Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Valentina Casarino

Spectral properties of weakly almost periodic cosine functions

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.3, p. 177–211.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_3_177_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:177-211 (1998)

Analisi funzionale. — *Spectral properties of weakly almost periodic cosine functions.* Nota di VALENTINA CASARINO, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — The spectral structure of the infinitesimal generator of a strongly continuous cosine function of linear bounded operators is investigated, under assumptions on the almost periodic behaviour of applications generated, in various ways, by C. Moreover, a first approach is presented to the analysis of connection between cosine functions and dynamical systems.

KEY WORDS: Cosine functions; Asymptotically almost periodic applications; Dynamical systems.

RIASSUNTO. — *Funzioni coseno debolmente quasi periodiche.* Si studia la struttura spettrale del generatore infinitesimale di una funzione coseno fortemente continua di operatori lineari limitati, sotto ipotesi sul comportamento quasi periodico di applicazioni generate, in diversi modi, da C. È, inoltre, presentato un primo approccio all'analisi del legame fra funzioni coseno e sistemi dinamici.

The notion of asymptotic almost periodicity, introduced by M. Fréchet in 1941 for scalar-valued functions defined on a half-line, was extended to general vector-valued maps and afterwards applied to functions generated, in various ways, by a semigroup $T : \mathbb{R}_+ \to \mathcal{L}(\mathcal{E})$ of linear bounded operators acting on a Banach space $\mathcal{E}$, by W. M. Ruess and W. H. Summers [25].

When applied to functions defined on the entire real line, this notion can often be reduced to the classical definition of almost periodicity; for example, E. Vesentini proved in [28] that, if $U : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ is a uniformly bounded, strongly asymptotically almost periodic group, then U is strongly almost periodic.

In the case of a cosine operator function, H. Henriquez proved [17] that every asymptotically almost periodic cosine function is almost periodic on $\mathbb{R}$, by showing firstly that C is almost periodic if, and only if, C is asymptotically almost periodic in the sense of Stepanov. Since an asymptotically almost periodic function is also asymptotically almost periodic in the sense of Stepanov, then the equivalence between asymptotical almost periodicity and almost periodicity for cosine functions follows.

This result can be improved: indeed, by using only cosine functional equation and some elementary properties of almost periodic functions, analogous results to that of H. Henriquez can be stated for a single orbit of C, that is for the map $t \mapsto C(t)x_0$, for some $x_0 \in \mathcal{E}$. In particular, it turns out that also in this case periodicity or almost periodicity in asymptotical sense entail, respectively, periodicity or almost periodicity on the entire real line.

This is particularly interesting at the light of the results obtained by E. Vesentini in the semigroups' framework; in [28] he described the constraints imposed on the

(*) Nella seduta del 24 aprile 1998.

spectrum of the infinitesimal generator of a strongly continuous semigroup T by very weak hypotheses on the almost periodic behaviour of the semigroup, for example by the existence of some $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$ for which the function $t \mapsto < T(t)x, \lambda >$ is asymptotically almost periodic. As in the semigroups' theory, it will be shown that the only assumption, that the map $t \mapsto < C(t)x, \lambda >$ is asymptotically almost periodic for some $x \in \mathcal{E}$ and some $\lambda \in \mathcal{E}'$, conditions the spectral structure of the infinitesimal generator of C. In particular, if C is uniformly bounded (so that the spectrum of X and the spectrum of the adjoint operator $X^{\odot}$ are contained in the real negative semiaxis), it turns out that the squares of all frequencies of the map $t \mapsto < C(t)x, \lambda >$, up to factor -1, belong to $\mathrm{p}\sigma(X^{\odot})$; *viceversa*, for every $\zeta \in \mathrm{p}\sigma(X^{\odot})$, there exist some $x \in \mathcal{E}$ and some $\lambda \in \mathcal{E}'$, such that $< x, \lambda > \neq 0$ and $\sqrt{-\zeta}$ is a frequency of the almost periodic map $t \mapsto < C(t)x, \lambda >$. This result is more exhaustive than the analogous one for semigroups; in semigroups' framework, it can be proved only that the set of all frequencies of an asymptotically almost periodic map $t \mapsto < T(t)x, \lambda >$ is contained in the intersection $i\mathbb{R} \cap \big(\mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)\big)$. From a technical point of view, the more complete information obtained for a uniformly bounded cosine function, generated by X, depends on the inclusion $\mathrm{p}\sigma(X) \subset \mathrm{p}\sigma(X')$, proved in Theorem 2.4.

It has to be observed that, though every infinitesimal generator X of a strongly continuous cosine function C generates also a strongly continuous semigroup T, only few results are in this case retrievable from the semigroups' theory, since the weak or strong almost periodicity of C doesn't imply the same property for the corresponding semigroup.

Since the mean ergodic theorem for cosine functions constitutes the main technical tool in the proofs, several extensions of this theorem are discussed below. Moreover, almost all of the results proved in this paper hold, with only small changes in the proof, when the non constant function $t \mapsto < C(t)x, \lambda >$ is assumed to be, for some $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, asymptotically almost periodic in the sense of Stepanov (for more details on this class of almost periodic functions, for which continuity fails, and only measurability and integrability in the sense of Lebesgue are required, see [9]).

In the last section, a first approach to the analysis of connection between dynamical systems and cosine functions is presented. Starting from a dynamical system $(\Phi\,, K)$ (K being a Hausdorff compact space and Φ being a continuous flow defined on it), a cosine function C is built on a Banach space intrinsecally associated to the system. In particular, the constraints, imposed on the spectral structure of the infinitesimal generator of C by the existence of asymptotically stable points and of periodic or almost periodic orbits for Φ, are investigated.

1. Preliminaries and notations

Throughout the paper $\mathcal{E}$ will denote a complex Banach space and $\mathcal{E}'$ the topological dual of $\mathcal{E}$. For $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, $< x, \lambda >$ will denote the value of λ in x.

A function $f : \mathbb{R} \to \mathcal{E}$ is called *periodic* if there exists a real number $\tau > 0$ such that $f(t + \tau) = f(t)$ for every $t \in \mathbb{R}$. A function $f : \mathbb{R}_+ \to \mathcal{E}$ is said to be *asymptotically* *periodic* if there is some $K \geq 0$ such that the restriction of f to $[K, +\infty)$ is periodic.

Let $\mathbb{J}$ be either $\mathbb{R}$ or $\mathbb{R}_+$. A subset Λ of $\mathbb{J}$ is called *relatively dense* in $\mathbb{J}$ if there exists a number $l > 0$ such that every interval of length l in $\mathbb{J}$ contains at least one number from Λ.

A real number $\tau \in \mathbb{R} \setminus \{0\}$ is called an ε-period for $f : \mathbb{R} \to \mathcal{E}$ if

$$||f(t+\tau) - f(t)|| \leq \varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{1.1}$$

The function f is called *almost periodic* if, for every $\varepsilon > 0$, the set of all ε-periods is relatively dense in $\mathbb{R}$.

Given a function $f : \mathbb{J} \to \mathcal{E}$, the ω-translate of f is defined by $f_\omega(t) = f(\omega + t)$; $H(f) = \{f_\omega : \omega \in \mathbb{J}\}$ will denote the set of all translates of f.

Let $\mathcal{C}_b(\mathbb{J}, \mathcal{E})$ denote the Banach space of all continuous bounded functions from $\mathbb{J}$ to $\mathcal{E}$ endowed with the uniform norm.

S. Bochner characterized continuous almost periodic functions defined on $\mathbb{R}$:

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathcal{E}$ be a continuous function. For f to be almost periodic it is necessary and sufficient that $H(f)$ is relatively compact in $\mathcal{C}_b(\mathbb{R}, \mathcal{E})$.*

For every almost periodic function f the limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t f(s)ds \tag{1.2}$$

exists in $\mathcal{E}$. For every almost periodic function f and every $\theta \in \mathbb{R}$ the function $t \mapsto$ $\mapsto e^{-i\theta t} f(t)$ is almost periodic. Hence, the corresponding limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{-i\theta s} f(s)ds \tag{1.3}$$

exists; it can be proved that it does not vanish for at most a countable set of values of θ. These values are said the *frequencies* of f, and the corresponding values of the limit are called the *Fourier coefficients* of f. The following property of almost periodic functions will be used:

LEMMA 1.2. *Let $\mathcal{E}, \mathcal{F}$ be Banach space. If $\phi : \mathbb{J} \to \mathcal{E}$ is almost periodic and $\psi : \mathcal{E} \to \mathcal{F}$ is continuous on $\overline{\mathcal{R}(\phi)}$, then $\psi \circ \phi : \mathbb{J} \to \mathcal{F}$ is almost periodic.*

A function $f : \mathbb{R}_+ \to \mathcal{E}$ is called *asymptotically almost periodic* if $H(f)$ is relatively compact in $\mathcal{C}_b(\mathbb{R}_+, \mathcal{E})$.

If f is asymptotically almost periodic, the limit (1.3) exists for every $\theta \in \mathbb{R}$ and vanishes for all values of θ, with the possible exception of an at most countable set. The values of θ for which the limit is non-zero and the corresponding values of (1.3) are called again the frequencies and the Fourier coefficients of f.

The following decomposition theorem, due, in the general case, to W. M. Ruess and to W. H. Summers, holds:

THEOREM 1.3. *A function* $f \in \mathcal{C}_b(\mathbb{R}_+, \mathcal{E})$ *is asymptotically almost periodic if, and only if, one of the following two equivalent conditions is satisfied*:

1) *there exist a unique almost periodic function* $g \in \mathcal{C}_b(\mathbb{R}, \mathcal{E})$ *and a unique* $h \in \mathcal{C}_b(\mathbb{R}_+, \mathcal{E})$, *vanishing at infinity, such that* $f = h + g_{|\mathbb{R}_+}$;

2) *for every* $\varepsilon > 0$ *there exist* $\Lambda > 0$ *and* $K \geq 0$ *such that every interval of lenght* Λ *contains some* τ *for which*

$$||f(t+\tau) - f(t)|| \leq \varepsilon$$

holds whenever $t, t+\tau \geq K$.

The functions g and h are called, respectively, the *principal term* and the *correction term* of f.

2. Some spectral properties of cosine functions

Let $\mathcal{L}(\mathcal{E})$ be the Banach algebra of the linear bounded operators on $\mathcal{E}$.

A *cosine operator function* is an application from the real line to $\mathcal{L}(\mathcal{E})$ satisfying the relations:

1) $C(t+s) + C(t-s) = 2C(t)C(s)$ for every $t, s \in \mathbb{R}$;

2) $C(0) = I$.

A cosine function C is said to be strongly continuous if $\lim_{t\to 0} C(t)x = x$ for every $x \in \mathcal{E}$.

If C is a strongly continuous cosine function, then $S : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ defined by

$$S(t)x = \int_0^t C(s)xds \text{ for every } x \in \mathcal{E} \text{ and } t \in \mathbb{R}$$

is called the *sine function* associated to C.

Throughout this paper $X : \mathcal{D}(X) \subset \mathcal{E} \to \mathcal{E}$ will denote the infinitesimal generator of a strongly continuous cosine function of linear bounded operators acting on a complex Banach space $\mathcal{E}$. $\rho(X)$, $\sigma(X)$, $\mathrm{p}\sigma(X)$, $\mathrm{r}\sigma(X)$ and $c\sigma(X)$ will denote, respectively, the resolvent set, spectrum, point spectrum, residual spectrum and continuous spectrum of the operator X.

Set $C'(t) = C(t)'$ for every $t \in \mathbb{R}$. C' defines a weak-star continuous cosine function on $\mathcal{E}'$, which is not, in general, strongly continuous. Consider now the closed subspace

$$\mathcal{E}^{\odot} = \{\lambda \in \mathcal{E}' : \lim_{t\to 0} C'(t)\lambda = \lambda\}.$$

From the definition it follows that $\mathcal{E}^{\odot}$ is $C'(t)$-invariant; moreover, $\mathcal{E}^{\odot}$ is weak-star dense in $\mathcal{E}'$ and $\mathcal{E}^{\odot} = \overline{D(X')}$. Set $C^{\odot}(t) = C'(t)_{|\mathcal{E}^{\odot}}$ for every $t \in \mathbb{R}$: $C^{\odot}$ defines a strongly continuous cosine function on the Banach space $\mathcal{E}^{\odot}$, which will be called the *adjoint cosine function*.

If $\mathcal{E}$ is reflexive, then $\mathcal{E}^{\odot}$ is weak dense, and therefore, as a consequence of the Hahn-Banach theorem, dense in norm. Since $\mathcal{E}^{\odot}$ is closed, it holds:

PROPOSITION 2.1. *If $\mathcal{E}$ is reflexive, C' is a strongly continuous cosine function on $\mathcal{E}'$.*

If $X^{\odot}$ is the infinitesimal generator of $C^{\odot}$, it can be shown that $X^{\odot}$ is the part of X' in $\mathcal{E}^{\odot}$, *i.e.* $X^{\odot}$ is the restriction of X' to the linear space

$$\mathcal{D}(X^{\odot}) = \{\lambda \in D(X') : X'\lambda \in \mathcal{E}^{\odot}\}.$$

E. Vesentini has recently proved that, if X is a linear operator with dense domain in $\mathcal{E}$, then $\mathrm{p}\sigma(X') = k\sigma(X)$, where the compression spectrum $k\sigma$ is defined by

$$k\sigma(X) = \{\zeta \in \mathbb{C} : \overline{\mathcal{R}(X - \zeta I)} \neq \mathcal{E}\}.$$

Moreover, he proves also that, if X generates a strongly continuous semigroup, then

$$\mathrm{p}\sigma(X^{\odot}) = \mathrm{p}\sigma(X') = k\sigma(X). \tag{2.1}$$

Since, if X is the infinitesimal generator of a strongly continuous cosine function, X is also the generator of a strongly continuous semigroup, then (2.1) *a fortiori* holds when X generates a strongly continuous cosine function. The analogous relation for $\mathrm{p}\sigma(C^{\odot}(t))$ e $\mathrm{p}\sigma(C'(t))$, $t \in \mathbb{R}$, requires, on the contrary, a direct proof.

PROPOSITION 2.2. *If C is a strongly continuous cosine function, then*

$$\mathrm{p}\sigma(C^{\odot}(t)) = \mathrm{p}\sigma(C'(t)) \textit{ for every } t \in \mathbb{R}.$$

PROOF. Since $C^{\odot}(t)$ is the restriction of $C'(t)$ to $\mathcal{E}^{\odot}$, then $\mathrm{p}\sigma(C^{\odot}(t)) \subset \mathrm{p}\sigma(C'(t))$ for every $t \in \mathbb{R}$. Conversely, let $\zeta \in \mathbb{C}$ be such that $C'(t)\lambda = \zeta\lambda$, for some $\lambda \in \mathcal{E}' \setminus \{0\}$. Let $\tau \in \mathbb{R}$ be such that τ^2 is in $r(X)$, and therefore in $r(X')$. Then $(\tau^2 I - X')^{-1}\lambda =$ $= [(\tau^2 I - X)^{-1}]'\lambda \in \mathcal{D}(X') \subset \mathcal{E}^{\odot}$. For every $x \in \mathcal{E}$ it holds:

$$< x, C'(t)(\tau^2 I - X')^{-1}\lambda > =< (\tau^2 I - X)^{-1} C(t)x, \lambda >=$$

$$=< C(t)(\tau^2 I - X)^{-1}x, \lambda >=< x, (\tau^2 I - X')^{-1} C'(t)\lambda > ,$$

whence $C'(t)(\tau^2 I - X')^{-1}\lambda = (\tau^2 I - X')^{-1} C'(t)\lambda$ for every $t \in \mathbb{R}$.

Hence $C'(t)(\tau^2 I - X')^{-1}\lambda = \zeta(\tau^2 I - X')^{-1}\lambda$, whence the thesis follows since $C^{\odot}(t) = C'(t)$ on $\mathcal{E}^{\odot}$. □

B. Nagy proved [22] that

$$\cosh(t\sqrt{\mathrm{p}\sigma(X)}) = \mathrm{p}\sigma(C(t)) \quad \text{for every } t \in \mathbb{R},$$

which, combined with (2.1), yields the following

PROPOSITION 2.3. *If X is the infinitesimal generator of a strongly continuous cosine function C, then*

$$\cosh(t\sqrt{k\sigma(X)}) = k\sigma(C(t)) \quad \textit{for all } t \in \mathbb{R}.$$

If X is the infinitesimal generator of a strongly continuous, uniformly bounded semigroup of linear bounded operators on $\mathcal{E}$, J. van Neerven [23] has recently proved that

$$\mathrm{p}\sigma(X) \cap i\mathbb{R} \subseteq \mathrm{p}\sigma(X') \cap i\mathbb{R}.$$

An analogous, stronger result will now be obtained in the framework of cosine functions.

Consider the space $BUC(\mathbb{R})$ consisting of all complex-valued, bounded, uniformly continuous functions, endowed with the sup-norm.

Fix $\theta \in \mathbb{R}$. Let Ψ_o be a functional in $BUC(\mathbb{R})'$ such that

$$< \cos(\bullet\theta)\,,\,\Psi_o >= 1.$$

For every $n = 1\,,2\,,\ldots$ define

$$< f, \psi_n > = \frac{1}{n}\int_0^n < \frac{f(\cdot + s) + f(\cdot - s)}{2}\,,\,\Psi_o > ds \quad \text{if } \theta = 0 \text{ and}$$

$$< f, \psi_n > = \frac{2}{n}\int_0^n \cos(\theta s) < \frac{f(\cdot + s) + f(\cdot - s)}{2}\,,\,\Psi_o > ds \quad \text{if } \theta \neq 0.$$

ψ_n belong to $BUC(\mathbb{R})'$ and, moreover, they satisfy $||\psi_n|| \leq 2||\Psi_o||$ for every $n = 1\,,2\,,\ldots$, if $\theta \neq 0$ and $||\psi_n|| \leq ||\Psi_o||$ for every $n = 1\,,2\,,\ldots$, if $\theta = 0$.

It holds

$$< 1\,,\psi_n >= \frac{1}{n}\int_0^n < \frac{1(\cdot + s) + 1(\cdot - s)}{2}\,,\,\Psi_o > ds = 1$$

for $\theta = 0$, while, for every $n = 1\,,2\,,\ldots$ and $\theta \neq 0$, one gets:

$$< \cos(\theta\cdot)\,,\,\psi_n > = \frac{2}{n}\int_0^n \cos(\theta s) < \frac{\cos\theta(\cdot + s) + \cos\theta(\cdot - s)}{2}\,,\,\Psi_o > ds =$$

$$= \frac{2}{n}\int_0^n \cos(\theta s) < \cos(\theta\cdot)\cos(\theta s)\,,\,\Psi_o > ds =$$

$$= \frac{2}{n}\int_0^n \cos^2(\theta s) < \cos(\theta\cdot)\,,\,\Psi_o > ds =$$

$$= \frac{2}{n}\int_0^n \cos^2(\theta s)ds = 1 + \frac{\sin(\theta n)\cos(\theta n)}{\theta n}.$$

Since the unit ball of a dual Banach space is weak* compact, there exists a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k} \rightharpoonup \Psi$. If $\theta \neq 0$, the weak*-cluster point Ψ is such that

$$< \cos(\theta\cdot)\,,\,\Psi >= \lim_{k\to+\infty} < \cos(\theta\cdot)\,,\,\psi_{n_k} >= \lim_{k\to+\infty}\left(1 + \frac{\sin(\theta n_k)\cos(\theta n_k)}{\theta n_k}\right) = 1.$$

If $\theta = 0$ the equality $< 1\,,\Psi >= 1$ follows trivially from the definition of ψ_{n_k}.

Moreover, for every $f \in \mathcal{C}(K)$, $\theta \in \mathbb{R}$ and $t \in \mathbb{R}$ one has

$$
\begin{aligned}
&< \frac{f(\cdot + t) + f(\cdot - t)}{2}\,,\, \Psi >= \lim_{k\to+\infty} < \frac{f(\cdot + t) + f(\cdot - t)}{2}\,,\, \psi_{n_k} >= \\
&= \lim_{k\to+\infty} \frac{1}{2n_k}\int_0^{n_k} \cos(\theta s) < f(\cdot + t + s) + f(\cdot + t - s) + \\
&+ f(\cdot - t + s) + f(\cdot - t - s)\,,\, \Psi_o > ds = \\
&= \lim_{k\to+\infty} \frac{1}{2n_k}\Big[\int_t^{n_k+t} \cos[\theta(s-t)] < f(\cdot + s)\,,\, \Psi_o > ds + \\
&+ \int_{-t}^{n_k-t} \cos[\theta(s+t)] < f(\cdot + s)\,,\, \Psi_o > ds + \\
&- \int_t^{t-n_k} \cos[\theta(t-s)] < f(\cdot + s)\,,\, \Psi_o > ds + \\
&- \int_{-t}^{-t-n_k} \cos[\theta(s+t)] < f(\cdot + s)\,,\, \Psi_o > ds\Big] = \\
&\lim_{k\to+\infty} \frac{1}{2n_k}\Big[\int_{t-n_k}^{t+n_k} \cos[\theta(s-t)] < f(\cdot + s)\,,\, \Psi_o > ds + \\
&+ \int_{-t-n_k}^{-t+n_k} \cos[\theta(s+t)] < f(\cdot - s)\,,\, \Psi_o > ds\Big] = \\
&= \lim_{k\to+\infty} \frac{1}{2n_k}\cdot\cos(\theta t)\int_{t-n_k}^{t+n_k} \cos(\theta s) < f(\cdot + s) + f(\cdot - s)\,,\, \Psi_o > ds = \\
&\qquad\qquad = \lim_{k\to+\infty} \cos(\theta t) < f, \psi_{n_k} >= \cos(\theta t) < f, \Psi > .
\end{aligned}
$$

A functional $\Psi \in BUC(\mathbb{R})'$, such that $< \cos(\theta\cdot)\,,\, \Psi >= 1$ and fulfilling the property

$$< \frac{f(\cdot + t) + f(\cdot - t)}{2}\,,\, \Psi >= \cos(\theta t) < f, \Psi >$$

for every $f \in \mathcal{C}(K)$ and $t \in \mathbb{R}$ will be called a *cosine invariant mean*, in analogy to the definition of left or right invariant means.

THEOREM 2.4. *Let C be a strongly continuous uniformly bounded cosine function on a Banach space $\mathcal{E}$. Then*

$$\mathrm{p}\sigma(X) \subset \mathrm{p}\sigma(X') \text{ and } \mathrm{r}\sigma(X) \cup \mathrm{p}\sigma(X) = \mathrm{p}\sigma(X').$$

PROOF. Let $-\theta^2 \in \mathbb{R}_-$ belong to $\mathrm{p}\sigma(X)$. It will be assumed $\theta \neq 0$; the case in which $\theta = 0$ is very similar and easier.

Let $x_0 \in \mathcal{D}(X)$ be such that $Xx_0 = -\theta^2 x_0$. Suppose $\theta > 0$. That implies [10] that $C(t)x_0 = \cos(\theta t)x_0$ for every $t \in \mathbb{R}$.

Let $\lambda_0 \in \mathcal{E}' \setminus \{0\}$ be such that $< x_0\,,\, \lambda_0 >= 1$.

Consider a cosine invariant mean $\Psi \in BUC(\mathbb{R})'$; let $\mu_0 \in \mathcal{E}'$ be a form defined by

$$< x, \mu_0 >= \Psi\big(< C(\cdot)x, \lambda_0 > \big)$$

for all $x \in \mathcal{E}$. Observe that the map $t \mapsto < C(\cdot)x\,, \lambda >$ belongs to $BUC(\mathbb{R})$ for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$.

Then for all $x \in \mathcal{E}$ from the invariance property satisfied by Ψ it follows

$$\begin{aligned} < x\,, C'(t)\mu_0 > =& < C(t)x\,, \mu_0 >= \\ &= \Psi\big(< C(t)C(\cdot)x\,, \lambda_0 > \big) = \\ &= \Psi\big(< \frac{C(\cdot + t) + C(\cdot - t)}{2}\, x\,, \lambda_0 > \big) = \\ &\qquad = \cos(\theta t)\Psi(< C(\cdot)x\,, \lambda_0 >) = \cos(\theta t) < x\,, \mu_0 > . \end{aligned}$$

Thus $C'(t)\mu_0 = \cos(\theta t)\mu_0$ for every $t \in \mathbb{R}$, and therefore $\mu_0 \in \mathcal{D}(X')$ with $X'\mu_0 = = -\theta^2\mu_0$. Finally, μ_0 does not vanish identically, since

$$\begin{aligned} < x_0\,, \mu_0 > =& \Psi\big(< C(\cdot)x_0\,, \lambda_0 > \big) = \\ &= \Psi\big(< \cos(\theta\cdot)x_0\,, \lambda_0 > \big) = \Psi\big(\cos(\theta\cdot)\big) = 1\,, \end{aligned}$$

and that proves the first part of thesis. The last equality follows from the first one and from the inclusions

$$\mathrm{r}\sigma(X) \subset \mathrm{p}\sigma(X^{\odot}) \subset \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X). \quad \square$$

Theorem 2.4 implies, in particular, that, if $-\theta^2$ is, for some $\theta \in \mathbb{R}$, an eigenvalue of X, then the operator $X - \theta^2 I$, which is not injective, can neither be surjective.

3. Almost periodic orbits

A strongly continuous cosine function is said to be *strongly asymptotically periodic* or *strongly asymptotically almost periodic* if for every $x \in \mathcal{E}$ the function, from $\mathbb{R}_+$ to $\mathcal{E}$, $t \mapsto C_{|\mathbb{R}_+}(t)x$ is, respectively, asymptotically periodic or asymptotically almost periodic.

The function $t \mapsto < C(t)x\,, \lambda >$, for some $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, is said to be asymptotically almost periodic if its restriction to $\mathbb{R}_+$ is asymptotically almost periodic.

As remarked in the introduction, H. Henriquez proved [17] that every asymptotically almost periodic cosine function is almost periodic on $\mathbb{R}$. Now, the same equivalence will be shown for a single orbit of C.

Proposition 3.1. *Let C be a strongly continuous cosine function.*

1) *If the function $t \mapsto C(t)x_0$ is asymptotically periodic for some $x_0 \in \mathcal{E}$, then the map $t \mapsto C(t)x_0$ is periodic.*

2) *If the function $t \mapsto C(t)x_0$ is asymptotically almost periodic for some $x_0 \in \mathcal{E}$, then $t \mapsto C(t)x_0$ is almost periodic.*

Proof. By hypothesis there are some $K \geq 0$ and $\tau > 0$ such that the function $t \mapsto C(t)x_0$ is periodic on $[K, +\infty)$ with period τ. Let $t \in [0\,, +\infty)$ and let $s > K$ be fixed. Then:

$$\begin{aligned} C(t)x_0 &= 2C(s)C(t+s)x_0 - C(t+2s)x_0 \\ C(t+\tau)x_0 &= 2C(s)C(t+s+\tau)x_0 - C(t+2s+\tau)x_0. \end{aligned}$$

The second equation is equivalent to

$$C(t+\tau)x_0 = 2C(s)C(t+s)x_0 - C(t+2s)x_0\,,$$

whence the equality $C(t)x_0 = C(t+\tau)x_0$ for every $t \geq 0$ follows. Since C is even as function of t, the function $t \mapsto C(t)x_0$ is periodic.

2) For every $\varepsilon > 0$ there is some $K = K(\varepsilon\,, x)$ such that the restriction to $[K, +\infty)$ of the function $t \mapsto C(t)x$ is almost periodic. Fix $s > K$. The functional equation

$$C(t)x = 2C(s)C(t+s)x - C(t+2s)x$$

and a standard application of Lemma 1.2 show that it is possible to choose $K = 0$. Then, since C is even, the function $t \mapsto C(t)x$ is almost periodic. □

Point 1) of Proposition 3.1 shows, in particular, that the theory of asymptotically periodic cosine functions is reduced to that of periodic cosine functions, which is treated in [14].

In the weak framework, stronger assumptions must be assumed in order to get analogous results.

PROPOSITION 3.2. *Let C be a strongly continuous cosine function and let $\lambda_0 \in \mathcal{E}' \setminus \{0\}$.*

1) *If the function $t \mapsto < C(t)x\,, \lambda_0 >$ is asymptotically periodic for every $x \in \mathcal{D}(X)$, then $t \mapsto < C(t)x\,, \lambda_0 >$ is periodic for every $x \in \mathcal{D}(X)$.*

2) *If the function $t \mapsto < C(t)x\,, \lambda_0 >$ is asymptotically almost periodic for every $x \in \mathcal{D}(X)$, then $t \mapsto < C(t)x\,, \lambda_0 >$ is almost periodic for every $x \in \mathcal{D}(X)$.*

3) *If the function $t \mapsto < C(t)x\,, \lambda_0 >$ is asymptotically almost periodic for every $x \in \mathcal{D}(X)$ and if C is uniformly bounded, then $t \mapsto < C(t)x\,, \lambda_0 >$ is almost periodic for every $x \in \mathcal{E}$.*

PROOF. 1) If $x \in \mathcal{D}(X)$, then $C(t)x \in \mathcal{D}(X)$ for every $t \in \mathbb{R}$. To get the thesis, it suffices to apply the functional equation as in Proposition 3.1 and to recall that C is even.

2) The proof is very similar to that of point 1).

3) Let x be in $\mathcal{E}$. Then there is a sequence $\{x_n\} \subset \mathcal{D}(X)$, such that $x_n \to x$. Thus

$$| < C(t)x\,, \lambda > - < C(t)x_n\,, \lambda > | \leq M||\lambda|| \cdot ||x - x_n||\,,$$

if $||C(t)|| \leq M$ for every $t \in \mathbb{R}$, and therefore the function $t \mapsto < C(t)x\,, \lambda >$, as uniform limit on $\mathbb{R}$ of the sequence of almost periodic functions $< C(t)x_n\,, \lambda >$, is almost periodic. □

In particular, if C is a weakly asymptotically almost periodic cosine function, *i.e.* if the functions $t \mapsto < C(t)x\,, \lambda >$ are asymptotically almost periodic for all $x \in \mathcal{E}$ and all $\lambda \in \mathcal{E}'$, C is weakly almost periodic. As a consequence of Proposition 3 in [10], under the additional hypothesis that $\mathcal{E}$ is weakly sequentially complete, C is an almost periodic function.

4. Ergodic properties

Some ergodic properties of cosine functions will now be discussed, mainly in four situations: in the general case of a Banach space $\mathcal{E}$, in the framework of a weakly sequentially complete Banach space $\mathcal{E}$, in the case of a dual of a Banach space, and, finally, in the case in which the range of the infinitesimal generator is closed.

Let $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be a strongly continuous, uniformly bounded cosine function, *i.e.* $||C(t)|| \leq M$ for all $t \in \mathbb{R}$ and a finite $M \geq 1$.

Let θ be a real number. Set

$$\mathcal{F}_\theta = \{x \in \mathcal{E} : \lim_{t\to+\infty} \frac{1}{t} \int_0^t \cos(\theta s) C(s) x ds \text{ exists in } \mathcal{E}\}.$$

It is possible to prove that the linear space $\mathcal{F}_\theta$ is closed for all $\theta \in \mathbb{R}$. For every $x \in \mathcal{F}_\theta$ define

$$P_\theta = \begin{cases} \lim_{t\to+\infty} \frac{1}{t} \int_0^t C(s) x ds & \text{if } \theta = 0 \\ \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos(\theta s) C(s) x ds & \text{if } \theta \neq 0. \end{cases} \tag{4.1}$$

Observe that, for any $b \in \mathbb{R}$ and $x \in \mathcal{F}_\theta$:

$$\lim_{t\to+\infty} \frac{1}{t} \int_b^t \cos(\theta s) C(s) x ds = \lim_{t\to+\infty} \frac{1}{t} \left(\int_0^t - \int_0^b \right) \cos(\theta s) C(s) x ds =$$
$$= \lim_{t\to+\infty} \frac{1}{t} \int_0^t \cos(\theta s) C(s) x ds ,$$

and therefore for any $b \in \mathbb{R}$ and $x \in \mathcal{F}_\theta$ it results

$$P_\theta x = \begin{cases} \lim_{t\to+\infty} \frac{1}{t} \int_b^t C(s) x ds \text{ if } \theta = 0 \\ \lim_{t\to+\infty} \frac{2}{t} \int_b^t \cos(\theta s) C(s) x ds \text{ if } \theta \neq 0. \end{cases}$$

The following theorem collects some known facts about P_θ, which will be useful in the following and which are only recalled, and some new results.

Theorem 4.1. *Let $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be a strongly continuous, uniformly bounded cosine function on a complex Banach space $\mathcal{E}$. Let P_θ and $\mathcal{F}_\theta$ be defined as above. Then*

1) $C(t)\mathcal{F}_\theta \subset \mathcal{F}_\theta$ *and* $P_\theta C(t) = \cos(\theta t) P_\theta = C(t) P_\theta$ *on* $\mathcal{F}_\theta$;
2) P_θ *is a linear projection operator in* $\mathcal{F}_\theta$, *with* $||P_\theta|| \leq 2M$;
3) $\overline{\mathcal{R}(X + \theta^2 I)} \subset \ker P_\theta$;
4) $\mathcal{R} P_\theta = \ker(X + \theta^2 I)$ *and* $P_{\theta|\ker(X+\theta^2 I)} = I$;
5) $\mathcal{F}_\theta = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ *and* $\ker P_\theta = \overline{\mathcal{R}(X + \theta^2 I)}$.

PROOF. 1) It is proved in [10].

2) First of all, for every $x \in \mathcal{F}_0$ it holds $||P_0 x|| \leq M||x||$ and 1) implies

$$P_0{}^2 x = \lim_{t\to+\infty} \frac{1}{t} \int_0^t C(s) P_0 x ds = P_0 x\,;$$

if $\theta \neq 0$, then $||P_\theta x|| \leq 2M||x||$ for every $x \in \mathcal{F}_\theta$ and

$$P_\theta{}^2 x = \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos(\theta s) P_\theta C(s) x ds = \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos^2(\theta s) P_\theta x ds = P_\theta x,$$

so that for every $\theta \in \mathbb{R}$ P_θ is a projection in $\mathcal{F}_\theta$ and $P_\theta \in \mathcal{L}(\mathcal{F}_\theta; \mathcal{E})$.

3) For $\theta = 0$, it is proved in [16], for $\theta \neq 0$ a proof can be found in [24].

4) First of all, it shall be proved that $\ker(X + \theta^2 I) \subset \mathcal{R}P_\theta$. In the case $\theta = 0$ this fact was proved in [16]. It will be shown now in the case $\theta \neq 0$.

If $u \in \ker(X + \theta^2 I)$, I. Cioranescu proved that:

$$C(t)u = \cos(\theta t)u \quad \text{for all } t \in \mathbb{R} \text{ and for all } \theta \in \mathbb{R}. \tag{4.2}$$

Therefore, if $u \in \ker(X + \theta^2 I)$, with $\theta \neq 0$, it holds:

$$P_\theta u = \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos^2(\theta s) u ds = u.$$

If $y \in \mathcal{R}(P_\theta)$ with $y = P_\theta x$ for some $x \in \mathcal{F}_\theta$, the same computations as in [24] show that $(X + \theta^2 I)P_\theta x = 0$, *i.e.* $\mathcal{R}(P_\theta) \subset \ker(X + \theta^2 I)$.

5) From 3) and 4) it follows that $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ is contained in $\mathcal{F}_\theta$. To prove the converse inclusion, suppose, by contradiction, that there exists $x \in \mathcal{F}_\theta$, $x \notin \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$. By the Hahn-Banach theorem there is some $\lambda \in \mathcal{E}'$ such that $< x, \lambda > \neq 0$ and $\lambda = 0$ on $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$. Since $<(X + \theta^2 I)x, \lambda> = 0$ for all $x \in \mathcal{D}(X)$, one has $\lambda \in \mathcal{D}(X' + \theta^2 I)$ and $< x, X'\lambda + \theta^2\lambda >= 0$ for all $x \in \mathcal{D}(X)$, *i.e.* $X'\lambda = -\theta^2\lambda$.

Then $X'\lambda = -\theta^2\lambda \in \mathcal{D}(X')$, whence $\lambda \in \mathcal{D}(X^\odot)$ and $X^\odot\lambda = -\theta^2\lambda$.

Suppose $\theta > 0$. Since $C^\odot(t)\lambda = \cosh t\sqrt{-\theta^2}\lambda = \cos(\theta t)\lambda$, it results:

$$< P_\theta x, \lambda >= \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos(\theta s) < C(s)x, \lambda > ds =$$

$$= \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos(\theta s) < x, C^\odot(s)\lambda > ds = \lim_{t\to+\infty} \frac{2}{t} \int_0^t \cos^2(\theta s) < x, \lambda > ds =< x, \lambda > .$$

From 4) the equality $< P_\theta x, \lambda >= 0$ follows; this yields a contradiction, since $< x, \lambda > \neq 0$ by hypothesis.

Finally, since P_θ is a projector in $\mathcal{F}_\theta$, it holds: $\mathcal{F}_\theta = \mathcal{R}(P_\theta) \oplus \overline{\ker P_\theta}$. It has been proved that $\mathcal{R}(P_\theta) = \ker(X + \theta^2 I)$ and $\mathcal{F}_\theta = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$; it results, therefore, $\ker P_\theta = \overline{\mathcal{R}(X + \theta^2 I)}$. □

For a strongly continuous and uniformly bounded semigroup T generated by X, it is possible to prove the mean ergodic theorem for T and then to apply it to the semigroup $T(t) = e^{-i\theta t}T(t)$, generated by $X - i\theta I$, for every $\theta \in \mathbb{R}$, which is also uniformly bounded. If C is a strongly continuous, uniformly bounded cosine function generated by X, then $X + \theta^2 I$ generates a strongly continuous cosine function, whose uniform boundedness cannot be guaranteed; therefore, an explicit proof of the ergodic theorem for the cosine function generated by $X + \theta^2 I$ is necessary.

3) and 4) of Theorem 4.1 imply, in particular, that

$$\ker(X + \theta^2 I) \cap \overline{\mathcal{R}(X + \theta^2 I)} = \{0\}. \tag{4.3}$$

It is well known that, if X is the infinitesimal generator of a strongly continuous, uniformly bounded semigroup, then $\ker X \cap \overline{\mathcal{R}X} = \{0\}$. Observe that (4.3) isn't retrievable from the analogous result for the semigroups, since the semigroup generated by $X + \theta^2 I$ isn't uniformly bounded.

Let now $\mathcal{E}$ be a complex, weakly sequentially complete Banach space and let the strongly continuous cosine function $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be uniformly bounded. The integral $\int_0^t \cos(\theta s) < C(s)x, \lambda > ds$ is, for all θ and t in $\mathbb{R}$, $x \in \mathcal{E}$, $\lambda \in \mathcal{E}'$, a Riemann integral. Fix $\theta \in \mathbb{R}$. If the limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \cos(\theta s) < C(s)x, \lambda > ds \tag{4.4}$$

exists for all $\lambda \in \mathcal{E}'$, then there is some $Q_\theta x \in \mathcal{E}$ such that

$$< Q_\theta x, \lambda >= \begin{cases} \lim\limits_{t \to +\infty} \dfrac{1}{t} \displaystyle\int_0^t < C(s)x, \lambda > ds & \text{if } \theta = 0 \\ \lim\limits_{t \to +\infty} \dfrac{2}{t} \displaystyle\int_0^t \cos(\theta s) < C(s)x, \lambda > ds & \text{if } \theta \neq 0 \end{cases}$$

for every $\lambda \in \mathcal{E}'$. Let

$$\mathcal{G}_\theta = \{x \in \mathcal{E} : \lim_{t \to +\infty} \frac{1}{t} \int_0^t \cos(\theta s) < C(s)x, \lambda > ds \text{ exists for every } \lambda \in \mathcal{E}'\}.$$

The properties of Q_θ are illustrated in the following theorem.

THEOREM 4.2. *Let $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be a strongly continuous, uniformly bounded cosine function on a weakly sequentially complete complex Banach space $\mathcal{E}$. Let Q_θ and $\mathcal{G}_\theta$ be defined as above. Then for every $\theta \in \mathbb{R}$ it holds:*
1) *$C(t)\mathcal{G}_\theta \subset \mathcal{G}_\theta$ and $Q_\theta C(t) = \cos(\theta t) Q_\theta = C(t) Q_\theta$ on $\mathcal{G}_\theta$ for all $t \in \mathbb{R}$;*
2) *Q_θ is a linear projection operator in $\mathcal{G}_\theta$, with $||Q_\theta|| \leq 2M$;*
3) *$\overline{\mathcal{R}(X + \theta^2 I)} \subset \ker Q_\theta$, $\ker(X + \theta^2 I) \subset \mathcal{G}_\theta$ and $Q_{\theta\,|\ker(X+\theta^2 I)} = I$;*
4) *$\mathcal{R}Q_\theta = \ker(X + \theta^2 I)$;*
5) *$\mathcal{G}_\theta = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ and $\ker Q_\theta = \overline{\mathcal{R}(X + \theta^2 I)}$;*
6) *$P_\theta = Q_\theta$ on $\mathcal{G}_\theta$.*

PROOF. 1) If $x \in \mathcal{G}_\theta$, then

$$< C(\tau)Q_\theta x, \lambda >=< Q_\theta x, C(\tau)'\lambda >= \lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos(\theta s) < C(s)x, C(\tau)'\lambda > ds =$$

$$= \lim_{t\to+\infty} \frac{1}{t}\int_{-t}^t e^{-i\theta s} < C(\tau)C(s)x, \lambda > ds =$$

$$= \lim_{t\to+\infty} \frac{1}{2t}\int_{-t}^t e^{-i\theta s} < [C(s+\tau) + C(s-\tau)]x, \lambda > ds =$$

$$= \lim_{t\to+\infty} \frac{1}{2t}\Big[e^{i\theta\tau}\int_{\tau-t}^{\tau+t} e^{-i\theta u} < C(u)x, \lambda > du + e^{-i\theta\tau}\int_{-t-\tau}^{t-\tau} e^{-i\theta u} < C(u)x, \lambda > du\Big] =$$

$$= (e^{i\theta\tau} + e^{-i\theta\tau})/2 < Q_\theta x, \lambda >= \cos(\theta\tau) < Q_\theta x, \lambda >$$

for all $\lambda \in \mathcal{E}'$, and therefore

$$C(\tau)Q_\theta x = \cos(\theta\tau)Q_\theta x \tag{4.5}$$

for all τ and $\theta \in \mathbb{R}$. Analogously

$$< Q_\theta C(\tau)x, \lambda >= \lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos(\theta s) < C(s)C(\tau)x, \lambda > ds = \cos(\theta\tau) < Q_\theta x, \lambda >$$

and therefore, if the limit (4.4) exists for every $\lambda \in \mathcal{E}'$, then $Q_\theta C(\tau)x$ exists and $Q_\theta C(\tau)x = \cos(\theta\tau)Q_\theta x$.

2) It follows from 1) and the uniform boundedness of C.

3) On $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ P_θ exists, so that Q_θ exists also and it coincides with P_θ, whence 3) follows.

4) $Q_\theta x \in \mathcal{D}(X)$ and $(X + \theta^2 I)Q_\theta x = 0$, since, if $\tau \in \mathbb{R}\setminus\{0\}$ it results

$$(2/\tau^2)\big(C(\tau) - I\big)Q_\theta x = (2/\tau^2)\big(\cos(\theta\tau) - 1\big)Q_\theta x,$$

which tends to $-\theta^2 Q_\theta x$ when $\tau \to 0$. The other inclusion is given by 3).

5) It will now be shown that $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ is the set of all $x \in \mathcal{E}$ for which the limit (4.4) exists for all $\lambda \in \mathcal{E}'$. If there exists $x \in \mathcal{G}_\theta$, $x \notin \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$, then there is some $\lambda \in \mathcal{E}'$ such that $< x, \lambda >\neq 0$ and λ vanishes on $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$. Hence, exactly as in 5) of Theorem 4.1, it results $\lambda \in \mathcal{D}(X^\odot)$ with $X^\odot\lambda = -\theta^2\lambda$, and therefore

$$< Q_\theta x, \lambda >= \lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos^2(\theta s) < x, \lambda > ds =< x, \lambda >,$$

which yields a contradiction, since $Q_\theta x \in \ker(X + \theta^2 I)$ and λ vanishes on $\ker(X + \theta^2 I)$.

6) Obvious by 5). □

In virtue of the fact that the dual of a Banach space $\mathcal{E}'$ is always sequentially weak-star complete (as a consequence of the Banach-Steinhaus theorem), a version of the mean ergodic theorem on the dual space of $\mathcal{E}$ can be proved. Set

$$\mathcal{H}'_\theta = \left\{\lambda \in \mathcal{E}' : \lim_{t\to+\infty} \frac{1}{t}\int_0^t \cos(\theta s) < C(s)x, \lambda > ds \text{ exists for every } \quad x \in \mathcal{E}\right\}.$$

Observe that $\mathcal{H}'_\theta = \mathcal{H}'_{-\theta}$ for every $\theta \in \mathbb{R}$.

If $\lambda \in \mathcal{H}'_\theta$ for some $\theta \in \mathbb{R}$, then there is $R_\theta\lambda \in \mathcal{E}'$ such that:

$$< x, R_\theta\lambda >= \begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{t}\displaystyle\int_0^t < C(s)x, \lambda > ds & \text{if } \theta = 0 \\ \lim\limits_{t\to+\infty} \dfrac{2}{t}\displaystyle\int_0^t \cos(\theta s) < C(s)x, \lambda > ds & \text{if } \theta \neq 0 \end{cases}$$

for every $x \in \mathcal{E}$.

Theorem 4.3. *Let $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ be a strongly continuous, uniformly bounded cosine function on a complex Banach space $\mathcal{E}$. Let R_θ and $\mathcal{H}'_\theta$ be defined as above, for any real number θ. Then*

1) $R_\theta C(t)'\lambda = C(t)' R_\theta\lambda = \cos(\theta t)R_\theta\lambda$ *for every* $\lambda \in \mathcal{H}'_\theta$;
2) R_θ *is a linear projection operator in* $\mathcal{H}'_\theta$, *with* $||R_\theta\lambda|| \leq 2M||\lambda||$ *for every* $\lambda \in \mathcal{H}'_\theta$;
3) $\mathcal{R}R_\theta = \ker(X^\odot + \theta^2 I)$;
4) $\mathcal{H}'_\theta = \mathcal{R}(R_\theta) \oplus \ker R_\theta$ *and* $\mathcal{R}(R_\theta) \cap \ker R_\theta = \{0\}$;
5) $\ker(X^\odot + \theta^2 I) \oplus \overline{\mathcal{R}(X^\odot + \theta^2 I)} \subset \mathcal{H}'_\theta$ *and*

$$R_\theta\lambda = \begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{t}\displaystyle\int_0^t C^\odot(s)\lambda ds & \textit{if } \theta = 0 \\ \lim\limits_{t\to+\infty} \dfrac{2}{t}\displaystyle\int_0^t \cos(\theta s) C^\odot(s)\lambda ds & \textit{if } \theta \neq 0 \end{cases}$$

on $\ker(X^\odot + \theta^2 I) \oplus \overline{\mathcal{R}(X^\odot + \theta^2 I)}$.

Proof. 1) The proof is very similar to that of 1) in Theorem 4.2.

2) If $\lambda \in \mathcal{H}'_\theta$, $\theta \neq 0$, 1) implies that

$$< x, R_\theta\lambda >= \lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos(\theta s) < x, C'(s)R_\theta\lambda > ds =$$
$$= \lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos(\theta s) < C(s)x, R_\theta\lambda > ds \text{ for every } x \in \mathcal{E},$$

and therefore $R_\theta\lambda \in \mathcal{H}'_\theta$ and $R_\theta^2 = R_\theta$. A similar computation shows that, if $\lambda \in \mathcal{H}'_0$, then $R_0\lambda \in H'_0$ and $R_0^2 = R_0$.

Finally, the uniform boundedness of C yields the estimate $||R_\theta|| \leq 2M$ for every $\theta \in \mathbb{R}$.

3) Let $x \in \mathcal{D}(X)$ and $\lambda \in \mathcal{H}'_\theta$, for some $\theta \neq 0$. Then, for every $t \neq 0$ it results:

$$(2/t^2) < C(t)x - x, R_\theta\lambda >= (2/t^2)(\cos(\theta t) - 1) < x, R_\theta\lambda >,$$

and therefore, by passing to the limit for $t \to 0$, it results $< (X + \theta^2 I)x, R_\theta\lambda >= 0$ for every $x \in \mathcal{D}(X)$. Thus $R_\theta\lambda \in \mathcal{D}(X' + \theta^2 I)$ and $< x, (X' + \theta^2 I)R_\theta\lambda >= 0$ for all $x \in \mathcal{D}(X)$, whence $R_\theta\lambda \in \mathcal{D}(X^\odot)$ and $(X^\odot + \theta^2 I)R_\theta\lambda = 0$. *Viceversa*, if $(X^\odot + \theta^2 I)\lambda = 0$ for some $\lambda \in \mathcal{D}(X^\odot) \setminus \{0\}$, then, by supposing $\theta \neq 0$, $C^\odot(t)\lambda = \cos(\theta t)\lambda$ for every $t \in \mathbb{R}$, and therefore

$$\lim_{t\to+\infty} \frac{2}{t}\int_0^t \cos(\theta s) < C(s)x, \lambda > ds =< x, \lambda >$$

for every $x \in \mathcal{E}$. Thus $\lambda \in \mathcal{H}'_\theta$ and $R_\theta \lambda = \lambda$. An analogous computation for the case $\theta = 0$ concludes the proof of 3).

4) Obvious, since R_θ is a projector in $\mathcal{H}'_\theta$.

5) It is obvious, since $C^\odot$ is a strongly continuous, uniformly bounded cosine function on $\mathcal{E}^\odot$. □

It is worth noticing that, in the particular case of a reflexive Banach space $\mathcal{E}$, R_θ is the projection operator defined in $\mathcal{E}'$ by

$$R_\theta \lambda = \begin{cases} \lim\limits_{t \to +\infty} \dfrac{1}{t} \displaystyle\int_0^t C'(s) \lambda ds & \text{if } \theta = 0 \\ \lim\limits_{t \to +\infty} \dfrac{2}{t} \displaystyle\int_0^t \cos(\theta s) C'(s) \lambda ds & \text{if } \theta \neq 0. \end{cases}$$

COROLLARY 4.4. *If the limit* (4.4) *exists for every* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^\odot$, *then* $\mathcal{H}'_\theta = \mathcal{E}^\odot$ *and* $R_\theta \in \mathcal{L}(\mathcal{E}^\odot)$.

Finally, a situation will be illustrated, in which the operator P_θ, introduced at the beginning of this section, converges uniformly.

LEMMA 4.5. *For every* $\zeta \in \rho(X + \theta^2 I)$ *it holds*

$$[I - \zeta R(\zeta, X + \theta^2 I)]\mathcal{E} = \mathcal{R}(X + \theta^2 I).$$

PROOF. Since $[\zeta I - (X + \theta^2 I)]R(\zeta, X + \theta^2 I)y = y$ for every $\zeta \in \rho(X + \theta^2 I)$ and for every $y \in \mathcal{E}$, then

$$[I - \zeta R(\zeta, X + \theta^2 I)]y = -(X + \theta^2 I)R(\zeta, X + \theta^2 I)y.$$

Conversely, if $y = (X + \theta^2 I)x$ for some $x \in \mathcal{D}(X)$, then

$$y = [\zeta I - (X + \theta^2 I)]R(\zeta, X + \theta^2 I)y = [\zeta I - (X + \theta^2 I)](\zeta R(\zeta, X + \theta^2 I)x - x) =$$

$$= -[I - \zeta R(\zeta, X + \theta^2 I)][\zeta I - (X + \theta^2 I)]x. \qquad \square$$

THEOREM 4.6. *Let* $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$ *be a strongly continuous, uniformly bounded cosine function on a complex Banach space* $\mathcal{E}$, *generated by* X. *Let* θ *be a real number. If* $\mathcal{R}(X + \theta^2 I)$ *is closed, then* $\mathcal{E} = \ker(X + \theta^2 I) \oplus \mathcal{R}(X + \theta^2 I)$.

If, in addition, θ *is such that*

$$\lim_{n \to \infty} \frac{||\zeta^n R(\zeta, X + \theta^2 I)^n||}{n} = 0\,, \tag{4.6}$$

then the operator P_θ *defined by* (4.1) *converges uniformly on* $\mathcal{E}$.

PROOF. First of all, it will be shown that $\mathcal{E} = \ker(X + \theta^2 I) \oplus \mathcal{R}(X + \theta^2 I)$. Observe that

$$\ker(X' + \theta^2 I) \cap \mathcal{R}(X' + \theta^2 I) \subset \ker(X^\odot + \theta^2 I) \cap \mathcal{R}(X^\odot + \theta^2 I). \tag{4.7}$$

In fact, if $\lambda \in \ker(X' + \theta^2 I)$, then $X'\lambda = -\theta^2\lambda$, so that $\lambda \in \mathcal{D}(X^{\odot})$ and $X^{\odot}\lambda = -\theta^2\lambda$.

Let now $\mu \in \mathcal{D}(X')$ be such that $\lambda = (X' + \theta^2 I)\mu$. Since $\mu \in \mathcal{D}(X' + \theta^2 I)$ and $(X' + \theta^2 I)\mu \in \mathcal{D}(X' + \theta^2 I)$, μ belongs to $\mathcal{D}(X^{\odot} + \theta^2 I)$, and, moreover, $\lambda = (X^{\odot} + \theta^2 I)\mu$, whence (4.7) follows.

Now, by the closed range theorem of S. Banach, one deduces:

$$\big(\ker(X + \theta^2 I) \oplus \mathcal{R}(X + \theta^2 I)\big)^{\perp} \subset \ker(X + \theta^2 I)^{\perp} \cap \mathcal{R}(X + \theta^2 I)^{\perp} =$$

$$= \mathcal{R}(X' + \theta^2 I) \cap \ker(X' + \theta^2 I) = \{0\}\,,$$

since $X^{\odot}$ is the infinitesimal generator of a strongly continuous, uniformly bounded cosine function and therefore (4.3) can be applied. This implies $\mathcal{E} = \ker(X + \theta^2 I) \oplus \mathcal{R}(X + \theta^2 I)$.

Suppose now that (4.6) holds. If $u \in \ker(X + \theta^2 I)$, then point 4) of Theorem 4.1 implies $P_\theta u = u$. It will now be proved the uniform convergence of P_θ on $\mathcal{R}(X + \theta^2 I)$.

$X + \theta^2 I$ is the infinitesimal generator of a strongly continuous cosine function, which will be denoted by $\widetilde{C}$.

Since $\widetilde{C}(t)(\mathcal{R}(X + \theta^2 I)) \subset \mathcal{R}(X + \theta^2 I)$, then $\{\widetilde{C}(t)_{|\mathcal{R}(X+\theta^2 I)}\}$, $t \in \mathbb{R}$, defines a strongly continuous cosine function on the Banach space $\mathcal{R}(X + \theta^2 I)$, generated by the operator $Y_\theta = (X + \theta^2 I)_{|\mathcal{D}(X)\cap\mathcal{R}(X+\theta^2 I)}$.

It is easy to check that Y_θ is a closed operator. It will now be proved that Y_θ is one-to-one.

Suppose $Y_\theta y = 0$ for some $y \in \mathcal{D}(Y_\theta)$. Let ζ be a real number in $\rho(Y_\theta)$. Since $(\zeta I - Y_\theta)R(\zeta\,, Y_\theta)y = y$, then $[I - \zeta R(\zeta\,, Y_\theta)]y = 0$.

Lemma 4.5 entails that $\mathcal{R}(I - \zeta R(\zeta\,, X + \theta^2 I))$ is closed. Under this hypothesis, it is well known from the proof of the uniform ergodic theorem in the discrete case [19], that the operator $I - \zeta R(\zeta\,, X + \theta^2 I)$ is invertible, in virtue of (4.6), on $\mathcal{R}(X + \theta^2 I)$.

Since $[I - \zeta R(\zeta\,, Y_\theta)]y = 0$ implies $[I - \zeta R(\zeta\,, (X + \theta^2 I)_{|\mathcal{R}(X+\theta^2 I)})]y = 0$, then $y = 0$ by the consideration above, and therefore Y_θ is one-to-one.

It will now be proved that $\mathcal{R}(Y_\theta) = \mathcal{R}(X + \theta^2 I)$. First of all, observe that

$$[I - \zeta R(\zeta\,, X + \theta^2 I)]\mathcal{R}(X + \theta^2 I) \subset \mathcal{R}(Y_\theta)\,,$$

since, if $y = [I - \zeta R(\zeta\,, X + \theta^2 I)](X + \theta^2 I)x$ for some $x \in \mathcal{D}(X)$, then $y \in \mathcal{R}(X + \theta^2 I)$. Lemma 4.5 implies $[I - \zeta R(\zeta\,, X + \theta^2 I)]x \in \mathcal{R}(X + \theta^2 I)$, and therefore $y \in \mathcal{R}(Y_\theta)$. Now, since Y_θ is a restriction of $X + \theta^2 I$, it holds:

$$\mathcal{R}(X + \theta^2 I) \supseteq \mathcal{R}(Y_\theta) \supseteq [I - \zeta R(\zeta\,, X + \theta^2 I)]\mathcal{R}(X + \theta^2 I) \supseteq \mathcal{R}(X + \theta^2 I)\,,$$

where the last inclusion follows from the invertibility of $I - \zeta R(\zeta\,, X + \theta^2 I)$ on $\mathcal{R}(X + \theta^2 I)$. Thus $\mathcal{R}(Y_\theta) = \mathcal{R}(X + \theta^2 I)$ and therefore, by the closed graph theorem, the operator Y_θ^{-1}, which is closed since Y_θ is closed, is continuous on $\mathcal{R}(X + \theta^2 I)$.

Let $v \in \mathcal{R}(X + \theta^2 I)$; there exists some $z \in \mathcal{D}(Y_\theta)$ such that $v = (X + \theta^2 I)z = Y_\theta z$.

Since, if $z \in \mathcal{D}(X)$, then $S(s)z \in \mathcal{D}(X)$ and $\frac{d}{ds}C(s)z = XS(s)z = S(s)Xz$, one has:

$$\frac{1}{t}\int_0^t \cos(\theta s)\, C(s)\, v ds = \frac{1}{t}\int_0^t \cos(\theta s)\, C(s)(X + \theta^2 I) z ds =$$

$$= \frac{1}{t}\int_0^t \left\{ \frac{d}{ds}\left[\frac{d}{ds}C(s)z\cos(\theta s)\right] + \theta\sin(\theta s) XS(s)z + \cos(\theta s)\theta^2 C(s)z \right\} ds =$$

$$= \frac{1}{t}\big(XS(t)z\cos(\theta t)\big) + \frac{1}{t}\int_0^t \left[\theta\sin(\theta s)XS(s)z + \cos(\theta s)\theta^2 C(s)z\right] ds =$$

$$= \frac{1}{t}\big(XS(t)z\cos(\theta t)\big) + \frac{1}{t}\int_0^t \left[\frac{d}{ds}\big(C(s)z\cdot\theta\cdot\sin(\theta s)\big)\right] =$$

$$= \frac{1}{t}\big(XS(t)z\cos(\theta t)\big) + \frac{1}{t}\theta\sin(\theta t)\, C(t)z,$$

whence

$$||\frac{1}{t}\int_0^t \cos(\theta s)\, C(s)\, v ds|| \leq \frac{1}{t}||XS(t)z|| + \frac{|\theta|}{t}||C(t)z||.$$

Now, the Kallmann and Rota's inequality (1) [18] yields the estimate

$$\sup_{t\geq 0}||XS(t)z|| \leq 2\sup_{t\geq 0}||C(t)Xz||^{1/2}\cdot\sup_{t\geq 0}||C(t)z||^{1/2} \leq$$

$$\leq 2\sup_{t\geq 0}\left[||C(t)v|| + \theta^2||C(t)z||\right]^{1/2}\sup_{t\geq 0}||C(t)z||^{1/2},$$

(1) This inequality asserts, in principle, that, if a function is small and its second derivative is small, then its first derivative is necessarily small. For complex-valued functions, defined on $(0\,,\,+\infty)$, E. Landau proved firstly that:

$$||f'||^2_\infty \leq 4\,||f''||_\infty\cdot||f||_\infty;$$

afterwards Hardy, Littlewood and Pólya extended this result to the norm L^2. Finally, Kallman and Rota generalized, as much as possible, the Banach space norm, with respect to which the inequality holds; they proved, moreover, the following result: if T be a strongly continuous contraction semigroup on a Banach space, then, for every $x \in \mathcal{D}(X^2)$, it holds:

$$||Xx||^2 \leq 4\,||X^2x||\cdot||x||.$$

By applying this result to the infinitesimal generator of the translations group on $BUC\,(\mathbb{R};\mathcal{E})$ (*i.e.* on the space of all uniformly continuous, bounded maps on $\mathbb{R}$, with values in $\mathcal{E}$), one gets the following inequality [18]:

$$\big(\sup_{t\geq 0}||X\,S(t)x||\big)^2 \leq 4\,\big(\sup_{t\geq 0}||C(t)\,Xx||\big)\cdot\big(\sup_{t\geq 0}||C(t)x||\big) \text{ for every } x \in \mathcal{D}(X).$$

whence

$$||\frac{1}{t}\sup_{t\geq 0}\int_0^t \cos(\theta s)C(s)vds|| \leq \frac{2}{t}\sup_{t\geq 0}\left[||C(t)v|| + \theta^2||C(t)z||\right]^{1/2}\sup_{t\geq 0}||C(t)z||^{1/2} +$$

$$+\frac{|\theta|}{t}\sup_{t\geq 0}||C(t)z|| \leq \frac{M}{t}\left[2||Y_\theta^{-1}||^{1/2}\cdot\sqrt{1+\theta^2||Y_\theta^{-1}||} + |\theta|\cdot||Y_\theta^{-1}||\right]||v||\,,$$

and therefore P_θ converges to zero uniformly on $\mathcal{R}(X+\theta^2 I)$. □

REMARK 4.7. Observe that for $\theta = 0$ condition (4.6) is trivially satisfied.

PROPOSITION 4.8. *Let C be a strongly continuous cosine function on a Banach space $\mathcal{E}$. If $\mathcal{R}(X+\theta^2 I)$ is closed for some $\theta\in\mathbb{R}$, then $\mathcal{R}(X^\odot+\theta^2 I)$ is closed.*

PROOF. It suffices to prove that $\overline{\mathcal{R}(X^\odot+\theta^2 I)}\subset\mathcal{R}(X^\odot+\theta^2 I)$.

Let $\lambda\in\overline{\mathcal{R}(X^\odot+\theta^2 I)}$, *i.e.* $\lambda = \lim_{n\to\infty}(X^\odot+\theta^2 I)\mu_n$, with $\mu_n\in\mathcal{D}(X^\odot)$ for every $n\in\mathbb{N}$.

Thus $\lambda = \lim_{n\to\infty}(X'+\theta^2 I)\mu_n$ and, since $\mathcal{R}(X'+\theta^2 I)$ is closed by the closed range theorem, then there exists $\mu\in\mathcal{D}(X')$ such that $\lambda = (X'+\theta^2 I)\mu$. Now $\mu\in\mathcal{D}(X'+$ $+\theta^2 I)$ and $(X'+\theta^2 I)\mu\in\overline{\mathcal{D}(X'+\theta^2 I)}$, and therefore $\mu\in\mathcal{D}(X^\odot+\theta^2 I)$ and $\lambda =$ $=(X^\odot+\theta^2 I)\mu$. □

The uniform convergence of P_θ entails, in some cases, uniform convergence for the projection operators Q_θ and R_θ.

COROLLARY 4.9. *Let $\mathcal{E}$ be a weakly sequentially complete complex Banach space. Let C be a strongly continuous, uniformly bounded cosine function on $\mathcal{E}$, generated by X. Let θ be a real number such that $\mathcal{R}(X+\theta^2 I)$ is closed and condition* (4.6) *holds. Then the operator Q_θ converges uniformly on $\mathcal{E}$.*

From Corollary 4.4 and Proposition 4.8 the following result immediately follows:

COROLLARY 4.10. *If $\mathcal{R}(X+\theta^2 I)$ is closed for some $\theta\in\mathbb{R}$ and if the limit* (4.4) *exists for every $x\in\mathcal{E}$ and $\lambda\in\mathcal{E}^\odot$, then*

$$\mathcal{H}'_\theta = \ker(X^\odot+\theta^2 I)\oplus\mathcal{R}(X^\odot+\theta^2 I).$$

If, in addition, θ is such that

$$\lim_{n\to\infty}\frac{||\zeta^n R(\zeta\,,X^\odot+\theta^2 I)^n||}{n} = 0\,,$$

then the operator R_θ converges uniformly on $\mathcal{E}^\odot = \mathcal{H}'_\theta$.

If C is a uniformly continuous cosine function, then, from the equality $||C'(t)-I|| =$ $=||C(t)-I||$, the uniform continuity of C' follows, so that, in particular, C' is strongly continuous and $\mathcal{E}' = \mathcal{E}^\odot$. Thus Corollary 4.10 yields

COROLLARY 4.11. *If the following conditions hold*:
1) *C is a uniformly continuous cosine function*;
2) *$\mathcal{R}(X+\theta^2 I)$ is closed for some $\theta \in \mathbb{R}$*;
3) *the limit* (4.4) *exists for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^{\odot}$*;

then

$$\mathcal{E}' = \ker(X^{\odot} + \theta^2 I) \oplus \mathcal{R}(X^{\odot} + \theta^2 I).$$

Recall that every strongly continuous cosine function on a Grothendieck space with the Dunford-Pettis property is uniformly continuous [26]. Indeed, a little stronger result can be proved; in particular, that, if C is a strongly continuous cosine function on a Grothendieck space, then $\mathcal{E}' = \mathcal{E}^{\odot}$ and also $\mathcal{E}'' = \mathcal{E}^{++}$. Thus the condition that $\mathcal{E}$ is a Grothendieck space, combined with 2) and 3) of Corollary 4.10, suffices to decompose the dual space $\mathcal{E}'$ as $\ker(X^{\odot} + \theta^2 I) \oplus \mathcal{R}(X^{\odot} + \theta^2 I)$.

5. THE REFLEXIVE CASE

For a strongly continuous, uniformly bounded semigroup T generated by X it is well known [27] that, if the Banach space $\mathcal{E}$ is reflexive, then $\mathrm{r}\sigma(X) \cap i\mathbb{R} = \emptyset$.

An analogous result will now be shown for the intersection between the residual spectrum of the infinitesimal generator X of a strongly continuous, uniformly bounded cosine function and the real negative semiaxis. Since the spectrum of the infinitesimal generator of a uniformly bounded cosine function is entirely contained in the real negative semiaxis, this result will imply that the residual spectrum of X is empty.

Since the residual spectrum of X consists of all $\zeta \in \mathbb{C}$ for which the operator $\zeta I - X$ is invertible, but its range is not dense, if $-\theta^2$ belongs to $\mathrm{r}\sigma(X)$ for some $\theta \in \mathbb{R}$, then $\ker(X+\theta^2 I) = \{0\}$ and $\overline{\mathcal{R}(X+\theta^2 I)} \neq \mathcal{E}$, whence the following result follows:

LEMMA 5.1. *Let X be a linear, densely defined operator on $\mathcal{E}$.*
If $\ker(X+\theta^2 I) \oplus \overline{\mathcal{R}(X+\theta^2 I)} = \mathcal{E}$, then $-\theta^2 \notin \mathrm{r}\sigma(X)$.
If, in addition, X generates a uniformly bounded cosine function, then $\mathrm{r}\sigma(X) = \emptyset$.

Goldstein, Radin and Showalter [16] proved that, if $\mathcal{E}$ is reflexive and X is the infinitesimal generator of a strongly continuous, uniformly bounded semigroup, then

$$\mathcal{E} = \ker X \oplus \overline{\mathcal{R}X}. \tag{5.1}$$

Under the above assumptions, for every $\theta \in \mathbb{R}$ $X - i\theta I$ generates the strongly continuous, uniformly bounded semigroup $\{e^{-i\theta t} T(t)\}$, and therefore it holds also $\mathcal{E} = \ker(X - i\theta I) \oplus \overline{\mathcal{R}(X - i\theta I)}$. If X generates a strongly continuous, uniformly bounded cosine function, then it generates also a strongly continuous, uniformly bounded semigroup, so that (5.1) holds. However, since the semigroup generated by $X + \theta^2 I$ is not uniformly bounded, in order to prove a formula like (5.1) for $X + \theta^2 I$ a direct proof is necessary; the technique is, however, very similar to that of [16].

Lemma 5.2. *If X is the infinitesimal generator of a strongly continuous, uniformly bounded cosine function $C : \mathbb{R} \to \mathcal{L}(\mathcal{E})$, and if $\mathcal{E}$ is reflexive, then for every $\theta \in \mathbb{R}$ it results*

$$\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)} = \mathcal{E}.$$

Proof. By identifying $\mathcal{E}''$ with $\mathcal{E}$ one obtains the following equalities:

$$\overline{\mathcal{R}(X + \theta^2 I)}^{\perp} = \ker(X' + \theta^2 I)\,,$$

$$\overline{\mathcal{R}(X' + \theta^2 I)}^{\perp} = \ker(X + \theta^2 I)\,,$$

$$\overline{\mathcal{R}(X' + \theta^2 I)} = \ker(X + \theta^2 I)^{\perp}\,,$$

where $\overline{\mathcal{R}(X + \theta^2 I)}^{\perp}$ denotes the annihilator of $\overline{\mathcal{R}(X + \theta^2 I)}$.

Consider now

$$\left(\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}\right)^{\perp} \subset \ker(X + \theta^2 I)^{\perp} \cap \overline{\mathcal{R}(X + \theta^2 I)}^{\perp} =$$

$$= \overline{\mathcal{R}(X' + \theta^2 I)} \cap \ker(X' + \theta^2 I).$$

Since $\mathcal{E}$ is reflexive, then X' generates a strongly continuous and uniformly bounded cosine function. If $\ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)} \neq \mathcal{E}$, then $\overline{\mathcal{R}(X' + \theta^2 I)} \cap \ker(X' + \theta^2 I) \neq$ $\neq \{0\}$, which, combined with (4.3) applied to X', yields a contradiction. □

Lemma 5.2 and Theorem 4.6 imply the following

Proposition 5.3. *Let X be the infinitesimal generator of a strongly continuous, uniformly bounded cosine function C on a Banach space $\mathcal{E}$.*

If $\theta \in \mathbb{R}$ is such that $\mathcal{R}(X + \theta^2 I)$ is closed, then $-\theta^2 \notin \mathrm{r}\sigma(X)$.

6. Spectrum and frequencies

The mean ergodic theorems proved in n. 3 will now be applied in order to relate the spectrum of X and the frequencies of the asymptotical almost periodic functions associated to a cosine function C.

If X represents the infinitesimal generator of a strongly continuous, uniformly bounded cosine function, the following notations will be used:

$$\pm\sqrt{-\mathrm{p}\sigma(X)} = \{\zeta \in \mathbb{R} : -\zeta^2 \in \mathrm{p}\sigma(X)\} \text{ and}$$

$$\pm\sqrt{\big(-\mathrm{p}\sigma(X)\big) \cup \big(-\mathrm{r}\sigma(X)\big)} = \{\zeta \in \mathbb{R} : -\zeta^2 \in \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)\}.$$

Theorem 6.1. *Let C be a strongly continuous, uniformly bounded cosine function.*

1) *If there are $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$ such that the function $t \mapsto < C(t)x, \lambda >$ is a non constant, asymptotically almost periodic function, then the set of frequencies of this function is contained in $\pm\sqrt{\big(-\mathrm{p}\sigma(X^{\odot})\big)}$.*

2) *Conversely, for every* $-\theta^2 \in \mathrm{p}\sigma(X^{\odot})$, *with* $\theta \in \mathbb{R}$, *there are* $x \in \mathcal{D}(X)$ *and* $\lambda \in \mathcal{E}'$, *such that* $< x, \lambda > \neq 0$ *and* θ *and* $-\theta$ *are frequencies of the periodic function* $t \mapsto$ $\mapsto < C(t)x, \lambda >$.

Proof. 1) Let $\theta \in \mathbb{R}$ be a frequency of some asymptotically almost periodic function $t \mapsto < C(t)x, \lambda >$. If $-\theta^2 \in \mathrm{c}\sigma(X) \cup \rho(X)$ for some $\theta \in \mathbb{R} \setminus \{0\}$, then $\overline{\mathcal{R}(X+\theta^2 I)} = \mathcal{E}$. Since for every $\theta \in \mathbb{R}$ $\ker(X+\theta^2 I) \cap \overline{\mathcal{R}(X+\theta^2 I)} = \{0\}$, it results $\ker(X+\theta^2 I) = \{0\}$ and therefore also $\mathcal{R}(P_\theta) = \{0\}$. Now:

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \cos(\theta s) < C(s)x, \lambda > ds = < \lim_{t\to+\infty} \frac{1}{t}\int_0^t \cos(\theta s) C(s)x ds, \lambda > = \frac{1}{2} < P_\theta x, \lambda > = 0,$$

and therefore θ cannot be a frequency of the function $t \mapsto < C(t)x, \lambda >$. An analogous computation for $\theta = 0$ shows that the set of frequencies of this function is contained in $\pm\sqrt{\big(-\mathrm{p}\sigma(X)\big) \cup \big(-\mathrm{r}\sigma(X)\big)}$, and therefore, by Theorem 2.4, it is contained in $\pm\sqrt{\big(-\mathrm{p}\sigma(X^{\odot})\big)}$.

2) If $\theta \in \mathbb{R}$ is such that $-\theta^2 \in \mathrm{p}\sigma(X^{\odot})$, then there is some $\lambda \in \mathcal{D}(X^{\odot}) \setminus \{0\}$ such that $X^{\odot}\lambda = -\theta^2\lambda$, and, therefore, if $\theta > 0$ $C^{\odot}(t)\lambda = \cos(\theta t)\lambda$ for every $t \in \mathbb{R}$. Hence the function $t \mapsto C^{\odot}(t)\lambda$ is either constant (if $\theta = 0$) or periodic of period $2\pi/\theta$.

Since the set $\{y \in \mathcal{E} :< y, \lambda > \neq 0\}$ is open and non empty and since $\mathcal{D}(X)$ is dense in $\mathcal{E}$, there exists $x \in \mathcal{D}(X)$ such that $< x, \lambda > \neq 0$. Now the map $t \mapsto$ $\mapsto < C(t)x, \lambda > = < x, C(t)'\lambda > = < x, C^{\odot}(t)\lambda >$ is either a scalar valued periodic function with period $2\pi/\theta$, if $\theta \neq 0$, or a constant. □

In virtue of Theorem 2.4 and of Lemma 5.2, the first part of Theorem 6.1 can be slightly improved when $\mathcal{E}$ is reflexive:

Theorem 6.2. *If* $\mathcal{E}$ *is reflexive,* C *is uniformly bounded and for some* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$ *the function* $t \mapsto < C(t)x, \lambda >$ *is a non constant asymptotically almost periodic function, then the set of the frequencies of this function is a subset of* $\pm\sqrt{-\mathrm{p}\sigma(X)}$.

Theorem 4.6 also yields

Corollary 6.3. *If* C *is uniformly bounded, if for some* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}'$ *the function* $t \mapsto < C(t)x, \lambda >$ *is a non constant, asymptotically almost periodic function and if* $\theta \in \mathbb{R}$ *is a frequency of this map such that* $\mathcal{R}(X+\theta^2 I)$ *is closed, then* $\theta \in \pm\sqrt{-\mathrm{p}\sigma(X)}$.

Suppose now C weakly asymptotically almost periodic (and therefore weakly almost periodic). By a routine application of the Banach-Steinhaus theorem, C is uniformly bounded. Moreover the limit

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \cos(\theta s) < C(s)x, \lambda > ds$$

exists for all $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$. By the mean ergodic theorem in weakly sequentially complete Banach spaces, one gets

$$\mathcal{E} = \ker(X+\theta^2 I) \oplus \overline{\mathcal{R}(X+\theta^2 I)}$$

for every $\theta \in \mathbb{R}$.

Thus it holds:

THEOREM 6.4. *If $\mathcal{E}$ is weakly sequentially complete and C is a strongly continuous, weakly asymptotically almost periodic function, then for every $\theta \in \mathbb{R}$ $\mathcal{E} = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$.*

In that case, Lemma 5.1 entails that $\mathrm{r}\sigma(X) = \emptyset$.

Suppose the function $t \mapsto < C(t)x, \lambda >$ be asymptotically almost periodic (and therefore almost periodic) for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^{\odot}$. From Corollary 4.4 the following result follows:

PROPOSITION 6.5. *If C is uniformly bounded and $t \mapsto < C(t)x, \lambda >$ is asymptotically almost periodic for every $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}^{\odot}$, then the frequencies of $t \mapsto < C(t)x, \lambda >$ are contained in $\pm\sqrt{-\mathrm{p}\sigma(X^{\odot})}$.*

The definition of index of a complex number with respect to an operator yields a criterion, establishing if a real number can be a frequency for some asymptotically almost periodic map, associated to a cosine function C.

A complex number ζ is said to be of index ν (where ν is a positive integer) with respect to a linear operator X in case $(X - \zeta I)^{\nu+1} \equiv 0$ implies $(X - \zeta I)^{\nu} \equiv 0$ and there is x_0 such that $(X - \zeta I)^{\nu} x_0 = 0$ and $(X - \zeta I)^{\nu-1} x_0 \neq 0$.

ζ has index zero if , by definition, $X - \zeta I$ has an inverse. If no such integer ν exists, ζ is said to be of infinite index.

THEOREM 6.6. *Let C be a strongly continuous and uniformly bounded cosine function on a complex Banach space $\mathcal{E}$ and let $\mathcal{E} = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$ for some $\theta \in \mathbb{R}$. Then $-\theta^2$ is of index 0 or 1 with respect to $X + \theta^2 I$.*

If $-\theta^2$ is of index 0, then $\pm\theta$ cannot be a frequency for any asymptotically almost periodic function $t \mapsto < C(t)x, \lambda >$, with $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$.

If $-\theta^2$ has index 1, then $-\theta^2$ belongs to $\mathrm{p}\sigma(X)$ and θ or $-\theta$ is a frequency of $t \mapsto C(t)x$, for some $x \in \mathcal{E}$.

PROOF. If $(X + \theta^2 I)^2 x = 0$ for some $x \in \mathcal{D}(X^2)$, then $(X + \theta^2 I)x \in \ker(X + \theta^2 I) \cap \mathcal{R}(X + \theta^2 I)$, so that from (4.3) $(X + \theta^2 I)x = 0$ follows.

Suppose now that $-\theta^2$ is of index 0. Then $X + \theta^2 I$ has an inverse, and therefore $\ker(X + \theta^2 I) = \{0\}$. Since $\mathcal{E} = \ker(X + \theta^2 I) \oplus \overline{\mathcal{R}(X + \theta^2 I)}$, then $-\theta^2$ doesn't belong to $\mathrm{r}\sigma(X)$, and therefore $-\theta^2 \in \mathrm{c}\sigma(X) \cup \rho(X)$. Theorem 6.1 yields now the thesis.

If $-\theta^2$ has index 1, then $-\theta^2 \in \mathrm{p}\sigma(X)$ and there is some $x \in \mathcal{E}$ such that θ or $-\theta$ is a frequency of a periodic function $t \mapsto C(t)x$. $\square$

7. HARMONIC ANALYSIS OF COSINE FUNCTIONS

It will now be shown that, in some cases, it is possible to bound from below the distance between eigenvalues of X and X'.

LEMMA 7.1. *Let C be a strongly continuous cosine function on a Banach space $\mathcal{E}$, generated by X. Let ζ be a complex number. If:*

(i) *there exist* $x_0 \in \mathcal{D}(X)$ *and* $\lambda_0 \in \mathcal{D}(X^{\odot})$ *such that* $< x_0 , \lambda_0 > \neq 0$, *and either* x_0 *is an eigenvector of* X *with eigenvalue* ζ *or* λ_0 *is an eigenvector of* $X^{\odot}$ *with eigenvalue* ζ ;

(ii) *the function* $t \mapsto < C(t)x_0 , \lambda_0 >$ *is asymptotically almost periodic with inclusion length* Λ ;

then ζ *is a real negative number and, if* $\zeta = -\theta^2$ *for some* $\theta \in \mathbb{R}$, *then*

$$\theta^2 > \pi^2/\Lambda^2. \tag{7.1}$$

Proof. By hypothesis, the function $t \mapsto < C(t)x_0 , \lambda_0 >$ is asymptotically almost periodic. *i.e.* for every $\varepsilon > 0$ there are $\Lambda = \Lambda(x_0 , \lambda_0 , \varepsilon)$ and $K = K(x_0 , \lambda_0 , \varepsilon) \geq 0$ such that, for all $s \geq 0$, the interval $[s, s + \Lambda]$ contains some τ for which:

$$| < C(t + \tau)x_0 , \lambda_0 > - < C(t)x_0 , \lambda_0 > | \leq \varepsilon \tag{7.2}$$

whenever $t, t + \tau \geq K$.

For $\varepsilon > 0$ and $s \in [0 , + \infty)$ let $\tau \in [s, s + \Lambda]$ be such that (7.2) holds, *i.e.*:

$$| \cosh[(t + \tau)\sqrt{\zeta}] - \cosh(t\sqrt{\zeta})| \leq \frac{\varepsilon}{| < x_0 , \lambda_0 > |} \quad \text{for all } t \geq K. \tag{7.3}$$

Setting $\sqrt{\zeta} = \alpha + i\beta$, for some real α and β, then the condition (7.3) is equivalent to:

$$\begin{aligned} e^{2(t+\tau)\alpha} + e^{-2(t+\tau)\alpha} + e^{2t\alpha} + e^{-2t\alpha} + 2\cos[2(t + \tau)\beta] + \\ + 2\cos(2t\beta) - 2e^{\tau\alpha}\{e^{2t\alpha}\cos(\tau\beta) + \cos[(2t + \tau)\beta]\} + \\ -2e^{-\tau\alpha}\{e^{-2t\alpha}\cos(\tau\beta) + \cos[(2t + \tau)\beta]\} \leq 4\varepsilon^2| < x_0 , \lambda_0 > |^{-2} \end{aligned}$$

which yields:

$$\begin{aligned} |e^{2(t+\tau)\alpha} - 2e^{\tau\alpha}\{e^{2t\alpha}\cos(\tau\beta) + \cos[(2t + \tau)\beta]\}| \leq 4\varepsilon^2| < x_0 , \lambda_0 > |^{-2} + \\ + 2e^{-\tau\alpha}\{e^{-2t\alpha}\cos(\tau\beta) + \cos[(2t + \tau)\beta]\} + \\ - 2e^{-2(t+\tau)\alpha} - e^{2t\alpha} - e^{-2t\alpha} - 2\cos[2\beta(t + \tau)] - 2\cos[2t\beta]. \end{aligned}$$

Now suppose $\alpha > 0$; if $t \geq K$ is fixed and s (and therefore also τ) tends to $+ \infty$, the right member of the inequality is bounded, while, since for τ sufficiently great one has:

$$e^{2(t+\tau)\alpha} \geq 2e^{\tau\alpha}\left|e^{2t\alpha}\cos(\tau\beta) + \cos[(2t + \tau)\beta]\right|,$$

the left one increases to infinity; this yields a contradiction. The same argument works also in the case $\alpha < 0$, so that $\alpha = 0$ and therefore $\zeta \in \mathbb{R}_-$.

Set $\zeta = -\theta^2$, for some $\theta \in \mathbb{R}$. There is no loss of generality by taking $\theta > 0$. Then (7.3) becomes

$$| \cos[(t + \tau)\theta] - \cos(t\theta)| \leq \varepsilon| < x_0 , \lambda_0 > |^{-1} \text{ for all } t, t + \tau \geq K.$$

If N is the least positive integer for which $2N\pi/\theta > K$, the last inequality yields

$$| \cos\tau\theta - 1| \leq \varepsilon| < x_0 , \lambda_0 > |^{-1},$$

which, exactly as in [28], by choosing $0 < \varepsilon < \sqrt{2}| < x_0 , \lambda_0 > |$, leads to the estimate $\theta > \pi/\Lambda$, whence $\theta^2 > \pi^2/\Lambda^2$. $\square$

Let now the infinitesimal generator of a strongly continuous cosine function C be such that for any $\varphi \in \mathbb{R}$ the operator $X + \varphi^2 I$ generates a strongly continuous cosine function C_φ for which every application

$$t \mapsto < C_\varphi(t)x , \lambda >$$

is asymptotically almost periodic for every $x \in \mathcal{D}(X)$ and $\lambda \in \mathcal{D}(X^\odot)$.

This entails that for every $\varphi \in \mathbb{R}$ and for every $\varepsilon > 0$ an inclusion length $\Lambda = \Lambda(\varepsilon , x , \lambda , \varphi)$ is determined. The following result shall be proved:

Proposition 7.2. *Let C be a strongly continuous, uniformly bounded cosine function on $\mathcal{E}$, generated by X, fulfilling condition i) of Lemma 7.1. Suppose, moreover:*

1) *for every φ such that $-\varphi^2 \in \mathrm{p}\sigma(X^\odot)$ the map $t \mapsto < C_\varphi(t)x , \lambda >$ is asymptotically almost periodic for every $x \in \mathcal{D}(X)$ and $\lambda \in \mathcal{D}(X^\odot)$;*

2) *for some $0 < \varepsilon < \sqrt{2}| < x_0 , \lambda_0 > |$ it holds*

$$\sup\{\Lambda(\varepsilon , x_0 , \lambda_0 , \varphi) : \varphi \in \mathrm{p}\sigma(X^\odot)\} <+ \infty.$$

Then $\mathrm{p}\sigma(X^\odot)$ (and therefore also $\mathrm{p}\sigma(X)$) has no accumulation point.

Proof. Since i) of Lemma 7.1 holds, $-\theta^2 + \varphi^2$ is an eigenvalue of $X^\odot + \varphi^2$ with eigenvector λ_0. Since, moreover, the map $t \mapsto < C_\varphi(t)x_0 , \lambda_0 >$ is asymptotically almost periodic, from Lemma 7.1 the constraint $\varphi^2 - \theta^2 < 0$ follows.

Now, by applying Lemma 7.1 to $X^\odot + \varphi^2$, whose point spectrum is the image of $\mathrm{p}\sigma(X^\odot)$ under the translation by φ^2, the thesis follows. □

Observe that, if the set $\mathrm{p}\sigma\,(X^\odot)$ has no accumulation point, then $\overline{\mathrm{p}\sigma\,(X^\odot)} = \overline{\mathrm{p}\sigma(X) \cup (\mathrm{r}\sigma(X))}$ is discrete. That implies that also $\mathrm{p}\sigma(X) \cup (\mathrm{r}\sigma(X))$ is discrete, and, in particular, that it is at most countable. It will be seen in the following which constraints on the spectral structure of X are imposed by the enumerability of the set $\mathrm{p}\sigma(X) \cup (\mathrm{r}\sigma(X))$.

From now on, it will be assumed that $\mathrm{p}\sigma(X) \cup (\mathrm{r}\sigma(X))$ is at most enumerable; in particular, under that hypothesis the harmonic analysis of cosine functions can be studied; the main tool is the convergence theorem that Harald Bohr proved at the end of his book [7]: if $f : \mathbb{R} \to \mathbb{C}$ is an almost periodic function with Fourier spectrum $\{\theta_n\}$ linearly independent over the rational numbers, then the Fourier series

$$\sum_{n=0}^{+\infty} a_n e^{i\theta_n \bullet}$$

of f converges to f uniformly on $\mathbb{R}$.

Let C be a strongly continuous cosine function such that C is weakly asymptotically almost periodic, and therefore weakly almost periodic. If θ_n is a frequency for the function $t \mapsto < C(t)x , \lambda >$, for some $x \in \mathcal{E}$ and $\lambda \in \mathcal{E}'$, then also $-\theta_n$ is a frequency for the same map and the Fourier coefficients corresponding to the frequencies are equal. From what has been proved in n. 6, if $\{\theta_0 , \theta_1 , ...\}$ is an ordering of the set $\left(\sqrt{-\mathrm{p}\sigma(X) \cup (-\mathrm{r}\sigma(X))}\right) \cap \mathbb{R}_+$, then the Fourier series of the almost periodic function

$t \mapsto < C(t)x, \lambda >$ is, for every $x \in \mathcal{E}$ and every $\lambda \in \mathcal{E}^{\odot}$:

$$\sum_{n=0}^{+\infty} < x, R_{\theta_n}\lambda > \cos(\theta_n t), \tag{7.4}$$

so that the following result holds:

COROLLARY 7.3. *If the set* $\sqrt{-\mathrm{p}\sigma(X) \cup (-\mathrm{r}\sigma(X))}$ *is linearly indipendent over the rational numbers, then the Fourier series* (7.4) *converges uniformly on* $\mathbb{R}$ *to the almost periodic function* $t \mapsto < C(t)x, \lambda >$ *for every* $x \in \mathcal{E}$ *and* $\lambda \in \mathcal{E}^{\odot}$.

A. G. Baskakov [7] showed with a counterexample that the convergence theorem of Harald Bohr cannot be extended, in its original form, to the case of functions with values in infinite-dimensional Hilbert spaces. Nevertheless, E. Vesentini showed that, in the case of a group of linear bounded operators on a Banach space $\mathcal{E}$, the original proof of H. Bohr can be adapted to the vectorial framework, at least when $\mathcal{E}$ is reflexive.

In the case of cosine functions, suppose $\mathcal{E}$ reflexive and C strongly almost periodic. The Fourier series of $t \mapsto C(t)x$ is given by

$$\sum_{n=1}^{+\infty} \cos\theta_n t P_{\theta_n} x \text{ for every } x \in \mathcal{E}.$$

Suppose that the set $\sqrt{-\mathrm{p}\sigma(X)}$ is linearly independent over the rationals. Then the same calculations as in [28] lead to the estimate

$$\sum_{n=0}^{N} ||P_{\theta_n} x|| < 2M||x||,$$

which, combined with the uniqueness theorem for Banach space-valued functions, yields:

THEOREM 7.4. *If* $\mathcal{E}$ *is reflexive,* C *is strongly almost periodic and the set* $+\sqrt{-\mathrm{p}\sigma(X)}$ *is linearly independent over the rationals, then the Fourier series*

$$\sum_{n=1}^{+\infty} \cos\theta_n t P_{\theta_n} x$$

converges uniformly on $\mathbb{R}$ *to* $t \mapsto C(t)x$.

8. UNIFORM ALMOST PERIODICITY

Let the uniformly bounded cosine function C be such that the family of functions $\{t \mapsto < C(t)x, \lambda >: x \in \mathcal{E}, \lambda \in \mathcal{E}^{\odot}\}$ is uniformly asymptotically almost periodic, *i.e.* for every $\varepsilon > 0$ there is some $\Lambda > 0$ such that for every $s \geq 0$ the interval $[s, s + \Lambda]$ contains some τ for which

$$| < C(t+\tau)x, \lambda > - < C(t)x, \lambda > | \leq \varepsilon$$

for every $x \in \mathcal{E}$, $\lambda \in \mathcal{E}^{\odot}$, $t, t + \tau \geq K$, for some $K = K(\varepsilon, x, \lambda)$.

A similar reasoning as in n.2 shows that if C is uniformly bounded and such that the family $\{t \mapsto < C(t)x, \lambda >: \ x \in \mathcal{E}\,, \ \lambda \in \mathcal{E}^{\odot}\}$ is uniformly asymptotically almost periodic, then this family is uniformly almost periodic.

If $\theta \in \mathbb{R}$ is a frequency and τ_{ε} is an ε-period of $t \mapsto < C(t)x, \lambda >$, then, as a consequence of Theorem 6.1, $-\theta^2 \in \mathrm{p}\sigma(X^{\odot})$. Thus there is some $\mu \in \mathcal{D}(X^{\odot}) \setminus \{0\}$ such that

$$|\cos(\tau_{\varepsilon}\theta) - 1| \leq \varepsilon / | < y, \mu > |^{-1} \tag{8.1}$$

is fulfilled for every $x \in \mathcal{E}$.

(8.1) yields:

$$|e^{i\tau_{\varepsilon}\theta} - 1|^2 = |e^{i\tau_{\varepsilon}\theta} + e^{-i\tau_{\varepsilon}\theta} - 2| = 2|\cos(\tau_{\varepsilon}\theta) - 1| \leq 2\varepsilon| < y, \mu > |^{-1}. \tag{8.2}$$

A subset of real numbers Λ is called harmonious if for each $\varepsilon > 0$ the set $\cap_{\lambda \in \Lambda}\{\tau : |e^{i\tau\lambda} - 1| \leq \varepsilon\}$ is relatively dense in $\mathbb{R}$.

From (8.2) and from the fact that the set of all ε-periods is relatively dense, it follows the following

THEOREM 8.1. *If C is a uniformly bounded cosine function and if the family $\{t \mapsto < C(t)x, \lambda >: \ x \in \mathcal{E}\,, \ \lambda \in \mathcal{E}^{\odot}\}$ is uniformly asymptotically almost periodic, the set $\sqrt{-\mathrm{p}\sigma(X^{\odot})}$ (and, therefore, also the set $\sqrt{-\mathrm{p}\sigma(X)}$) is harmonious.*

Suppose now $\mathcal{E}$ be weakly sequentially complete; let C be a weakly asymptotically almost periodic cosine function, and therefore strongly almost periodic, such that $\sqrt{-\mathrm{p}\sigma(X)}$ is a harmonious set. I. Cioranescu has proved that an almost periodic cosine function such that $\sqrt{-\mathrm{p}\sigma(X)}$ is harmonious is uniformly almost periodic. Therefore the following result holds:

THEOREM 8.2. *Let $\mathcal{E}$ be weakly sequentially complete and let the strongly continuous cosine function C be weakly asymptotically almost periodic. If $\sqrt{-\mathrm{p}\sigma(X)}$ is a harmonious set, then the function $t \mapsto C(t)$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{E})$ is almost periodic. Moreover, its spectrum consists of simple poles of the resolvent function of X.*

In the particular context of Hilbert spaces, in analogy to what E. Vesentini proved [28] for strongly continuous semigroups, it is possible to weaken the hypothesis on the harmoniousity of the point spectrum of X, in order to prove the uniform almost periodicity of C.

Let $\mathcal{H}$ be a complex Hilbert space, endowed with an inner product $(\ |\ )$ and let C be a strongly continuous cosine function such that the function $t \mapsto (C(t)x|x)$ is asymptotically almost periodic for all $x \in \mathcal{H}$.

The identity

$$\begin{aligned} 4\big(C(t)x|y\big) = & \big(C(t)(x+y)|(x+y)\big) + \big(C(t)(x-y)|(x-y)\big) + \\ & + i\Big(\big(C(t)(x+iy)|(x+iy)\big) - \big(C(t)(x-iy)|(x-iy)\big)\Big) \end{aligned}$$

shows that C is weakly asymptotically almost periodic and therefore strongly almost periodic, *i.e.* the functions $t \mapsto C(t)x$ are a.p. for all $x \in \mathcal{H}$ and the set of eigenvectors of the infinitesimal generator X is total in $\mathcal{H}$, that is it spans a dense subset in $\mathcal{H}$.

H. Fattorini [11] investigated the structure of almost periodic cosine functions in Hilbert spaces and he proved the representation

$$C(t)x = \sum_{\theta \geq 0} \cos t\sqrt{\theta} P_\theta x$$

for every $x \in \mathcal{H}$. The spectral projections P_θ are mutually orthogonal. If $\mathcal{H}$ is separable, the projectors P_θ don't vanish only for at most a countable set of values of θ and, therefore, by the Baire theorem there is a dense subset $\mathcal{D} \subset \mathcal{H}$ such that, if $x \in \mathcal{D}$, then $P_\theta x \neq 0$ for all θ, *i.e.*:

$$(P_\theta x|x) = (P_\theta^2 x|x) = (P_\theta x|P_\theta x) = ||P_\theta x||^2 > 0.$$

Assume, as in [28], the following condition:

j_1) *there is an open, non empty set* $\mathcal{A} \subset \mathcal{H}$ *such that, if* $x \in \mathcal{A}$, *then the set of the frequencies of the almost periodic function* $t \mapsto (C(t)x|x)$ *is harmonious.*

Since $\mathcal{A} \cap \mathcal{D} \neq \emptyset$, $\sqrt{-\mathrm{p}\sigma(X)}$ is a harmonious set. By Theorem 6.2 for every $x \in \mathcal{H}$ the set of frequencies of the $\mathcal{E}$-valued almost periodic functions $t \mapsto C(t)x$ is contained in $\pm\sqrt{-\mathrm{p}\sigma(X)}$, which is a harmonious set. Bart and Goldberg proved that if Λ is an harmonious set, then every bounded subset of the Banach space of the vector-valued almost periodic functions, whose frequencies are contained in Λ, is uniformly almost periodic. Since C is uniformly bounded, the set

$$\{t \mapsto C(t)x : \ ||x|| \leq 1\}$$

is bounded, hence, by the result above, it is uniformly almost periodic, *i.e.* the function $t \mapsto C(t)$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H})$ is a.p.

Therefore the following result can be stated:

THEOREM 8.3. *Let* $\mathcal{H}$ *be a separable Hilbert space. If* $C : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ *is a strongly continuous, uniformly bounded cosine function such that* $t \mapsto (C(t)x|x)$ *is asymptotically almost periodic for all* $x \in \mathcal{H}$ *and if* j_1) *holds, then*

1) *the function* $t \mapsto C(t)$ *from* $\mathbb{R}$ *to* $\mathcal{L}(\mathcal{H})$ *is an almost periodic function*;
2) *the set* $\sqrt{-\mathrm{p}\sigma(X)}$ *is harmonious*;
3) *every point of* $\sigma(X)$ *is a simple pole of the resolvent operator.*

9. DYNAMICAL SYSTEMS AND COSINE FUNCTIONS

Let K be a compact metric space. Let $\Phi : \mathbb{R} \times K \to K$ be a continuous flow, *i.e.* for every $t \in \mathbb{R}$ the mapping Φ_t defined by $\Phi_t(x) = \Phi(t, x)$ for every $x \in K$ is continuous and satisfies:

1) $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for every $s, t \in \mathbb{R}$;

2) $\Phi_0(x) = x$ for every $x \in K$.

Φ defines a strongly continuous cosine operator function $C : \mathbb{R} \to \mathcal{L}(\mathcal{C}(K))$, acting on the Banach space $\mathcal{C}(K)$ of all complex-valued continuous functions on K, endowed with the sup-norm. C is expressed by:

$$C(t)f = (f \circ \Phi_t + f \circ \Phi_{-t})/2$$

for every $f \in \mathcal{C}(K)$ and $t \in \mathbb{R}$.

Some definitions of topological dynamics will now be collected.

Let $x \in K$. The set

$$O(x) = \{\Phi_t(x) : t \in \mathbb{R}\}$$

is called the *orbit* of x. The set

$$O^+(x) = \{\Phi_t(x) : t \in \mathbb{R}_+\}$$

is called the *forward orbit* of x.

If, given $x \in K$, there exists $\tau > 0$ for which $\Phi_\tau(x) = x$ and $\Phi_t(x) \neq x$ for every $t \in (0\,,\tau)$, x is said to be a *periodic point* of Φ with period τ. If $\Phi_t(x) = x$ for every $t \geq 0$, x is called a *fixed point* of Φ.

If, given $x \in K$, there is some $t_0 > 0$ for which $\Phi_{t_0}(x)$ is a periodic point of Φ, x is called a *preperiodic point* or an *eventually periodic point*.

A point $x \in K$ is said to be *asymptotically periodic* for the restriction of Φ to $\mathbb{R}_+$ if there exists a periodic point $y \in K$ for which

$$\lim_{t \to +\infty} d(\Phi_t(x)\,, \Phi_t(y)) = 0.$$

The point $x \in K$ is called *asymptotically stable* for the restriction of the flow Φ to $\mathbb{R}_+$ if, for every $\varepsilon > 0$ and $K > 0$ there exists some $\tau > K$ for which

$$d(\Phi_\tau(x)\,, x) \leq \varepsilon.$$

The *non-wandering set* $\Omega(\Phi)$ of Φ consists of all points $x \in K$ such that, for every neighbourhood U of x and every $K > 0$, there exists some $\tau \geq K$ for which

$$\Phi_\tau(U) \cap U \neq \emptyset.$$

Let now d be a distance defining the metric topology of K. Consider the restriction of the flow Φ to $\mathbb{R}_+$.

A point $x \in K$ is said to be *asymptotically almost periodic* for that restriction if for every $\varepsilon > 0$ there are $t_0 \geq 0$ and $l > 0$, for which every interval of length l in $\mathbb{R}_+$ contains some τ such that

$$d(\Phi_{t+\tau}(x)\,, \Phi_t(x)) < \varepsilon$$

for all $t \geq t_0$.

The point x is said to be *almost periodic* for Φ if the condition above holds with $t_0 = 0$.

The previous definitions can be partially related [29, 30]:

(1) every periodic point of Φ belongs to $\Omega(\Phi)$;

(2) every preperiodic point is asymptotically periodic;

(3) all fixed points of Φ and all almost periodic points are asymptotically stable;

(4) every asymptotically stable point is non-wandering and the converse is true if Φ is C-*contractive*, *i.e.* if there exists some $C > 0$ such that

$$d(\Phi_t(u)\,,\,\Phi_t(v)) \leq Cd(u\,,\,v)$$

for all $t \in \mathbb{R}$ and $u\,,\,v \in K$.

Let $x \in K$ be such that the maps

$$t \mapsto < C(t)f,\, \delta_x >= (f(\Phi_t(x)) + f(\Phi_{-t}(x)))/2$$

are asymptotically almost periodic on $\mathbb{R}_+$ for every $f \in \mathcal{C}(K)$. In virtue of Proposition 3.2, the applications above are almost periodic on $\mathbb{R}$ for all $f \in \mathcal{C}(K)$. For every $\theta \in \mathbb{R}$ the limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \cos(\theta s) \frac{f(\Phi_s(x)) + f(\Phi_{-s}(x))}{2}\, ds$$

exists for all $f \in \mathcal{C}(K)$, so that $\delta_x \in \mathcal{H}'_\theta$.

As observed in n. 3, the weak-star sequential completeness of $\mathcal{C}(K)'$ entails the existence of a linear projector $R_\theta : \mathcal{H}'_\theta \to \mathcal{C}(K)'$; in particular, let $R_\theta \delta_x \in \mathcal{C}(K)'$ be defined by

$$< f,\, R_\theta \delta_x >= \begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{t} \displaystyle\int_0^t < C(s)f,\, \delta_x > ds & \text{if } \theta = 0 \\ \lim\limits_{t\to+\infty} \dfrac{2}{t} \displaystyle\int_0^t \cos(\theta s) < C(s)f,\, \delta_x > ds & \text{if } \theta \neq 0 \end{cases}$$

for every $f \in \mathcal{C}(K)$.

Observe that $R_\theta \delta_x$ can be identified with a Borel measure on K, *i.e.*

$$\int f\, dR_\theta \delta_x = \begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{t} \displaystyle\int_0^t < C(s)f,\, \delta_x > ds & \text{if } \theta = 0 \\ \lim\limits_{t\to+\infty} \dfrac{2}{t} \displaystyle\int_0^t \cos(\theta s) < C(s)f,\, \delta_x > ds & \text{if } \theta \neq 0. \end{cases}$$

The following invariance result can be stated also by invoking properties of R_θ, illustrated in Theorem 4.3. Here a more direct proof is presented.

LEMMA 9.1. *Fix $\theta \in \mathbb{R}$. For every $t \in \mathbb{R}$, $\lambda \in \mathcal{H}'_\theta$ and $f \in \mathcal{C}(K)$ it results*

$$< \frac{f \circ \Phi_t + f \circ \Phi_{-t}}{2}\,,\, R_\theta \lambda >= \cos(\theta t) < f,\, R_\theta \lambda > .$$

PROOF. Let f be in $\mathcal{C}(K)$, $t \in \mathbb{R}$ and $\lambda \in \mathcal{H}'_\theta$. If $\theta \neq 0$, it holds:

$$< \frac{f \circ \Phi_t + f \circ \Phi_{-t}}{2}, R_\theta \lambda >=$$
$$= \lim_{a \to +\infty} \frac{1}{a} \int_0^a \cos(\theta s) < \left[\frac{f \circ \Phi_t + f \circ \Phi_{-t}}{2} \circ \Phi_s + \frac{f \circ \Phi_t + f \circ \Phi_{-t}}{2} \circ \Phi_{-s} \right], \lambda > ds =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \int_0^a \cos(\theta s) < f \circ \Phi_{t+s} + f \circ \Phi_{s-t} + f \circ \Phi_{t-s} + f \circ \Phi_{-t-s}, \lambda > ds =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \left[\int_t^{a+t} \cos[\theta(s-t)] < f \circ \Phi_s, \lambda > ds + \int_{-t}^{a-t} \cos[\theta(s+t)] < f \circ \Phi_s, \lambda > ds + \right.$$
$$\left. - \int_t^{t-a} \cos[\theta(t-s)] < f \circ \Phi_s, \lambda > ds - \int_{-t}^{-t-a} \cos[\theta(s+t)] < f \circ \Phi_s, \lambda > ds \right] =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \left[\int_{t-a}^{t+a} \cos[\theta(s-t)] < f \circ \Phi_s, \lambda > ds + \right.$$
$$\left. + \int_{-t-a}^{-t+a} \cos[\theta(s+t)] < f \circ \Phi_s, \lambda > ds \right] =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \left[\int_{t-a}^{t+a} \cos[\theta(s-t)] < f \circ \Phi_s + f \circ \Phi_{-s}, \lambda > ds \right] =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \left[\cos(\theta t) \int_{t-a}^{t+a} \cos(\theta s) < f \circ \Phi_s + f \circ \Phi_{-s}, \lambda > ds + \right.$$
$$\left. + \sin(\theta t) \int_{t-a}^{t+a} \sin(\theta s) < f \circ \Phi_s + f \circ \Phi_{-s}, \lambda > ds \right] =$$
$$= \lim_{a \to +\infty} \frac{1}{2a} \left[\cos(\theta t) \int_{-a}^{a} \cos(\theta s) < f \circ \Phi_s + f \circ \Phi_{-s}, \lambda > ds + \right.$$
$$\left. + \sin(\theta t) \int_{-a}^{a} \sin(\theta s) < f \circ \Phi_s + f \circ \Phi_{-s}, \lambda > ds \right] =$$
$$= \lim_{a \to +\infty} \frac{1}{a} \cos(\theta t) \int_{-a}^{a} \cos(\theta s) < C(s) f, \lambda > ds = \cos(\theta t) < f, R_\theta \lambda >,$$

where the last two equalities hold, since $s \mapsto \sin(\theta s)$ and $s \mapsto \cos(\theta s)$ are, respectively, an odd and an even function.

A similar, easier computation for the case $\theta = 0$ concludes the proof. □

In particular, for every $f \in \mathcal{C}(K)$, $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$ it holds

$$< \frac{f \circ \Phi_t + f \circ \Phi_{-t}}{2}, R_\theta \delta_x >= \cos(\theta t) < f, R_\theta \delta_x > .$$

Observe that $R_\theta \delta_x$ is different from zero if, and only if, there exists some $f \in \mathcal{C}(K) \setminus \{0\}$ such that the application $t \mapsto < f, R_\theta \delta_x >$ is asymptotically almost periodic, with θ as a frequency. Theorem 6.1 entails that $-\theta^2 \in \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)$.

In the dynamical systems' language, if the flow Φ satisfies the invariance property

$$< f \circ \Phi_{t_0}, \mu >=< f, \mu >$$

for every $f \in \mathcal{C}(K)$, for some $t_0 \in \mathbb{R} \setminus \{0\}$ and for only one Borel probability measure, Φ_{t_0} is said to be *uniquely ergodic*. Lemma 9.1 suggests a different invariance property for Φ, from which some informations about the largest number of periodic orbits of Φ will be deduced.

Theorem 9.2. *Let Φ be a continuous flow on the compact Hausdorff space K. If, for some $t_0 \in \mathbb{R} \setminus \{0\}$, there exists only one Borel probability measure μ on K satisfying the following invariance property:*

$$< \frac{f \circ \Phi_{t_0} + f \circ \Phi_{-t_0}}{2}, \mu >=< f, \mu >$$

for every $f \in \mathcal{C}(K)$, then Φ admits at most two periodic orbits.

Proof. First of all, observe that, if the maps $t \mapsto < C(t)f, \delta_x >$ are asymptotically almost periodic on $\mathbb{R}_+$ for every $f \in \mathcal{C}(K)$, for some $x \in K$, then $R_0\delta_x$ is a Borel probability measure, such that

$$< \frac{f \circ \Phi_s + f \circ \Phi_{-s}}{2}, R_0\delta_x >=< f, R_0\delta_x > ,$$

for every $s \in \mathbb{R}$, $f \in \mathcal{C}(K)$. Moreover, the support of $R_0\delta_x$ is the closure of the orbit of x, $\overline{O(x)}$.

Suppose that there are x_1 and x_2 in K, $x_1 \neq x_2$, such that $\Phi_{\tau_i}(x_i) = x_i$, for some $\tau_i > 0$, $i = 1, 2$. Then the maps $t \to < C(t)f, \delta_{x_i} >$, $i = 1, 2$, are periodic for every $f \in \mathcal{C}(K)$ and the support of $R_0\delta_{x_i}$ is $O(x_i)$, $i = 1, 2$. By hypothesis it results $R_0\delta_{x_1} = R_0\delta_{x_2} = \mu$, and therefore $O(x_1) = O(x_2)$. □

It will now be shown which constraints on the point spectrum of the infinitesimal generator X of C are imposed by topological transitivity and by density of periodic points of the flow Φ.

Recall that a flow Φ is said to be *topologically transitive* if there exists some $x_0 \in K$, for which the orbit $\overline{O(x_0)}$ is dense in K; Φ is said to be *one-sided topologically transitive* if $\overline{0^+(x_0)} = K$ for some $x_0 \in K$.

Proposition 9.3. *Let $\Phi : \mathbb{R} \times K \to K$ be a continuous flow on a compact Hausdorff space K. If the restriction of Φ to $\mathbb{R}_+$ is topologically one-sided transitive and if the set of its periodic points is dense in K, then, if there exist eigenvalues ζ of X for which $dim_{\mathbb{C}}\ker(X - \zeta I) = 1$, then they are rational multiples of some point in $\mathbb{R}_-$.*

Proof. Suppose that the set of eigenvalues ζ of X for which $dim_{\mathbb{C}}\ker(X - \zeta I) = 1$ is not empty. If $\zeta = -\theta^2$ for some $\theta \in \mathbb{R}$, if $f \in \ker(X + \theta^2 I)$ and x_0 is a periodic point of X of period τ, then it results

$$f(x_0) = \frac{f(\Phi_\tau(x_0)) + f(\Phi_{-\tau}(x_0))}{2} = \cos(\tau\theta)f(x_0) ,$$

and therefore either $f(x_0) = 0$ or $\cos(\tau\theta) = 1$, *i.e.* $\tau\theta = 2n_0\pi$ for some $n_0 \in \mathbb{Z}$.

If $-\lambda^2 \in \mathbb{R}_-$ is another eigenvalue of X, such that $dim_{\mathbb{C}}\ker(X - \lambda I) = 1$, with eigenfunction g, then it results:

$$g(x_0) = \frac{g(\Phi_\tau(x_0)) + g(\Phi_{-\tau}(x_0))}{2} = \cos(\tau\lambda)g(x_0)\,,$$

so that either $g(x_0) = 0$ or $\tau\lambda = 2m_0\pi$ for some $m_0 \in \mathbb{Z}$.

From the hypothesis that Φ is topologically transitive, it follows that the sets $\{x \in K : f(x) \neq 0\}$ and $\{x \in K : g(x) \neq 0\}$ are open, dense subsets of K, so that $f(x)g(x) \neq 0$ on a dense set of K, and therefore $\theta/\lambda = m_0/n_0$, implying the thesis. $\square$

Observe that, in the particular case in which $\sqrt{\mathrm{p}\sigma(X)} = \mathrm{p}\sigma\sqrt{X}$, the assumption that $dim_{\mathbb{C}}\ker(X - \zeta I) = 1$ is verified by every eigenvalue ζ of X. Indeed, the operator $\sqrt{X}$ is the infinitesimal generator of the strongly continuous group U on $\mathcal{C}(K)$, expressed by

$$U(t)f = f \circ \Phi_t \text{ for every } f \in \mathcal{C}(K)\,,$$

and E. Vesentini has proved [29] that, if k is an eigenvalue of $\sqrt{X}$, then $dim_{\mathbb{C}}\ker(\sqrt{X} - -kI) = 1$.

Theorem 9.4. *Let* $\Phi : \mathbb{R} \times K \to K$ *be a continuous flow. Let the following conditions hold*:

1) *there is some* $x \in K$ *for which the maps* $t \mapsto < C(t)f, \delta_x >$ *are asymptotically almost periodic (and therefore almost periodic) for every* $f \in \mathcal{C}(K)$;

2) *there exists* $t_0 \neq 0$ *such that every eigenfunction of* $C(t_0)$ *generates a dense subspace of* $C(t_0)$.

Then, if θ *is a frequency of the application* $t \mapsto < C(t)f, \delta_x >$, *for some* $f \in \mathcal{C}(K)$, *there exists* $n_0 \in \mathbb{Z}$ *such that the point* $-(\theta + 2\pi n_0/s)^2$ *belongs to* $\mathrm{p}\sigma(X)$.

Proof. Let $\theta \in \mathbb{R}$ be frequency of the almost periodic application

$$t \mapsto (f(\Phi_t(x)) + f(\Phi_{-t}(x)))/2$$

for some $f \in \mathcal{C}(K)$. Thus Theorem 6.1 implies that $-\theta^2 \in \mathrm{p}\sigma(X) \cup \mathrm{r}\sigma(X)$.

Suppose that that there is s, which can be assumed without loss of generality, greater than 0, such that

$$\cos(\theta s) \notin \mathrm{p}\sigma(C(s)).$$

Let now $\zeta \in \mathrm{p}\sigma(X)$ and let g be a corresponding eigenvector, different from zero. The invariance property of Lemma 9.1 and the spectral mapping theorem for cosine operator functions imply:

$$\begin{aligned}\cos(t\theta) < g, R_\theta\delta_x > &= < \frac{g(\Phi_t(x)) + g(\Phi_{-t}(x))}{2}, R_\theta\delta_x > = \\ &= \cosh(t\sqrt{\zeta}) < g, R_\theta\delta_x >\,,\end{aligned}$$

whence $(\cos t\theta - \cosh t\sqrt{\zeta}) < g, R_\theta\delta_x > = 0$ for every $t \in \mathbb{R}$. Hence either $\zeta = -\theta^2$ or $< g, R_\theta\delta_x > = 0$. Observe that $-\theta^2 \notin \mathrm{p}\sigma(X)$, since, by hypothesis, there is some

$s \in \mathbb{R} \setminus \{0\}$ for which $\cos(\theta s) \notin \mathrm{p}\sigma(C(s))$. Thus $< g, R_\theta \delta_x >= 0$, this equality holding for every eigenfunction of X. Now, condition 2) implies that $R_\theta \delta_x$ vanishes on a dense subspace of $\mathcal{C}(K)$, and therefore $R_\theta \delta_x = 0$. That is absurd and therefore $\cos(\theta s) \in$ $\in \mathrm{p}\sigma(C(s))$ for every $s \in \mathbb{R}$. Now fix $s \in \mathbb{R}$. The spectral mapping theorem proved by B. Nagy shows that there exists some $n_0 \in \mathbb{Z}$ such that the point $-(\theta + 2\pi n_0/s)^2$ belongs to $\mathrm{p}\sigma(X)$. □

Theorem 2.4 implies the following result.

COROLLARY 9.5. *Under the same hypotheses of Theorem 9.4, if* θ *is a frequency of the application* $t \mapsto < C(t)f, \delta_x >$, *for some* $f \in \mathcal{C}(K)$, *there exists* $n_0 \in \mathbb{Z}$ *such that the point* $-(\theta + 2\pi n_0/s)^2$ *belongs to* $\mathrm{p}\sigma(X')$.

It is well known [29] that, if Φ is a C – *contractive* flow and if the point x is asymptotically almost periodic for the restriction of Φ to $\mathbb{R}_+$, then x is an almost periodic point for the flow Φ. Thus Proposition 3.2 entails that the maps $t \mapsto$ $\mapsto < C(t)f, \delta_x >$ are almost periodic for every $f \in \mathcal{C}(K)$. A direct proof of this fact is given in the following

LEMMA 9.6. *Let* $x \in K$ *be an asymptotically almost periodic point for the restriction to* $\mathbb{R}_+$ *of a continuous, C-contractive flow* Φ. *Then the maps*

$$t \mapsto \frac{f \circ \Phi_t(x) + f \circ \Phi_{-t}(x)}{2}$$

from $\mathbb{R}$ *to* $\mathbb{C}$ *are almost periodic for every* $f \in \mathcal{C}(K)$.

PROOF. Consider

$$\begin{aligned} |f(\Phi_{t+\tau}(x)) + & f(\Phi_{-t+\tau}(x)) - f(\Phi_t(x)) - f(\Phi_{-t}(x))| \leq \\ & \leq |f(\Phi_{t+\tau}(x)) - f(\Phi_t(x))| + |f(\Phi_{-t+\tau}(x)) - f(\Phi_{-t}(x))|. \end{aligned}$$

Since K is compact, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x_1, x_2 \in K$ with $d(x_1, x_2) < \delta$, then $|f(x_1) - f(x_2)| < \varepsilon$. The first term in the inequality above is obviously less then an arbitrarily small $\delta > 0$ if $t \geq 0$, since x is an almost periodic point for Φ; since, moreover, $d(\Phi_{-t+\tau}(x), \Phi_{-t}(x)) \leq Cd(\Phi_\tau(x), \Phi(x))$, also the second one is less then δ, so that the thesis follows. □

LEMMA 9.7. *Let* $x \in K$ *be such that the maps*

$$t \mapsto \frac{f \circ \Phi_t(x) + f \circ \Phi_{-t}(x)}{2}$$

from $\mathbb{R}$ *to* $\mathbb{C}$ *are almost periodic for every* $f \in \mathcal{C}(K)$. *Then* x *is an asymptotically stable point* (*and therefore* $x \in \Omega(\Phi)$).

PROOF. If x is not asymptotically stable, then there exists $\varepsilon_0 > 0$ and $K_0 > 0$ such that for every $t \geq K_0$ it holds

$$d(\Phi_t(x), x) > \varepsilon_0.$$

Let B the ball centered in x, with radius $\min(\varepsilon_0\,,\,\varepsilon_0/C)$. Take a function $f \in \mathcal{C}(K)$ such that Supp $f \subset B$ and there exists some $y \in B$ for which $f(y) \neq 0$.

Observe that

$$\begin{aligned} d(\Phi_{-t}(x)\,,\,x) \geq \frac{1}{C} d(\Phi_t \circ \Phi_{-t}(x)\,,\,\Phi_t(x)) = \\ = \frac{1}{C} d(\Phi_t(x)\,,\,x) > \frac{\varepsilon_0}{C} \end{aligned}$$

for every $t \geq K$. Thus

$$\lim_{a \to +\infty} \frac{1}{a} \int_0^a \cos(\theta t) \frac{f \circ \Phi_t(x) + f \circ \Phi_{-t}(x)}{2} dt = 0$$

for every $\theta \in \mathbb{R}$, entailing that the frequencies of the almost periodic function $t \mapsto$ $\mapsto (f \circ \Phi_t(x) + f \circ \Phi_{-t}(x))/2$ vanish. Then f is constant, yielding a contradiction. □

COROLLARY 9.8. *If the cosine function C is weakly almost periodic, then every point of K is asymptotically stable.*

REFERENCES

[1] L. AMERIO - G. PROUSE, *Almost periodic functions and functional equations*. Van Nostrand, New York 1971.

[2] W. ARENDT - C. J. K. BATTY, *Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half-line*. To appear.

[3] W. ARENDT - C. J. K. BATTY, *Almost periodic solutions of first and second order Cauchy problem*. To appear.

[4] W. ARENDT - A. GRABOSCH - G. GREINER - U. GROH - H. P. LOTZ - U. MOUSTAKAS - R. NAGEL (ed.) - F. NEUBRANDER - U. SCLOTTERBECK, *One-Parameter Semigroups of Positive Operators*. Lecture Notes in Mathematics, n.1184, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1986.

[5] J. BANKS - J. BROOKS - G. CAIRNS - G. DAVIS - P. STACEY, *On Devaney's definition of chaos*. Amer. Math. Monthly, 99, 1992, 332-334.

[6] H. BART - S. GOLDBERG, *Characterization of almost periodic strongly continuous groups and semigroups*. Math. Ann., 236, 1978, 105-116.

[7] A. G. BASKAKOV, *Harmonic analysis of cosine and exponential operator-valued functions*. Math. USSR Sb., 52,1985, 63-90.

[8] H. BOHR, *Almost periodic functions*. Chelsea, New York 1947.

[9] V. CASARINO, *Spectral properties of weakly asymptotically almost periodic semigroups in the sense of Stepanov*. Rend. Mat. Acc. Lincei, s. 9, vol. 8, 1997, 167-181.

[10] I. CIORANESCU, *Characterization of almost periodic strongly continuous cosine operator functions*. Journal of Math. Anal. and Appl., 116, 1986, 222-229.

[11] H. FATTORINI, *Uniformly bounded cosine functions in Hilbert space*. Indiana. Univ. Math. J., 20, 1970, 411-425.

[12] H. FATTORINI, *Second order linear differential equations*. North-Holland, Amsterdam 1985.

[13] M. FRÉCHET, *Les fonctions asymptotiquement presque-périodiques continues*. C. R. Acad. Sci. Paris, 213, 1941, 520-522.

[14] E. GIUSTI, *Funzioni coseno periodiche*. Boll. Unione Mat. Ital., 22, 1967, 478-485.

[15] J. A. GOLDSTEIN, *Semigroups of operators and applications*. Oxford University Press, Oxford 1985.

[16] J. A. GOLDSTEIN - C. RADIN - R. E. SHOWALTER, *Convergence rates of ergodic limits for semigroups and cosine functions*. Semigroup Forum, 16, 1978, 89-95.

[17] H. R. HENRIQUEZ, *On Stepanov-almost periodic semigroups and cosine functions of operators*. Journal of Math. Anal. and Appl., 146, 1990, 420-433.

[18] R. R. Kallman - G. Rota, *On the inequality* $\|f'\|^2 \leq 4\|f''\| \cdot \|f\|$. In: O. Shisha (ed.), *Inequalities II*. Academic Press, New York 1970, 187-192.
[19] M. Lin, *On the uniform ergodic theorem*. Proc. of the Amer. Math. Soc., 43, 1974, 337-340.
[20] M. Lin, *On the uniform ergodic theorem II*. Proc. of the Amer. Math. Soc., 46, 1974, 217-225.
[21] Y. Meyer, *Algebraic numbers and harmonic analysis*. North-Holland, Amsterdam 1972.
[22] B. Nagy, *On cosine operator functions in Banach spaces*. Acta Sci. Math. (Szeged), 36, 1974, 281-290.
[23] J. M. A. M. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*. Birkhäuser Verlag, Basel 1996.
[24] S. I. Piskarev, *Periodic and almost periodic cosine operator functions*. Math. USSR Sb., 46, 1983, 391-402.
[25] W. M. Ruess - W. H. Summers, *Compactness in spaces of vector-valued continuous functions and asymptotic almost periodicity*. Math. Nachr., 135, 1988, 7-33.
[26] S. Y. Shaw, *On W^*-continuous cosine operator functions*. Journal of Funct. Anal., 66, 1986, 73-95.
[27] E. Vesentini, *Introduction to continuous semigroups*. Scuola Normale Superiore, Pisa 1996.
[28] E. Vesentini, *Spectral properties of weakly asymptotically almost periodic semigroups*. Advances in Math., 128, 1997, 217-241.
[29] E. Vesentini, *Periodicity and almost periodicity in Markov lattice semigroups*. Ann. Scuola Normale, 26, Pisa 1998, 829-839.
[30] P. Walters, *An Introduction to Ergodic Theory*. Springer-Verlag, New York 1982.

Pervenuta il 5 marzo 1998,
in forma definitiva il 24 marzo 1998.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24 - 10129 Torino
casarino@calvino.polito.it