g \equiv f$ for this class of materials; material models (*e.g.* geomaterials) for which $g \not\equiv f$ are referred to as nonassociative material models [1, 8, 11, 14].

Within associative plasticity, a fundamental inequality is available which can be interpreted either as Drucker's stability postulate [5], or as the maximum (plastic) dissipation theorem of Mises, Taylor and Hill [7, 11]. This inequality, which is referred to as the maximum dissipation theorem (or principle) in the following, provides a firm theoretical basis to a great deal of developments of the classical plasticity theory, *e.g.* assessment of

(*) Pervenuta in forma definitiva all'Accademia il 19 settembre 1997.

solution uniqueness, limit plastic and shakedown analyses, steady-state stabilized response under periodic loads, convergence questions.

For materials not obeying the associative normality rule, the maximum dissipation theorem does not hold, nor something similar to it is available. This lack of a basic inequality, as powerful as the above one, has likely prevented the nonassociative plasticity theory from being developed as the associative plasticity theory has been. For instance, just to mention a few significant points, fundamental theorems analogous to those of the classical limit plastic analysis are not available for such materials, and the limit loads of a structure made of this material can only be loosely bracketed by limit loads computed by applying the classical theorems to fictitious associative materials with suitably chosen yield functions according to the methods of Radenkovic [18], de Josselin [4], Palmer [16], Salençon [19]. Similar considerations can be made for shakedown analysis and for the evaluation of the limit shakedown loads using methods of Pycko and Maier [17], Corigliano *et al.* [2]. Additionally, whereas variational statements to characterize the solution uniqueness and stability can be established within associative plasticity basing on appropriate positive definite, or positive semi-definite, functionals [9], such procedures are no longer valid for nonassociative plasticity, where only weaker statements have been proposed, *e.g.* by Mróz [15], Maier [12].

The present paper has the purpose to provide, within nonassociative plasticity and under the restriction of *strictly convex* potential function g, a suitable central inequality as a basis to establish a maximum *reduced* dissipation theorem. The concept of reduced (plastic) dissipation (D^{rd}) is introduced as the difference between the *actual* dissipation (D) and the *auxiliary* dissipation (D^*), *i.e.* $D^{rd} = D - D^*$. D^{rd} turns out to be a one-degree homogeneous function of plastic strain rates ($\dot{\varepsilon}^p$), and its first partial derivative at any regular point $\dot{\varepsilon}^p \neq \mathbf{0}$ equals the *reduced stress*, $\boldsymbol{\sigma}^{rd}$, this stress being the difference between the *actual stress* ($\boldsymbol{\sigma}$) and the *auxiliary stress* ($\boldsymbol{\sigma}^*$), *i.e.* $\boldsymbol{\sigma}^{rd} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^*$, (but the derivatives of D and D^* differ from $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^*$ by an additive *nondissipative* stress, $\mathbf{s}$, *i.e.* $\partial D/\partial\dot{\varepsilon}^p = \boldsymbol{\sigma} + \mathbf{s}$, $\partial D^*/\partial\dot{\varepsilon}^p = \boldsymbol{\sigma}^* + \mathbf{s}$). All three dissipation functions, D^{rd}, D and D^*, are *nonnegative* and, in general, *nonconvex*.

The above theorem will be shown to constitute a variational statement of the nonassociative constitutive plastic flow laws, in the sense that the related Kuhn-Tucker conditions provide these flow laws, but cast in an equivalent associative-type format; that is, the material behaviour is described in the enlarged space of the actual and auxiliary stresses, by two distinct sets of associative plastic flow laws. One of these is stress-guided, *i.e.* it operates in the *actual* stress $\boldsymbol{\sigma}$-space with the (actual) yield function $f(\boldsymbol{\sigma}) \leq 0$ and some (fictitious) *conventional* plastic strain rates, $\dot{\varepsilon}^{pc}$, normal to $f(\boldsymbol{\sigma}) = 0$; the other is strain-guided, *i.e.* it operates in the (fictitious) *auxiliary* stress $\boldsymbol{\sigma}^*$-space with some auxiliary (linear) yield function $f^*(\boldsymbol{\sigma}^*) \leq 0$ and the *actual* plastic strain rates, $\dot{\varepsilon}^p$, normal to $f^*(\boldsymbol{\sigma}^*) = 0$. A *liaison* is placed between the two above stress spaces through the requirement that the conventional plastic strain rate, $\dot{\varepsilon}^{pc}$, satisfies a specific *conjugation rule* (centred upon the Hessian matrix, $\mathbf{G}$, of the potential g). In case of associative plasticity, the proposed theorem will be shown to coincide with the classic maximum dissipation theorem, with $D^{rd} = D$, $D^* = 0$ and $\dot{\varepsilon}^{pc} = \dot{\varepsilon}^p$.

The outline of the paper is the following. After some preliminary considerations in Section 2, the central inequality of this work is established in Section 3. Section 4 is devoted to the derivation of the maximum reduced dissipation principle and to the related Kuhn-Tucker conditions with their mechanical interpretations. Some essential properties of the reduced dissipation are established in Section 5. Section 6 reports a simple application and conclusions are drawn in Section 7.

Notation. A compact notation will be used, with vectors and tensors denoted by bold face symbols. The dot ($\cdot$) and colon ($:$) products between vectors and tensors denote the simple and double index saturation operations, as for instance: $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ji}$, $\boldsymbol{\sigma} : \mathbf{A} : \boldsymbol{\sigma} = A_{ijhk}\sigma_{ji}\sigma_{kh}$, $(\mathbf{n}\cdot\boldsymbol{\sigma})_i = \sigma_{ji}n_j$, $(\mathbf{A} : \boldsymbol{\sigma})_{ij} = A_{ijhk}\sigma_{kh}$, where the indices denote Cartesian components and the repeated index summation rule is applied. The upper dot means time derivative, *e.g.* $\dot{a} = \partial a/\partial t$. The symbol $:=$ means equality by definition. Other symbols will be defined in the text, where they appear for the first time.

2. Preliminary considerations

A rate-independent perfectly plastic material not obeying the associated normality rule can be characterized by constitutive equations as

$$\dot{\varepsilon}^p = \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma}) , \qquad \mathbf{q}(\boldsymbol{\sigma}) := \partial g/\partial\boldsymbol{\sigma} \tag{1a}$$

$$f(\boldsymbol{\sigma}) \leq 0 , \quad \dot{\lambda} \geq 0 , \quad \dot{\lambda} f(\boldsymbol{\sigma}) = 0 \tag{1b}$$

where $\dot{\lambda}$ is the plastic (or consistency) coefficient, $f = f(\boldsymbol{\sigma})$ is the yield function and $g = g(\boldsymbol{\sigma})$ is the plastic potential function. The following assumptions are made as for the latter functions:

- $f(\boldsymbol{\sigma})$ is convex and smooth and the yield surface $f(\boldsymbol{\sigma}) = 0$ contains the origin, *i.e.* $f(\mathbf{0}) < 0$.
- $g(\boldsymbol{\sigma})$ is strictly convex and smooth, (at least) twice differentiable with respect to all its arguments, the latter arguments being coincident with the arguments of $f(\boldsymbol{\sigma})$.
- The gradient $\mathbf{q}(\boldsymbol{\sigma})$ is vanishing at the stress origin, *i.e.* $\mathbf{q}(\mathbf{0}) = \mathbf{0}$, (hence $\mathbf{q}(\boldsymbol{\sigma}) \neq \mathbf{0}$ $\forall\boldsymbol{\sigma} \neq \mathbf{0}$, by the strict convexity of g).
- $g(\boldsymbol{\sigma})$ takes finite values at any point $\boldsymbol{\sigma}$ located at a finite distance $|\boldsymbol{\sigma}|$ from the origin and satisfies the following properties:

$$\forall\boldsymbol{\sigma} : f(\boldsymbol{\sigma}) \geq 0 \Rightarrow g(\boldsymbol{\sigma}) \geq 0 \tag{2a}$$

$$C_g \subseteq C_f , \quad \text{with } C_g = C_f \text{ only if } g \equiv f, \tag{2b}$$

where C_g and C_f denote the convex sets of stress points enclosed by the reference equipotential surface, $g(\boldsymbol{\sigma}) = 0$, and the yield surface, $f(\boldsymbol{\sigma}) = 0$, respectively, *i.e.*

$$C_g := \{\boldsymbol{\sigma} : \ g(\boldsymbol{\sigma}) \leq 0\} , \ C_f := \{\boldsymbol{\sigma} : \ f(\boldsymbol{\sigma}) \leq 0\} . \tag{2c}$$

– $g(\boldsymbol{\sigma})$ being specified to within an additive constant, the reference equipotential surface, $g(\boldsymbol{\sigma}) = 0$, is allowed either to find itself in internal contact with the yield surface, $f(\boldsymbol{\sigma}) = 0$, or to degenerate into a single point, *i.e.* the origin $\boldsymbol{\sigma} = \mathbf{0}$ (in which case it is $g(\boldsymbol{\sigma}) > 0 \ \forall\, \boldsymbol{\sigma} \neq \mathbf{0}$), or even to be located somewhere in between, inside the yield surface.

According to eqs. ($1a$, b), the plastic strain rate vector $\dot{\varepsilon}^p$ is normal to the equipotential surface $g(\boldsymbol{\sigma}) =$ const passing through the stress point $\boldsymbol{\sigma}$, whereas the loading/unloading rule is governed by ($1b$) through the yield function. Since g intervenes in ($1a$, b) with its gradient $\mathbf{q}$, any additive constant to g is without consequences to the material behaviour description. On choosing $g(\boldsymbol{\sigma}) \equiv f(\boldsymbol{\sigma})$, it is $C_g = C_f$, and a material model with associated normality rule is generated.

An insight to ($1a$, b) enables one to state the following. If $\dot{\varepsilon}^p \neq \mathbf{0}$, it is $\dot{\lambda} > 0$ and $f(\boldsymbol{\sigma}) = 0$. Due to the assumptions made on g, the equation system ($1a$) can be solved for $\boldsymbol{\sigma}$ to have $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\dot{\varepsilon}^p/\dot{\lambda})$. The latter equation, with $\dot{\varepsilon}^p$ fixed and $\dot{\lambda} > 0$ variable, is the parametric equation of a line $\Lambda(\dot{\varepsilon}^p)$ in the stress space, locus of points $\boldsymbol{\sigma}$ where the gradient $\mathbf{q}(\boldsymbol{\sigma})$ is parallel to $\dot{\varepsilon}^p$, fig. 1. Λ departs from the stress origin (where $\dot{\lambda} =+ \infty$) and progresses on one side as $\dot{\lambda}$ decreases, intersecting the reference equipotential surface, $g(\boldsymbol{\sigma}) = 0$, at some point M_0 and then the yield surface, $f(\boldsymbol{\sigma}) = 0$, at some point M. (The gradient magnitude $|\mathbf{q}(\boldsymbol{\sigma})|$ at points $\boldsymbol{\sigma} \in \Lambda(\dot{\varepsilon}^p)$ increases with $|\boldsymbol{\sigma}|$, while remaining finite). As the line $\Lambda(\dot{\varepsilon}^p)$ only depends on the direction of $\dot{\varepsilon}^p$ and not on its magnitude, and since $\boldsymbol{\sigma}$ must be on the yield surface, $f = 0$, it follows that the stress $\boldsymbol{\sigma}$ corresponding to a given $\dot{\varepsilon}^p$ is the intersection M of $\Lambda(\dot{\varepsilon}^p)$ with the yield surface (fig. 1), and that $\boldsymbol{\sigma}$ turns out to be a zero-degree (positively) homogeneous function of $\dot{\varepsilon}^p$. The yield condition $f(\boldsymbol{\sigma}(\dot{\varepsilon}^p/\dot{\lambda})) = 0$ provides the $\dot{\lambda}$ value corresponding to point M as a one-degree homogeneous function, $\dot{\lambda}(\dot{\varepsilon}^p)$, by which $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\dot{\varepsilon}^p)$ can be obtained. On the other hand, for $\dot{\varepsilon}^p = \mathbf{0}$, it is $\dot{\lambda} = 0$ and $f(\boldsymbol{\sigma}) \leq 0$, whereas $\Lambda(\dot{\varepsilon}^p)$ is indeterminate.

Equations ($1a$, b) comply with the second principle of thermodynamics [10], as in fact the relevant plastic dissipation $D = \boldsymbol{\sigma}(\dot{\varepsilon}^p) : \dot{\varepsilon}^p$ is always nonnegative, *i.e.*

$$D(\dot{\varepsilon}^p) = \left(\dot{\lambda}\boldsymbol{\sigma} : \mathbf{q}(\boldsymbol{\sigma})\right)\Big|_{\dot{\varepsilon}^p} \geq \dot{\lambda}(\dot{\varepsilon}^p) g\left(\boldsymbol{\sigma}(\dot{\varepsilon}^p)\right) \geq 0\,, \tag{3}$$

where eq. ($2a$) and the inequality $g(\boldsymbol{\sigma}) < \boldsymbol{\sigma} : \mathbf{q}(\boldsymbol{\sigma}) \ \forall \boldsymbol{\sigma} \neq \mathbf{0}$ have been used; additionally, the equality sign between the second and third members holds if, and only if, $\dot{\varepsilon}^p = \mathbf{0}$. $D(\dot{\varepsilon}^p)$ is homogeneous to degree one like for associative plasticity, but it is (in general) nonconvex and its partial derivatives (where they exist) do not provide the stresses $\boldsymbol{\sigma}(\dot{\varepsilon}^p)$, as it will be shown later on. Equation (3) suggests one to consider the *reduced* dissipation function

$$D^{rd} := D(\dot{\varepsilon}^p) - D^*(\dot{\varepsilon}^p) \geq 0 \tag{4}$$

where D^* is the *auxiliary* dissipation, defined as

$$D^* = D^*(\dot{\varepsilon}^p) := \dot{\lambda}(\dot{\varepsilon}^p) g\left(\boldsymbol{\sigma}(\dot{\varepsilon}^p)\right). \tag{5}$$

Both D^{rd} and D^* are nonnegative, vanish for $\dot{\varepsilon}^p = \mathbf{0}$, and are degree-one homogeneous.

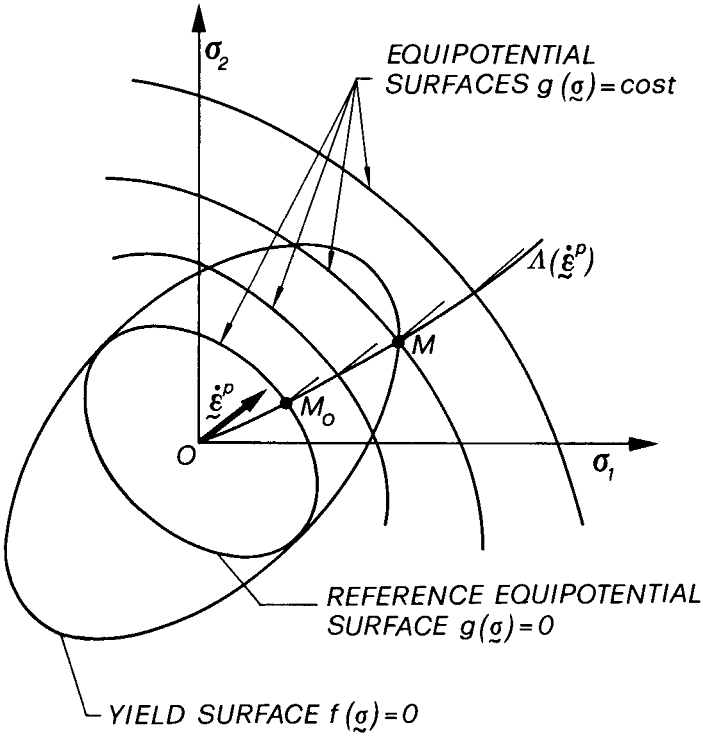

Fig. 1. – Yield surface $f(\sigma_1, \sigma_2) = 0$ and equipotential surfaces $g(\sigma_1, \sigma_2) = \text{const}$ in two-dimensional stress space.

In next Section, D^{rd} will be shown to play a variational role for the nonassociative flow laws ($1a, b$). Note that, with the choice $g \equiv f$, one has $D^{rd} = D$, $D^* = \dot{\lambda} f = 0$.

3. Central inequalities

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ be two distinct stress states of the material. By the convexity of $g(\boldsymbol{\sigma})$, one can write

$$g(\boldsymbol{\sigma}') - g(\boldsymbol{\sigma}) \geq \mathbf{q}(\boldsymbol{\sigma}) : (\boldsymbol{\sigma}' - \boldsymbol{\sigma}) \tag{6}$$

which holds for any pair $(\boldsymbol{\sigma}, \boldsymbol{\sigma}')$, with the equality sign only if $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$. Assume that the latter stress $\boldsymbol{\sigma}$ and some $\dot{\lambda} > 0$ and $\dot{\varepsilon}^p$ satisfy eqs. ($1a, b$). Thus, multiplying both sides of (6) by $\dot{\lambda}$ gives the inequality

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}') : \dot{\varepsilon}^p - [g(\boldsymbol{\sigma}) - g(\boldsymbol{\sigma}')]\dot{\lambda} \geq 0 \tag{7}$$

which holds for arbitrary $\boldsymbol{\sigma}'$. Note that the equality sign in (7) holds if, and only if, either $\boldsymbol{\sigma} = \boldsymbol{\sigma}'$, or $\dot{\varepsilon}^p = \mathbf{0}$ (hence $\dot{\lambda} = 0$), or both.

Inequality (7) can be further transformed by distinguishing two cases as to the way $\boldsymbol{\sigma}'$ is chosen:

i) $\boldsymbol{\sigma}' \in C_g$, *i.e.* $g(\boldsymbol{\sigma}') \leq 0$, in which case $\dot{\lambda} g(\boldsymbol{\sigma}') \leq 0$. Then, inequality (7) implies the following:

$$(8)\qquad (\boldsymbol{\sigma} - \boldsymbol{\sigma}') : \dot{\varepsilon}^p - \dot{\lambda} g(\boldsymbol{\sigma}) \geq 0\,,$$

which, since $\dot{\lambda} g(\boldsymbol{\sigma}) \geq 0$, can be rewritten as

$$(9)\qquad (\boldsymbol{\sigma} - \boldsymbol{\sigma}') : \dot{\varepsilon}^p \geq 0 \qquad \forall\, \boldsymbol{\sigma}' \in C_g.$$

Inequality (9) coincides with Drucker's inequality for $g(\boldsymbol{\sigma}) \leq 0$ taken as the relevant yield function.

ii) $\boldsymbol{\sigma}' \notin C_g$, *i.e.* $g(\boldsymbol{\sigma}') > 0$. Then, let $(\boldsymbol{\sigma}'\,,\, \dot{\lambda}')$ belong to the set $K(\dot{\varepsilon}^p)$ defined as:

$$(10)\qquad K(\dot{\varepsilon}^p) := \left\{ (\boldsymbol{\sigma}'\,,\, \dot{\lambda}') :\ f(\boldsymbol{\sigma}') \leq 0\,,\ \dot{\lambda}' \geq 0\,,\ \dot{\lambda}' \mathbf{q}(\boldsymbol{\sigma}') = \dot{\varepsilon}^p \right\}.$$

Equation (10) means that $\boldsymbol{\sigma}'$ lays on the $\Lambda(\dot{\varepsilon}^p)$ line, somewhere in between the origin O and M, fig. 1, (right on M if $\dot{\lambda}' = \dot{\lambda}$). Since $|\mathbf{q}(\boldsymbol{\sigma}')| < |\mathbf{q}(\boldsymbol{\sigma})|$ for $|\boldsymbol{\sigma}'| \neq |\boldsymbol{\sigma}|$, by the equality $\dot{\lambda}'|\mathbf{q}(\boldsymbol{\sigma}')| = \dot{\lambda}|\mathbf{q}(\boldsymbol{\sigma})|$ it is $\dot{\lambda}' > \dot{\lambda}$, hence $\dot{\lambda}' g(\boldsymbol{\sigma}') > \dot{\lambda} g(\boldsymbol{\sigma}')$, and inequality (7) implies:

$$(11)\qquad (\boldsymbol{\sigma} - \boldsymbol{\sigma}') : \dot{\varepsilon}^p - [\dot{\lambda} g(\boldsymbol{\sigma}) - \dot{\lambda}' g(\boldsymbol{\sigma}')] \geq 0 \quad \forall\, (\boldsymbol{\sigma}'\,,\, \dot{\lambda}') \in K^+(\dot{\varepsilon}^p)$$

where

$$(12)\qquad K^+(\dot{\varepsilon}^p) := \left\{ (\boldsymbol{\sigma}'\,,\, \dot{\lambda}') \in K(\dot{\varepsilon}^p) :\ g(\boldsymbol{\sigma}') > 0 \right\}\,,$$

which means that $\boldsymbol{\sigma}'$ lays on the $\Lambda(\dot{\varepsilon}^p)$ line somewhere in between M_0 and M, fig. 1, (the extreme M_0 being not allowed).

It is worth noting that for $g \equiv f$ inequality (9) transforms into Drucker's inequality of associative plasticity, whereas inequality (11) just disappears because the set $K^+(\dot{\varepsilon}^p)$ turns out to be empty (*i.e.* $M_0 \equiv M$ in fig. 1).

For subsequent use, it is convenient to assemble inequalities (9) and (11) into a single one. To this purpose, the Macauley operator is introduced, *i.e.* $\langle x \rangle = (x + |x|)/2$ for any scalar x, together with the step function $H(x)$ defined as

$$(13)\qquad H(x) := \begin{cases} 1 \text{ for } x > 0\,, \\ 0 \text{ for } x \leq 0\,. \end{cases}$$

With this notation, inequalities (9) and (11) can be given the following unified format:

$$(14)\qquad (\boldsymbol{\sigma} - \boldsymbol{\sigma}') : \dot{\varepsilon}^p - \left[\dot{\lambda} g(\boldsymbol{\sigma}) - \dot{\lambda}' \langle g(\boldsymbol{\sigma}') \rangle \right] \geq 0 \quad \forall\, (\boldsymbol{\sigma}'\,,\, \dot{\lambda}') \in K_C$$

where

$$(15)\qquad K_C := \left\{ (\boldsymbol{\sigma}'\,,\, \dot{\lambda}') :\ f(\boldsymbol{\sigma}') \leq 0\,,\ \dot{\lambda}' \geq 0\,,\ [\dot{\varepsilon}^p - \dot{\lambda}' \mathbf{q}(\boldsymbol{\sigma}')] H(g(\boldsymbol{\sigma}')) = \mathbf{0} \right\}.$$

The set in which $\boldsymbol{\sigma}'$ is allowed to vary in eq. (14) is constituted by the union of the convex set C_g enclosed by the reference equipotential surface, and the segment $M_0 M \subset \Lambda(\dot{\varepsilon}^p)$, fig. 1. It can be easily verified that inequality (14) identifies: either with (9) for $\boldsymbol{\sigma}' \in C_g$, in which case $\langle g(\boldsymbol{\sigma}') \rangle = H(g(\boldsymbol{\sigma}')) = 0$, the first constraint of (15) is satisfied due to property (2b), whereas $\dot{\lambda}\, g(\boldsymbol{\sigma}) > 0$; or even with (11) for $\boldsymbol{\sigma}' \notin C_g$, in which case $\langle g(\boldsymbol{\sigma}')\rangle = g(\boldsymbol{\sigma}')$ and $H(g(\boldsymbol{\sigma}')) = 1$.

4. Maximum reduced dissipation principle

Inequality (14), remembering (4) and (5), can be given the form

$$D^{rd} = \boldsymbol{\sigma} : \dot{\varepsilon}^p - \dot{\lambda} g(\boldsymbol{\sigma}) \geq \boldsymbol{\sigma}' : \dot{\varepsilon}^p - \dot{\lambda}' \langle g(\boldsymbol{\sigma}')\rangle\,, \quad \forall (\boldsymbol{\sigma}'\,, \dot{\lambda}') \in K_C \tag{16}$$

where $D^{rd} = D^{rd}(\dot{\varepsilon}^p)$ is the *reduced* dissipation corresponding to $\dot{\varepsilon}^p$. Since K_C includes the pair $\boldsymbol{\sigma}\,, \dot{\lambda}$ pertaing to D^{rd} of (16), it follows that inequality (16) expresses a maximum principle for $D^{rd}(\dot{\varepsilon}^p)$, namely (dropping the primes for simplicity):

$$D^{rd}(\dot{\varepsilon}^p) = \max_{(\boldsymbol{\sigma}\,,\dot{\lambda})} \psi(\boldsymbol{\sigma}\,, \dot{\lambda}) \equiv \boldsymbol{\sigma} : \dot{\varepsilon}^p - \dot{\lambda}\langle g(\boldsymbol{\sigma})\rangle \quad \text{s.t.} \;\; (\boldsymbol{\sigma}\,, \dot{\lambda}) \in K_C \tag{17}$$

where «s.t.» stands for «subject to». Either (16) and (17) will be referred to as the maximum reduced dissipation principle (or theorem) in the following.

Since $\dot{\lambda} = |\dot{\varepsilon}^p|/|\mathbf{q}(\boldsymbol{\sigma})|$ by the third constraint of eq. (15) assumed to be effective, the function $\psi = \psi(\boldsymbol{\sigma}\,, \dot{\lambda})$ of eq. (17) can be written in terms of $\boldsymbol{\sigma}$ as:

$$\psi = \hat{\psi}(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} : \dot{\varepsilon}^p - \frac{|\dot{\varepsilon}^p|\langle g(\boldsymbol{\sigma})\rangle}{|\mathbf{q}(\boldsymbol{\sigma})|} \qquad (\dot{\varepsilon}^p \text{ fixed}) \tag{18}$$

which, as long as $g(\boldsymbol{\sigma}) \leq 0$, is linear. For $g(\boldsymbol{\sigma}) > 0$, the derivative of $\hat{\psi}$ reads:

$$\frac{\partial \hat{\psi}}{\partial \boldsymbol{\sigma}} = \frac{|\dot{\varepsilon}^p| g(\boldsymbol{\sigma})}{|\mathbf{q}(\boldsymbol{\sigma})|^3} \mathbf{G} : \mathbf{q}(\boldsymbol{\sigma}) \tag{19}$$

where $\mathbf{G}$ is the (positive definite) Hessian matrix of g, *i.e.* $\mathbf{G} := \partial^2 g/(\partial\boldsymbol{\sigma} \otimes \partial\boldsymbol{\sigma})$. On differentiating the identity $\dot{\lambda}\mathbf{q}(\boldsymbol{\sigma}) = \dot{\varepsilon}^p$, with $\dot{\varepsilon}^p \neq \mathbf{0}$ fixed, one has

$$\mathrm{d}\boldsymbol{\sigma} = -\frac{\mathrm{d}\dot{\lambda}}{\dot{\lambda}} \mathbf{G}^{-1} : \mathbf{q}\,, \qquad |\mathrm{d}\boldsymbol{\sigma}| = \frac{|\mathrm{d}\dot{\lambda}|}{\dot{\lambda}} (\mathbf{q} : \mathbf{G}^{-1} : \mathbf{G}^{-1} : \mathbf{q})^{1/2} \tag{20}$$

such that the direction cosines of the tangent to $\Lambda(\dot{\varepsilon}^p)$ are given by:

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{|\mathrm{d}\boldsymbol{\sigma}|} = \frac{1}{Q} \mathbf{G}^{-1} : \mathbf{q} \tag{21a}$$

where the condition $|d\dot{\lambda}| = -d\dot{\lambda}$ has been accounted for and moreover

$$Q := (\mathbf{q} : \mathbf{G}^{-1} : \mathbf{G}^{-1} : \mathbf{q})^{1/2}. \tag{21b}$$

It follows that the tangential derivative of $\hat{\psi}$ along the curve $\Lambda(\dot{\varepsilon}^p)$ is:

$$\frac{\partial \hat{\psi}}{\partial \boldsymbol{\sigma}} : \frac{\mathrm{d}\boldsymbol{\sigma}}{|\mathrm{d}\boldsymbol{\sigma}|} = \frac{|\dot{\varepsilon}^p| g(\boldsymbol{\sigma})}{Q|\mathbf{q}|} > 0 \quad \forall\, \boldsymbol{\sigma} : g(\boldsymbol{\sigma}) > 0. \tag{22}$$

Therefore, the function $\hat{\psi}$ turns out to be monotonically increasing all along the line $\Lambda(\dot{\varepsilon}^p)$ from the value $\hat{\psi} = 0$ at the origin $\boldsymbol{\sigma} = \mathbf{0}$, and takes its constrained maximum value at point M, the intersection of $\Lambda(\dot{\varepsilon}^p)$ with the yield surface $f(\boldsymbol{\sigma}) = 0$, fig. 1.

It is of interest to derive the Kuhn-Tucker conditions of problem (17). To this aim, one can apply the Lagrange multiplier method by introducing the augmented function:

$$L = -\boldsymbol{\sigma} : \dot{\varepsilon}^p + \dot{\lambda}\langle g(\boldsymbol{\sigma})\rangle + \dot{\lambda}^c f(\boldsymbol{\sigma}) + \boldsymbol{\sigma}^* : \left[\dot{\varepsilon}^p - \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma})\right] H(g(\boldsymbol{\sigma})) \tag{23}$$

where $\dot{\lambda}^c \geq 0$ and $\boldsymbol{\sigma}^*$ are the relevant scalar and stress-like Lagrange multipliers. Following known procedures of mathematical programming [3, 13], the necessary Kuhn-Tucker conditions of problem (17) read:

$$\dot{\varepsilon}^{pc} := \dot{\lambda}^c \partial f / \partial \boldsymbol{\sigma} = \left[1 - H(g(\boldsymbol{\sigma}))\right] \dot{\varepsilon}^p + \dot{\lambda}\mathbf{G} : \boldsymbol{\sigma}^* H(g(\boldsymbol{\sigma})) \tag{24a}$$

$$f(\boldsymbol{\sigma}) \leq 0\,, \quad \dot{\lambda}^c \geq 0\,, \quad \dot{\lambda}^c f(\boldsymbol{\sigma}) = 0 \tag{24b}$$

$$\left[\dot{\varepsilon}^p - \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma})\right] H(g(\boldsymbol{\sigma})) = \mathbf{0} \tag{24c}$$

$$f^*(\boldsymbol{\sigma}^*) := \mathbf{q}(\boldsymbol{\sigma}) : \boldsymbol{\sigma}^* - \langle g(\boldsymbol{\sigma})\rangle \leq 0\,, \quad \dot{\lambda} \geq 0\,, \quad \dot{\lambda} f^* = 0. \tag{24d}$$

Here, $\dot{\varepsilon}^{pc}$ denotes some fictitious plastic strain rate tensor (with $\dot{\lambda}^c$ being the related plastic coefficient), which obeys the normality rule with respect to the yield surface $f = 0$ (for this reason it is referred to as the *conventional* plastic strain rate in the following). Also, $f^*(\boldsymbol{\sigma}^*) = 0$ denotes some fictitious yield surface of the $\boldsymbol{\sigma}^*$ stress space (superposed to the $\boldsymbol{\sigma}$ space), that is a plane orthogonal to $\mathbf{q}(\boldsymbol{\sigma})$ and located at a distance $d^* = \langle g(\boldsymbol{\sigma})\rangle / |\mathbf{q}(\boldsymbol{\sigma})|$ from the origin $\boldsymbol{\sigma}^* = \mathbf{0}$. As long as $g(\boldsymbol{\sigma}) > 0$, eq. (24*c*) provides the assigned plastic strain rate $\dot{\varepsilon}^p = \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma})$ as normal to the yield plane $f^* = 0$, *i.e.* $\dot{\varepsilon}^p = \dot{\lambda}\partial f^* / \partial \boldsymbol{\sigma}^*$, $\dot{\lambda}$ being the relevant plastic coefficient; also, the condition $\dot{\lambda} f^* = 0$ gives $\boldsymbol{\sigma}^* : \dot{\varepsilon}^p = \dot{\lambda} g(\boldsymbol{\sigma}) = D^*$, which explains the physical meaning of $\boldsymbol{\sigma}^*$ as the stress producing, through the actual plastic strain rate $\dot{\varepsilon}^p$, the auxiliary dissipation D^* (for this reason, $\boldsymbol{\sigma}^*$ is referred as the *auxiliary* stress in the following).

A better understanding of eqs. (24*a*-*d*) can be achieved by distinguishing two cases, namely:

a) Associative plasticity, i.e. $g \equiv f$, in which case $\langle g(\boldsymbol{\sigma})\rangle \equiv 0\,,\ H(g(\boldsymbol{\sigma})) \equiv 0$. Thus, eq. (17) transforms into

$$D^{rd}(\dot{\varepsilon}^p) = \max_{(\boldsymbol{\sigma})} \boldsymbol{\sigma} : \dot{\varepsilon}^p \quad \text{s.t. } f(\boldsymbol{\sigma}) \leq 0\,, \tag{25}$$

(the sign constraint $\dot{\lambda} \geq 0$ becoming meaningless), whereas the Kuhn-Tucker conditions (24*a*, *b*) take on the form:

$$\dot{\varepsilon}^{pc} := \dot{\lambda}^c \partial f / \partial \boldsymbol{\sigma} = \dot{\varepsilon}^p \tag{26a}$$

$$f(\boldsymbol{\sigma}) \leq 0\,, \quad \dot{\lambda}^c \geq 0\,, \quad \dot{\lambda}^c f(\boldsymbol{\sigma}) = 0. \tag{26b}$$

Also, the Kuhn-Tucker condition (24*c*) drops and (24*d*) becomes $\boldsymbol{\sigma}^* : \dot{\varepsilon}^p = D^* = 0$; that is, the auxiliary dissipation vanishes and the auxiliary stress becomes indeterminate, but normal to $\dot{\varepsilon}^p$. In other words, for $g \equiv f$, problem (17) identifies with the maximum dissipation theorem of associative plasticity, the conventional plastic strain rate

$\dot{\varepsilon}^{pc}$ identifies with the actual one, $\dot{\varepsilon}^{p}$, and the reduced dissipation coincides with the actual dissipation, $D^{rd} = D$.

b) *Nonassociative plasticity, i.e.* $g \not\equiv f$. Since the solution to problem (17) encopasses a stress point M on the yield surface, $f(\boldsymbol{\sigma}) = 0$, eqs. (24*a-d*) hold with $g(\boldsymbol{\sigma}) > 0$, (except when the surface $g(\boldsymbol{\sigma}) = 0$ is tangential to $f(\boldsymbol{\sigma}) = 0$ and for a particular $\dot{\varepsilon}^{p}$ such that M turns out to be a contact point). In order to simplify the discussion, let the plastic potential $g(\boldsymbol{\sigma})$ be chosen such as $g(\boldsymbol{\sigma}) > 0 \;\; \forall\, \boldsymbol{\sigma} \neq \mathbf{0}$ (what can always be done through a suitable additive constant). With this choice, problem (17) transforms into

$$D^{rd}(\dot{\varepsilon}^{p}) = \max_{(\boldsymbol{\sigma}\,,\dot{\lambda})} \left[\boldsymbol{\sigma} : \dot{\varepsilon}^{p} - \dot{\lambda} g(\boldsymbol{\sigma})\right] \text{ s.t. } (\boldsymbol{\sigma}\,,\dot{\lambda}) \in K(\dot{\varepsilon}^{p}) \tag{27}$$

and the Kuhn-Tucker conditions (24*a-d*) read:

$$\dot{\varepsilon}^{pc} := \dot{\lambda}^{c}\mathbf{p}(\boldsymbol{\sigma})\,, \qquad \big(\mathbf{p}(\boldsymbol{\sigma}) := \partial f/\partial\boldsymbol{\sigma}\big) \tag{28a}$$

$$f(\boldsymbol{\sigma}) \leq 0\,, \quad \dot{\lambda}^{c} \geq 0\,, \quad \dot{\lambda}^{c} f(\boldsymbol{\sigma}) = 0 \tag{28b}$$

$$\dot{\varepsilon}^{p} = \dot{\lambda}\partial f^{*}/\partial\boldsymbol{\sigma}^{*} = \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma}) \tag{29a}$$

$$f^{*} := \mathbf{q}(\boldsymbol{\sigma}) : \boldsymbol{\sigma}^{*} - g(\boldsymbol{\sigma}) \leq 0\,, \quad \dot{\lambda} \geq 0\,, \quad \dot{\lambda} f^{*} = 0 \tag{29b}$$

$$\dot{\varepsilon}^{pc} = \dot{\lambda}\partial g^{*}/\partial\boldsymbol{\sigma}^{*} = \dot{\lambda}\mathbf{G} : \boldsymbol{\sigma}^{*}\,, \quad g^{*} := (\boldsymbol{\sigma}^{*} : \mathbf{G} : \boldsymbol{\sigma}^{*})/2. \tag{30}$$

Equations (28)-(30) describe two distinct sets of associative flow laws operating in two superposed stress spaces, respectively. More precisely:

(*i*) In the actual stress space, $\boldsymbol{\sigma}$, eqs. (28*a*, *b*) express the conventional plastic strain rate tensor, $\dot{\varepsilon}^{pc}$, (with the pertinent plastic coefficient $\dot{\lambda}^{c}$), as normal to the actual yield surface, $f(\boldsymbol{\sigma}) = 0$;

(*ii*) In the superposed auxiliary stress space, $\boldsymbol{\sigma}^{*}$, eqs. (29*a*, *b*) express the actual plastic strain rate tensor, $\dot{\varepsilon}^{p}$, (with its actual plastic coefficient $\dot{\lambda}$), as normal to the auxiliary yield surface (hyperplane), $f^{*}(\boldsymbol{\sigma}^{*}) = 0$;

(*iii*) Equation (30) establishes a *liaison* between the two stress spaces with the requirement that $\dot{\varepsilon}^{pc}$ is normal to the Hessian ellipsoidal surface $g^{*} = \text{const}$ passing through $\boldsymbol{\sigma}^{*}$, and thus $\dot{\varepsilon}^{pc}$ and $\boldsymbol{\sigma}^{*}$ are conjugate of each other with respect to this ellipsoidal surface.

In other words, eqs. (28)-(30) describe the constitutive behaviour of a sort of (fictitious) composite material with two plastic constituents (or phases), these constituents being associative and sharing complementary aspects of the real material; that is, the stress-guided constituent, which is governed by the actual stress state but deforms in a fictitious way, and the strain-guided constituent, which deforms as the actual material but is governed by the (fictitious) auxiliary stress state.

The essential features of eqs. (28)-(30) are geometrically interpreted in fig. 2, where a set as $(\boldsymbol{\sigma}\,,\, \dot{\lambda}\,,\, \boldsymbol{\sigma}^{*}\,,\, \dot{\lambda}^{c})$ is assumed to be the/a solution to eqs. (28)-(30). The stress

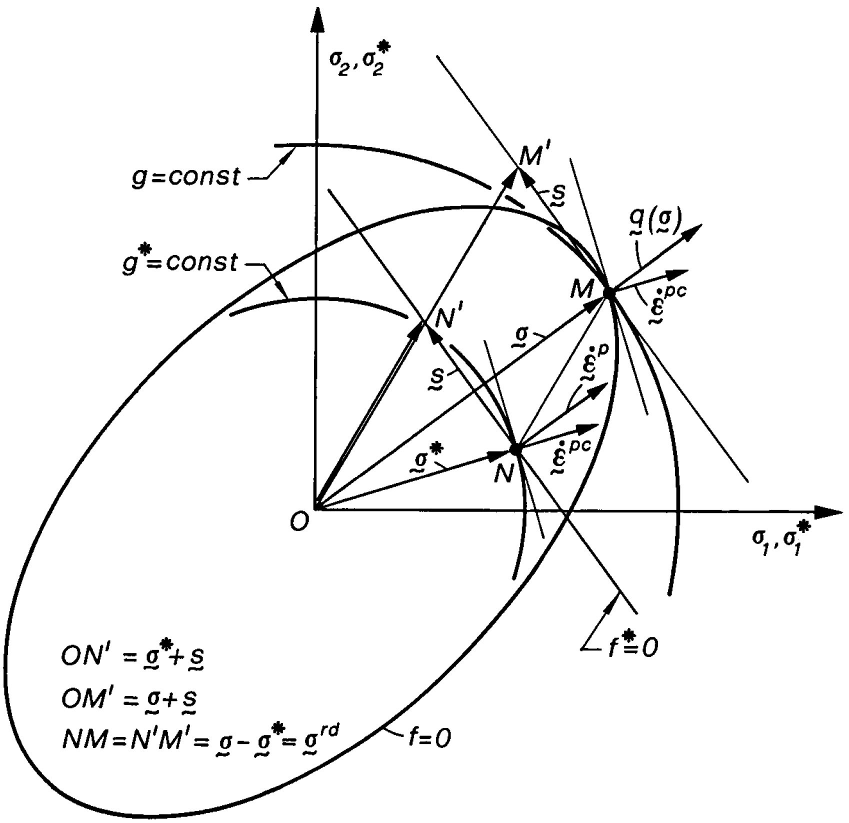

Fig. 2. – Non-associative material behaviour described as a fictitious composite material with two associative plastic constituents in the superposed two-dimensional stress and strain spaces.

vector $\boldsymbol{\sigma}$, drown from the origin O, touches the yield surface at point M, from where departs the gradient $\mathbf{q}(\boldsymbol{\sigma})$ of the equipotential surface $g(\boldsymbol{\sigma}) = \text{const}$ passing through M. According to eqs. ($29a$, b), the actual plastic strain rate, $\dot{\varepsilon}^p$ (which obeys a nonassociated normality rule in the $\boldsymbol{\sigma}$-space), on the contrary obeys an associated normality rule in the $\boldsymbol{\sigma}^*$-space, the relevant yield function being the hyperplane $f^* = \mathbf{q}(\boldsymbol{\sigma}) : \boldsymbol{\sigma}^* - g(\boldsymbol{\sigma}) = 0$, which is normal to the gradient $\mathbf{q}(\boldsymbol{\sigma})$ and is located at a distance $d^* = g(\boldsymbol{\sigma})/|\mathbf{q}(\boldsymbol{\sigma})|$ from the origin. The auxiliary stress, $\boldsymbol{\sigma}^*$, corresponding to $\dot{\varepsilon}^p \neq \mathbf{0}$, touches the hyperplane $f^* = 0$ at point N (fig. 2), from where $\dot{\varepsilon}^p$ departs parallel to $\mathbf{q}(\boldsymbol{\sigma})$. The conventional plastic strain, $\dot{\varepsilon}^{pc}$, departing from M normal to $f(\boldsymbol{\sigma}) = 0$, is parallel to the normal to the Hessian ellipsoidal surface $g^* = \text{const}$ passing through point N. (Note that, if g is a quadratic function, the Hessian matrix is constant and thus $g^*(\boldsymbol{\sigma}^*) \equiv g(\boldsymbol{\sigma}^*)$, *i.e.* the Hessian ellipsoidal surfaces identify with the equipotential surfaces $g(\boldsymbol{\sigma}) = \text{const}$, but they are embedded in the $\boldsymbol{\sigma}^*$-space).

Equations (28)-(30) imply that the nonassociative constitutive equations ($1a$, b) are satisfied, as in fact the following can be remarked.

α) If $\dot{\varepsilon}^p \neq \mathbf{0}$, it is $\dot{\lambda} > 0$ by ($29a$) and the second equation in ($29b$). Since the stress $\boldsymbol{\sigma}^*$ cannot vanish by (29b), (for $\boldsymbol{\sigma}^* = \mathbf{0}$ it would be $\dot{\lambda} g(\boldsymbol{\sigma}) = 0$, hence $g(\boldsymbol{\sigma}) = 0$, which is impossible), it follows from (30) that $\dot{\varepsilon}^{pc} \neq \mathbf{0}$, hence $\dot{\lambda}^c > 0$, $f(\boldsymbol{\sigma}) = 0$ by (28b). Equation $f(\boldsymbol{\sigma}) = 0$ and the equation system (29a) are then sufficient to uniquely determine $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\dot{\varepsilon}^p)$ and $\dot{\lambda} = \dot{\lambda}(\dot{\varepsilon}^p)$. Next, solving (30) for $\boldsymbol{\sigma}^*$ gives

$$\boldsymbol{\sigma}^* = (\dot{\lambda}^c/\dot{\lambda})\mathbf{G}^{-1} : \mathbf{p}(\boldsymbol{\sigma}) \tag{31}$$

and substituting the latter in the first (29b), which holds as an equality, yields:

$$\dot{\lambda}^c = \dot{\lambda} g(\boldsymbol{\sigma}) / \big(\mathbf{q}(\boldsymbol{\sigma}) : \mathbf{G}^{-1}(\boldsymbol{\sigma}) : \mathbf{p}(\boldsymbol{\sigma})\big)\,. \tag{32}$$

β) If $\dot{\varepsilon}^p = \mathbf{0}$, it is $\dot{\lambda} = 0$ by (29a), hence $\dot{\varepsilon}^{pc} = \mathbf{0}$ by (30) and $\dot{\lambda}^c = 0$, $f(\boldsymbol{\sigma}) \leq 0$ by (28a, b).

Since, therefore, the coeffients $\dot{\lambda}$ and $\dot{\lambda}^c$ are both either positive, in which case $f(\boldsymbol{\sigma}) = 0$, or vanishing, in which case $f(\boldsymbol{\sigma}) \leq 0$, it can be concluded that the Kuhn-Tucker conditions of eqs. (28)-(30) imply that eqs. (1a, b) are satisfied; that is, the complementarity conditions (28b) hold true also if $\dot{\lambda}^c$ is replaced by $\dot{\lambda}$. Thus, the fictitious composite associative material described by eqs. (28)-(30) is fully equivalent to the actual nonassociative material described by eqs. (1).

The above reasoning on the equivalence of eqs. (28)-(30) to eqs. (1) shows that eqs. (28)-(30) admit, for fixed $\dot{\varepsilon}^p \neq \mathbf{0}$, a unique solution and that therefore they are also sufficient conditions to the (nonconvex) problem (17). This result may perhaps be given a more rigorous justification within the framework of mathematical programming for generalized convex functions [13], but this point is not discussed here for brevity.

5. Some properties of the reduced dissipation

In this Section, nonassociative plasticity is considered, *i.e.* $g \not\equiv f$. The reduced dissipation of eq. (4) has been shown to be a homogeneous function of $\dot{\varepsilon}^p$ to degree one, like D and D^*. It will be shown by an example (in next Section) that D^{rd}, D, D^* are, in general, nonconvex. These three dissipation functions are differentiable for any $\dot{\varepsilon}^p \neq \mathbf{0}$, that is

$$\partial D/\partial \dot{\varepsilon}^p = \boldsymbol{\sigma} + \mathbf{s}\,, \quad \partial D^*/\partial \dot{\varepsilon}^p = \boldsymbol{\sigma}^* + \mathbf{s}\,, \tag{33}$$

and thus

$$\partial D^{rd}/\partial \dot{\varepsilon}^p = \boldsymbol{\sigma} - \boldsymbol{\sigma}^*\,, \tag{34}$$

where $\mathbf{s}$ is normal to $\dot{\varepsilon}^p$ and is thus *nondissipative*.

In order to prove the above, let $D = \boldsymbol{\sigma}(\dot{\varepsilon}^p) : \dot{\varepsilon}^p$ and $D^* = \boldsymbol{\sigma}^*(\dot{\varepsilon}^p) : \dot{\varepsilon}^p$ be differentiated with respect to $\dot{\varepsilon}^p$. One has:

$$\partial D/\partial \dot{\varepsilon}^p = \boldsymbol{\sigma} + \mathbf{Z} : \dot{\varepsilon}^p\,, \quad \mathbf{Z} := \big(\partial \boldsymbol{\sigma}/\partial \dot{\varepsilon}^p\big)^T \tag{35a}$$

$$\partial D^*/\partial \dot{\varepsilon}^p = \boldsymbol{\sigma}^* + \mathbf{Z}^* : \dot{\varepsilon}^p\,, \quad \mathbf{Z}^* := \big(\partial \boldsymbol{\sigma}^*/\partial \dot{\varepsilon}^p\big)^T\,. \tag{35b}$$

Because $f(\boldsymbol{\sigma}) = 0$ and $f^*(\boldsymbol{\sigma}^*) = 0$ as $\dot{\varepsilon}^p$ changes, differentiating these two identities with respect to $\dot{\varepsilon}^p$ one obtains the equalities:

$$\mathbf{Z} : \mathbf{p} = \mathbf{0} \tag{36a}$$

$$(\mathbf{Z} - \mathbf{Z}^*) : \mathbf{q}(\boldsymbol{\sigma}) = \mathbf{Z} : \mathbf{G} : \boldsymbol{\sigma}^* \tag{36b}$$

which hold for any $\dot{\varepsilon}^p \neq \mathbf{0}$. But, by eqns. (28*a*) and (30), it is $\mathbf{G} : \boldsymbol{\sigma}^* = (\dot{\lambda}^c/\dot{\lambda})\mathbf{p}$ and thus, by (36*a*), the r.h. side of (36*b*) vanishes, hence

$$(\mathbf{Z} - \mathbf{Z}^*) : \mathbf{q}(\boldsymbol{\sigma}) = \mathbf{0} \tag{37}$$

for any $\dot{\varepsilon}^p = \dot{\lambda}\mathbf{q}(\boldsymbol{\sigma}) \neq \mathbf{0}$. Equation (37) is equivalent to $\mathbf{Z} : \mathbf{q}(\boldsymbol{\sigma})\dot{\lambda} = \mathbf{Z}^* : \mathbf{q}(\boldsymbol{\sigma})\dot{\lambda}$, and thus one can write

$$\mathbf{s} := \mathbf{Z} : \dot{\varepsilon}^p = \mathbf{Z}^* : \dot{\varepsilon}^p. \tag{38}$$

This, through eqs. (35*a*, *b*), proves eq. (33), whereas eq. (34) is a straightforward consequence. Then, multiplying eq. (35*a*) by $\dot{\varepsilon}^p$ gives

$$(\partial D/\partial \dot{\varepsilon}^p) : \dot{\varepsilon}^p = \boldsymbol{\sigma} : \dot{\varepsilon}^p + \dot{\varepsilon}^p : \mathbf{Z} : \dot{\varepsilon}^p; \tag{39}$$

that is, D being homogeneous to degree one, it is

$$\dot{\varepsilon}^p : \mathbf{Z} : \dot{\varepsilon}^p = \mathbf{s} : \dot{\varepsilon}^p = 0 \tag{40}$$

for any $\dot{\varepsilon}^p \neq \mathbf{0}$, hence $\mathbf{s}$ is normal to $\dot{\varepsilon}^p$, fig. 2.

6. Example

A two-dimensional stress state is considered for a simple example. Let one take $f \equiv \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 - \sigma_y^2$, $g \equiv \sigma_1^2 + \sigma_2^2$. The yield surface is an ellipse, the equipotential surfaces are circles and circles are also the related Hessian equipotential surfaces ($\mathbf{G} = 2\mathbf{I}_2$). Using eqs. (28)-(30) and posing

$$\delta := (\dot{\varepsilon}_1^{p2} + \dot{\varepsilon}_2^{p2} + \dot{\varepsilon}_1^p\dot{\varepsilon}_2^p)^{1/2}, \tag{41}$$

one obtains:

$$\sigma_1 = \sigma_y\dot{\varepsilon}_1^p/\delta, \quad \sigma_2 = \sigma_y\dot{\varepsilon}_2^p/\delta, \tag{42}$$

$$\sigma_1^* = \sigma_y(2\dot{\varepsilon}_1^p - \dot{\varepsilon}_2^p)(\dot{\varepsilon}_1^{p2} + \dot{\varepsilon}_2^{p2})/4\delta^3 \tag{43a}$$

$$\sigma_2^* = \sigma_y(2\dot{\varepsilon}_2^p - \dot{\varepsilon}_1^p)(\dot{\varepsilon}_1^{p2} + \dot{\varepsilon}_2^{p2})/4\delta^3 \tag{43b}$$

$$D = \sigma_y(\dot{\varepsilon}_1^{p2} + \dot{\varepsilon}_2^{p2})/\delta, \quad D^{rd} = D^* = D/2. \tag{44}$$

With the positions:

$$\dot{\varepsilon}_1^p = r\cos\alpha, \quad \dot{\varepsilon}_2^p = r\sin\alpha \tag{45}$$

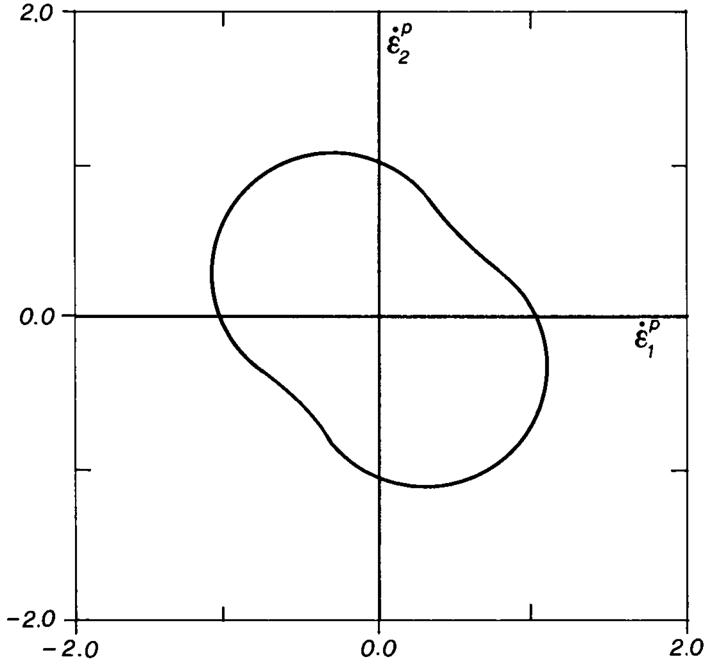

Fig. 3. – Plot representing the contour line $D(\dot{\varepsilon}_1^p, \dot{\varepsilon}_2^p)/\sigma_y = 1$ for a material having $f = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 + -\sigma_y^2 \leq 0$ as yield function and $g = \sigma_1^2 + \sigma_2^2$ as plastic potential function.

one obtains the contours $D/\sigma_y = c =$ const as the curves

$$\begin{aligned} \dot{\varepsilon}_1^p &= c\cos\alpha\sqrt{1-(1/2)\sin 2\alpha} \\ \dot{\varepsilon}_2^p &= c\sin\alpha\sqrt{1-(1/2)\sin 2\alpha} \end{aligned} \qquad (0 \leq \alpha \leq 2\pi) \tag{46}$$

Figure 3 is the plot of the contour $D/\sigma_y = c = 1$.

The nondissipative stresses s_1 and s_2 and matrix $\mathbf{Z}$ of Section 5 read:

$$s_1 = \sigma_y \dot{\varepsilon}_2^p(\dot{\varepsilon}_2^{p2} - \dot{\varepsilon}_1^{p2})/2\delta^3, \quad s_2 = \sigma_y \dot{\varepsilon}_1^p(\dot{\varepsilon}_1^{p2} - \dot{\varepsilon}_2^{p2})/2\delta^3 \tag{47}$$

$$\mathbf{Z} = \frac{\sigma_y}{2\delta^3}\begin{bmatrix} \dot{\varepsilon}_2^p(2\dot{\varepsilon}_2^p - \dot{\varepsilon}_1^p) & -\dot{\varepsilon}_2^p(2\dot{\varepsilon}_1^p - \dot{\varepsilon}_2^p) \\ -\dot{\varepsilon}_1^p(2\dot{\varepsilon}_2^p - \dot{\varepsilon}_1^p) & \dot{\varepsilon}_1^p(2\dot{\varepsilon}_1^p - \dot{\varepsilon}_2^p) \end{bmatrix}. \tag{48}$$

Equations (33), (34), (38) and (39) can be easily verified to be met.

7. Conclusions

The proposed maximum reduced dissipation principle can be regarded as a variational statement of the constitutive flow laws for nonassociative plasticity, in the sense that the related Kuhn-Tucker conditions are equivalent to these constitutive equations. It constitutes a generalization of the classical maximum dissipation principle of associative plasticity, which is recovered as soon as the relevant plastic potential identifies with the yield function.

The aforementioned Kuhn-Tucker conditions, which constitute a set of alternative constitutive equations for the nonassociative material, describe the material behaviour as that of a (fictitious) composite material with two plastic constituents, each constituent being associative within suitably enlarged stress and strain spaces. One constituent is governed by the actual stresses and experiences the (fictitious) conventional plastic strain rate vector, which is normal to the actual yield surface; the other constituent is governed by the (fictitious) auxiliary stresses and experiences the actual plastic strain rate vector, which is normal to the auxiliary yield (plane) surface, whereas a conjugancy rule places a liaison between the two stress spaces.

The description of the fictitious composite material model makes use of a number of new concepts and definitions that seem indeed to be too many. They however arise in a quite natural way and perhaps represent an inevitable cost to transform the nonassociative model into an equivalent composite model of associative nature.

The proposed principle holds under the essential hypothesis that the plastic potential function g be strictly convex. Such restriction seems not to place excessive limitations to the validity of the proposed principle, which can thus be regarded as a general statement for the nonassociative material behaviour. The equivalent associative-type constitutive equations, derived through this principle, need to be checked as for their effectiveness, particularly in relation to the following points.

a) Study of the conditions for uniqueness and stability of the material states in comparison to the related nonassociative contitutive equations. The passage to enlarged stress and strain spaces may likely carry a reacher phenomenology as for nonuniqueness and instabilities in the material behaviour.

b) In analogy to the classical maximum dissipation principle, which can be assumed as a starting point to derive the known theorems of limit plastic and shakedown analyses [6], the proposed principle is expected to provide new means to address the above analysis problems for materials not obeying the associated normality rule. Developements of this kind are under study at the present stage.

Extensions of the proposed principle to more general material models can be achieved without excessive difficulties, but this point is left open to future research work.

Acknowledgements

This paper has been completed with the financial support of the Ministero dell'Università e della Ricerca Scientifica e Tecnologica, MURST.

References

[1] W.-F. Chen - A. F. Saleeb, *Constitutive Equations for Engineering Materials*. Vol. 2: *Plasticity and Modelling*. Elsevier, Amsterdam 1994.

[2] A. Corigliano - G. Maier - S. Pycko, *Kinematic criteria of dynamic shakedown extended to nonassociative constitutive laws with saturation nonlinear hardening*. Rend. Mat. Acc. Lincei, s. 9, v. 6, 1995, 55-64.

[3] R. Courant - D. Hilbert, *Methods of Mathematical Physics*. Vol. 1, Interscience Publishers Inc., New York 1953.
[4] G. de Josselin de Jong, *Lower bound collapse theorem and lack of normality of strainrate to yield surface for soils*. In: J. Kravtchenko - P. M. Sirieys (eds.), *Reology and Soil Mechanics*. Springer-Verlag, Berlin 1966, 69-78.
[5] D. C. Drucker, *Plasticity*. In: J. N. Goodier - J.H. Hoff (eds.), *Structural Mechanics*. Pergamon Press, London 1960, 407-445.
[6] P. Fuschi - C. Polizzotto, *The shakedown load boundary of an elastic-perfectly plastic structure*. Meccanica, 30, 1995, 155-174.
[7] R. Hill, *The Mathematical Theory of Plasticity*. Oxford University Press, Oxford 1950.
[8] S. Kaliszky, *Plasticity. Theory and Engineering Applications*. Elsevier, Amsterdam 1989.
[9] W. T. Koiter, *General theorems of elastis-plastic solids*. In: J. S. Sneddon - R. Hill (eds.), *Progress in Solid Mechanics*. North Holland, Amsterdam 1960, 1, 167-221.
[10] J. Lemaitre - J. L. Chaboche, *Mécanique des Matériaux Solides*. Dunod, Paris 1985.
[11] J. Lubliner, *Plasticity Theory*. Macmillan Publishing Company, New York 1990.
[12] G. Maier, *A minimum principle for incremental elastoplasticity with non-associated flow laws*. J. Mech. Phys. Solids, 18, 1970, 319-330.
[13] O. L. Mangasarian, *Nonlinear Programming*. McGraw-Hill Book Company, New York 1969.
[14] J. B. Martin, *Plasticity: Fundamentals and General Results*. The MIT press, Cambridge, Ma., 1975.
[15] Z. Mróz, *Non-associated flow laws in plasticity*. J. de Mécanique, II, n. 1, 1963, 21-42.
[16] A. C. Palmer, *A limit theorem for materials with nonassociative flow laws*. J. de Mécanique, V, n. 2, 1966, 217-222.
[17] S. Pycko - G. Maier, *Shakedown theorems for some classes of nonassociative hardening elastic-plastic material models*. Int. J. Plasticity, 11, 1995, 367-395.
[18] D. Radenkovic, *Théorie des charges limites. Extension à la mécanique du sol*. In: J. Mandel (ed.), *Séminaire de Plasticitè.* (Ecole Polytechnique 1961), Magasin C.T.O., Paris 1962, 129-141.
[19] J. Salençon, *Théorie des charges limites*. In: *Seminaire de Plasticité et Viscoplasticité*. Ecole Polytechnique, 1972, Ediscience, Paris 1974, 205-229.

Pervenuta il 18 aprile 1997,
in forma definitiva il 19 settembre 1997.

Dipartimento di Ingegneria Strutturale e Geotecnica
Università degli Studi di Palermo
Viale delle Scienze - 90128 Palermo