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Time and space Sobolev regularity of solutions to homogeneous parabolic equations

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Analisi matematica. — *Time and space Sobolev regularity of solutions to homogeneous parabolic equations.* Nota di Gabriella Di Blasio, presentata (*) dal Corrisp. G. Da Prato.

Abstract. — We give necessary and sufficient conditions on the initial data such that the solutions of parabolic equations have a prescribed Sobolev regularity in time and space.

Key words: Parabolic equations; Sobolev regularity; Interpolation spaces.

Riassunto. — *Regolarità di Sobolev nel tempo e nello spazio per soluzioni di equazioni paraboliche.* In questo lavoro si caratterizzano i dati iniziali per cui le soluzioni di equazioni paraboliche hanno un'assegnata regolarità di Sobolev rispetto al tempo e allo spazio.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with regular boundary $\partial\Omega$ and let E be a second order uniformly elliptic operator in Ω; *i.e.*,

$$Eu := \sum_{i,j=1}^{n} (a_{i,j}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} (b_i(x) u)_{x_i} + c(x) u \ . \tag{1.1}$$

See [1, 2] for the precise assumptions on Ω and the coefficients of E. We want to study the Sobolev regularity, with respect to time and space, of the solutions of the following problem:

$$u_t(t, x) = Eu(t, x) \ , \quad (t, x) \in]0 , T] \times \Omega \tag{1.2}$$

$$u(t, x) = 0 \ , \qquad (t, x) \in]0 , T] \times \partial\Omega \tag{1.3}$$

$$u(0, x) = u_0(x) \ , \qquad x \in \Omega \ . \tag{1.4}$$

Here we consider second order operators and Dirichlet boundary conditions only for clarity of exposition. Using the same methods we could study operators of order $2m$ with more general boundary conditions.

It is known (see *e.g.* [8]) that for $p > 1$ the solution u of (1.2)-(1.4) satisfies

$$u_t \ , \ \ Eu \quad \in L^p(]0 , T[\times\Omega)$$

if and only if $u_0 \in H_*^{2-2/p,p}(\Omega)$, where $H_*^{\alpha,p}$ denote Sobolev or Besov spaces (see (2.4)).

In this paper we give necessary and sufficient conditions on u_0 such that the solutions of (1.2)-(1.4) have the Sobolev time regularity

$$\partial^k u/\partial t^k \in W^{\alpha,p}(0 , T; L^p(\Omega)) \tag{1.5}$$

for given $k \in \{0 , 1 , \cdots\}$, $0 < \alpha \leq 1$ and $1 \leq p < \infty$. Here $W^{\alpha,p}(0 , T; L^p(\Omega))$, if

(*) Nella seduta del 13 marzo 1998.

$\alpha < 1$, denotes the Sobolev space of functions $u \in L^p(]0\,,\,T[\times\Omega)$ satisfying

$$\int_0^T \int_0^T \|u(t\,,\cdot) - u(s\,,\cdot)\|^p_{L^p(\Omega)}\,|t-s|^{-1-\alpha p}\,ds\,dt < +\infty \ . \tag{1.6}$$

In addition we prove that property (1.5) is equivalent to a Sobolev space regularity for the solutions u of (1.2)-(1.4), *i.e.* we prove

$$\partial^k u/\partial t^k \in W^{\alpha,p}(0\,,\,T; L^p(\Omega)) \Leftrightarrow \partial^k u/\partial t^k \in L^p(0\,,\,T; H^{2\alpha,p}_*(\Omega)) \ . \tag{1.7}$$

Therefore we give also a characterization of the initial data such that problem (1.2)-(1.4) admits solutions with prescribed Sobolev space regularity.

2. Sobolev regularity

Let E and Ω be defined as in section 1 (all the results of this section, with obvious modifications, remain valid if we consider the case $\Omega = \mathbb{R}^n$). We study problem (1.2)-(1.4) in a L^p setting, $1 \le p < \infty$, with respect to time and space.

We say that a function $u \in L^p(]0\,,\,T[\times\Omega)$ is a *solution* of (1.2), (1.3) if for each $\epsilon > 0$ the function $t \to u(t\,,\cdot)$ is differentiable, in the sense of distributions, from $[\epsilon\,,\,T]$ into $L^p(\Omega)$ and $u_t \in L^p(]\epsilon\,,\,T[\times\Omega)$. In addition u satisfies $u(t\,,\cdot) \in H^{1,p}_0(\Omega)$ and $Eu(t\,,\cdot) \in L^p(\Omega)$ for a.e. $t \in]\epsilon\,,\,T[$ and $u_t(t\,,x) = Eu(t\,,x)$ for a.e. $(t\,,x) \in]\epsilon\,,\,T[\times\Omega$.

Given $u_0 \in Y$, for some function space Y, we say that a solution u of (1.2)-(1.3) satisfies (1.4) if $\lim_{t\to 0} u(t\,,\cdot) = u_0(\cdot)$ in Y. Here we are interested in characterizing those spaces Y of initial data for which the solutions of (1.2)-(1.4) exhibit Sobolev regularity in time up to 0, *i.e.*,

$$\partial^k u/\partial t^k \in W^{\alpha,p}(0\,,\,T; L^p(\Omega))$$

for given $k \ge 0$ and $0 < \alpha \le 1$ (see (1.6)).

We proceed as follows. We denote by Λ the realization of E, with homogeneous Dirichlet boundary conditions, in the space $L^p(\Omega)$,

$$\begin{cases} D(\Lambda) = \{u \in H^{1,p}_0(\Omega) : Eu \in L^p(\Omega)\} \\ \Lambda u = Eu \end{cases} \tag{2.1}$$

where Eu is understood in the sense of distributions. It is known that the operator Λ generates an analytic semigroup on $L^p(\Omega)$ (see *e.g.* [9]) and that, if $p > 1$, we have

$$D(\Lambda) = H^{1,p}_0(\Omega) \cap H^{2,p}(\Omega) \ . \tag{2.2}$$

Using (2.1) we can rewrite the initial boundary value problem (1.2)-(1.4) as a Cauchy problem in the space $L^p(\Omega)$ (see *e.g.* [9])

$$\begin{cases} u'(t) = \Lambda u(t) \quad , \quad t \in]0\,,\,T] \\ u(0) = u_0 \ . \end{cases} \tag{2.3}$$

As Λ generates an analytic semigroup we can study (2.3) using the theory of abstract parabolic equations (see Appendix). To do this we recall the characterization of some interpolation and extrapolation spaces generated by Λ. Without loss of generality we assume, for simplicity in notation, that there exists $\omega < 0$ such that each λ in the spectrum of Λ satisfies $\mathbb{R}e\,\lambda < \omega$.

Let $h = 1, 2$ and $0 < \theta < h$. We denote by $(L^p(\Omega), D(\Lambda^h))_{\theta/h,p}$ the real interpolation spaces of $L^p(\Omega)$ and $D(\Lambda^h)$, generated by the K-method (see *e.g.* [3, sec. 3.2]). For $h = 1$ we have the characterization

$$(2.4)\qquad \begin{aligned}&(L^p(\Omega), D(\Lambda))_{\theta,p} = H_*^{2\theta,p}(\Omega) :=\\ &:= \begin{cases} H^{2\theta,p}(\Omega)\ , & \theta \in]0, 1/2[\\ \{u \in B^{1,p}(\Omega)\ :\ \displaystyle\int_\Omega \frac{|u(x)|^p}{d(x,\partial\Omega)}dx < +\infty\}\ , & \theta = 1/2\\ H_0^{2\theta,p}(\Omega)\ , & \theta \in]1/2, 1[.\end{cases}\end{aligned}$$

Here $H^{2\theta,p}(\Omega)$ are the Sobolev spaces of fractional order, $B^{1,p}(\Omega)$ is the Besov space and $d(x, \partial\Omega)$ is the distance from x to $\partial\Omega$. For $p = 2$ we have

$$(2.5)\qquad H_*^{1,2}(\Omega) = H_0^{1,2}(\Omega)\ .$$

Property (2.4) is a consequence of (2.2) if $p > 1$ and it is proved in [5] if $p = 1$.

If $h = 2$ the description of the spaces $(L^p(\Omega), D(\Lambda^2))_{\theta/2,p}$, is more complicate. As an example let us consider the simple case $\Omega = \mathbb{R}^n$, $E = \Delta - I$ and $p > 1$. In this case we eliminate the boundary condition (1.3) and have $D(\Lambda) = H^{2,p}(\mathbb{R}^n)$. Hence

$$(L^p(\mathbb{R}^n), D(\Lambda^2))_{\theta/2,p} = (L^p(\mathbb{R}^n), H^{4,p}(\mathbb{R}^n))_{\theta/2,p} = Lip\,(2\theta, 4, p; L^p(\mathbb{R}^n))$$

where $Lip\,(2\theta, 4, p; L^p(\mathbb{R}^n))$ are the generalized Lipschitz spaces (see *e.g.* [3, Theorem 4.3.4]).

We now define the spaces $H_*^{2\theta,p}(\Omega)$ also for nonpositive exponent $-1 < \theta \le 0$. We set for $u \in L^p(\Omega)$

$$(2.6)\qquad \|u\|_{2\theta,p} := \begin{cases} \|\Lambda^{-1}u\|_{H_*^{2+2\theta,p}}\ , & \text{if}\ \ \theta \in]-1, 0[\\ \|\Lambda^{-1}u\|_{(L^p(\Omega), D(\Lambda^2))_{1/2,p}}\ , & \text{if}\ \ \theta = 0\end{cases}$$

and denote by $H_*^{2\theta,p}(\Omega)$ the completion of the space $\{u \in L^p(\Omega) : \|u\|_{2\theta,p} < +\infty\}$ with respect to $\|\cdot\|_{2\theta,p}$.

It can be seen that if $\theta < 0$ then $L^p(\Omega) \subseteq H_*^{2\theta,p}(\Omega)$ and hence $H_*^{2\theta,p}(\Omega)$ are called *extrapolation* spaces. The relationship between $L^p(\Omega)$ and $H_*^{0,p}(\Omega)$ is not known in general. For $p = 1, 2$ we have (see [6]) $H_*^{0,1}(\Omega) \subseteq L^1(\Omega)$ and

$$(2.7)\qquad H_*^{0,2}(\Omega) = L^2(\Omega)\ .$$

If $p \ne 1, 2$ the characterization of $H_*^{0,p}(\Omega)$ is known only in the case $\Omega = \mathbb{R}^n$ and we have $H_*^{0,p}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$ if $p \le 2$ and $L^p(\mathbb{R}^n) \subseteq H_*^{0,p}(\mathbb{R}^n)$, if $p \ge 2$ (see [10]). We refer to [6] and [7] for more details and proofs.

We now can apply the results of the Appendix to problem (2.3). From (2.4), (2.6) and Theorem 3.1 we obtain the following.

THEOREM 2.1. *Let* $0 < \alpha \le 1$. *The following properties are equivalent* (*i*) *Problem* (1.2)-(1.4) *admits a solution* $u \in W^{\alpha,p}(0, T; L^p(\Omega))$,

(*ii*) $u_0 \in H_*^{2\alpha-2/p,p}(\Omega))$.

In addition if (*i*) *or* (*ii*) *holds, then we also have* $u \in L^p(0, T; H_*^{2\alpha,p}(\Omega))$ *if* $\alpha < 1$ *or* $u \in L^p(0, T; D(\Lambda))$ *if* $\alpha = 1$.

From (2.4) and Theorem 3.2 we obtain the next result.

THEOREM 2.2. *Let* $0 < \alpha \leq 1$ *and* $n \in \{1, 2, \cdots\}$. *The following properties are equivalent*

(*i*) *Problem* (1.2)-(1.4) *admits a solution* u *satisfying* $\partial^n u/\partial t^n \in W^{\alpha,p}(0, T; L^p(\Omega))$

(*ii*) $u_0 \in D(\Lambda^{n-1})$ *and* $E^{n-1}u_0 \in H_*^{2+2\alpha-2/p,p}(\Omega)$, *if* $\alpha < 1/p$, $E^{n-1}u_0 \in (L^p(\Omega), D(\Lambda^2))_{1/2,p}$, *if* $\alpha = 1/p$; $u_0 \in D(\Lambda^n)$ *and* $E^n u_0 \in H_*^{2\alpha-2/p,p}(\Omega)$, *if* $\alpha > 1/p$.

In addition if (*i*) *or* (*ii*) *holds, then* $\partial^n u/\partial t^n \in L^p(0, T; H_*^{2\alpha,p}(\Omega))$, *if* $\alpha < 1$ *or* $\partial^n u/\partial t^n \in L^p(0, T; D(\Lambda))$, *if* $\alpha = 1$.

Finally in the case $p = 2$ and $\alpha = 2^{-1}$, using (2.5), (2.7) and Theorems 2.1, 2.2 we obtain

THEOREM 2.3. *For each* $k \geq 0$ *the following properties are equivalent.*

(*i*) *Problem* (1.2)-(1.4) *admits a solution* u *satisfying* $\partial^k u/\partial t^k \in W^{1/2,2}(0, T; L^2(\Omega))$

(*ii*) $u_0 \in D(\Lambda^k)$.

In addition if (*i*) *or* (*ii*) *holds, then we also have* $\partial^k u/\partial t^k \in L^2(0, T; H_0^{1,2}(\Omega))$.

3. APPENDIX

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of an analytic semigroup on a Banach space X and consider the following abstract parabolic equation

$$u'(t) = Au(t) \ , \ t > 0 \ . \tag{3.1}$$

By a solution of (3.1) in the sense of L^p, $1 \leq p < \infty$, we mean a function $u \in W^{1,p}(\epsilon, T; X) \cap L^p(\epsilon, T; D(A))$ satisfying $u'(t) = Au(t)$ for a.e. $t \in]\epsilon, T[$ and for each $\epsilon > 0$.

We want to characterize those spaces Y of initial data for which the solutions of (3.1) satisfying the initial condition

$$\lim_{t\to 0} u(t) = x \quad \text{in } Y \tag{3.2}$$

exhibit Sobolev regularity up to 0 for given $k \geq 0$ and $0 < \alpha \leq 1$, *i.e.*

$$u \in W^{k+\alpha,p}(0, T; X) \ .$$

Here $W^{k+\alpha,p}(0, T; X)$, if $\alpha < 1$, is the space of functions $u \in W^{k,p}(0, T; X)$ satisfying

$$\int_0^T \int_0^T \|u^{(k)}(t) - u^{(k)}(s)\|_X^p \, |t-s|^{-1-\alpha p} \, dt \, ds < +\infty \ .$$

The characterization of such initial data requires the introduction of two families of spaces, called interpolation and extrapolation spaces. For simplicity we assume, without loss of generality for our pourposes, that A generates a semigroup of negative type. Hence there exists $A^{-1} \in \mathcal{L}(X)$.

For given $h = 1\,,2$ and $0 < \theta < h$ we denote by $(X, D(A^h))_{\theta/h,p}$ the real interpolation spaces of X and $D(A^h)$, generated by the K-method (see *e.g.* [3, sec. 3.2]), and by $\|\cdot\|_{\theta,h,p}$ the corresponding norm. We set

$$\|x\|_{\theta,p} := \begin{cases} \|A^{-1}x\|_{1+\theta,1,p}\,, & \text{if } \theta \in]-1\,,0[\\ \|A^{-1}x\|_{1,2,p}\,, & \text{if } \theta = 0 \\ \|x\|_{\theta,1,p}\,, & \text{if } \theta \in]0\,,1[\end{cases} \tag{3.3}$$

and define $X_{\theta,p}$ as the completion of the space $\{\, x \in X : \|x\|_{\theta,p} < +\infty \,\}$ with respect to $\|\cdot\|_{\theta,p}$. Then $X_{\theta,p} = (X, D(A))_{\theta,p}$ if $\theta \in]0\,,1[$ whereas if $\theta < 0$ we have $X \subseteq X_{\theta,p}$. For this reason $X_{\theta,p}$, with $\theta < 0$, are called *extrapolation* spaces (see [4]). If $\theta = 0$, the space $X_{0,p}$ may be an intermediate or extrapolation space. In the case where X is a Hilbert space and A is selfadjoint, we have

$$X_{0,2} = X\,. \tag{3.4}$$

See [6] and [7, Sect. 3] for a detailed description and proofs of the properties of these spaces.

The following theorems characterize the space Y in (3.2) for which the solutions of (3.1), (3.2) have Sobolev regularity.

THEOREM 3.1. *Let* $0 < \alpha \le 1$. *The following properties are equivalent.*

(*i*) *Problem* (3.1), (3.2) *admits a solution* $u \in W^{\alpha,p}(0\,,T;X)$

(*ii*) $x \in X_{\alpha-1/p,p}$.

In addition if (*i*) *or* (*ii*) *holds, then we also have* $u \in L^p(0\,,T;X_{\alpha,p})$ *if* $\alpha < 1$ *and* $u \in L^p(0\,,T;D(A))$ *if* $\alpha = 1$.

PROOF. See [7, Theorem 5.9]. □

THEOREM 3.2. *Let* $0 < \alpha \le 1$. *For each* $n \ge 1$ *the following properties are equivalent.*

(*i*) *Problem* (3.1), (3.2) *admits a solution satisfying* $u \in W^{n+\alpha,p}(0\,,T;X)$

(*ii*) $A^{n-1}x \in X_{1+\alpha-1/p,p}$ *if* $\alpha < 1/p$, $A^{n-1}x \in X_{1,2,p}$ *if* $\alpha = 1/p$, $A^n x \in X_{\alpha-1/p,p}$ *if* $\alpha > 1/p$.

In addition if (*i*) *or* (*ii*) *holds, then we also have* $u^{(n)} \in L^p(0\,,T;X_{\alpha,p})$ *if* $\alpha < 1$ *and* $u^{(n)} \in L^p(0\,,T;D(A))$ *if* $\alpha = 1$.

PROOF. See [7, Theorem 5.10]. □

Finally, if X is a Hilbert space and A is selfadjoint the results for the case $p = 2$ and $\alpha = 2^{-1}$ may be written as follows.

THEOREM 3.3. *Let X be a Hilbert space and let A be selfadjoint. Then for each* $k \ge 0$ *the following properties are equivalent.*

(*i*) *Problem* (3.1), (3.2) *admits a solution satisfying* $u \in W^{k+1/2,2}(0\,,T;X)$

(*ii*) $x \in D(A^k)$.

In addition if (*i*) *or* (*ii*) *holds, then we also have* $u^{(k)} \in L^2(0\,,T;X_{1/2,2})$.

Proof. The results follow from Theorems 3.1, 3.2 and from (3.3), (3.4). □

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