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A nilpotency condition for finitely generated soluble groups

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Teoria dei gruppi. — *A nilpotency condition for finitely generated soluble groups.* Nota di COSTANTINO DELIZIA, presentata (*) dal Socio G. Zappa.

ABSTRACT. — We prove that if $k > 1$ is an integer and G is a finitely generated soluble group such that every infinite set of elements of G contains a pair which generates a nilpotent subgroup of class at most k, then G is an extension of a finite group by a torsion-free k-Engel group. As a corollary, there exists an integer n, depending only on k and the derived length of G, such that $G/Z_n(G)$ is finite. For $k < 4$, such n depends only on k.

KEY WORDS: Commutators; Nilpotency condition; Infinite set.

RIASSUNTO. — *Una condizione di nilpotenza per gruppi risolubili finitamente generati.* Sia $k > 1$ un intero; si considerano gruppi G risolubili finitamente generati tali che ogni insieme infinito di elementi di G contiene due elementi che generano un sottogruppo nilpotente di classe al più k, e si prova che un tale gruppo deve essere estensione di un gruppo finito tramite un gruppo k-Engel senza torsione. Da ciò segue che esiste un intero n, funzione soltanto di k e della lunghezza derivata di G, tale che $G/Z_n(G)$ è finito. Si dimostra anche che per $k < 4$ tale n dipende soltanto da k.

1. INTRODUCTION

We say that a group G satisfies the condition $(\mathcal{N}, \infty)$ if every infinite set of elements of G contains a pair which generates a nilpotent subgroup. If $k > 1$ is an integer, we say that G satisfies the condition $(\mathcal{N}_k, \infty)$ if every infinite set of elements of G contains a pair which generates a nilpotent subgroup of class at most k.

In [5] J. C. Lennox and J. Wiegold proved that a finitely generated soluble group G satisfies the condition $(\mathcal{N}, \infty)$ if and only if it is finite-by-nilpotent, that is, by a well-known theorem of P. Hall (see [2]), if and only if $G/Z_n(G)$ is finite for a suitable n.

Moreover it is very easy to show that if $G/Z_k(G)$ is finite for some group G then G satisfies the condition $(\mathcal{N}_k, \infty)$. For, every infinite set of elements of G contains two elements which are equivalent modulo $Z_k(G)$, so they generate a nilpotent subgroup of class at most k.

We are interested in the following question:
given an integer $k > 1$, is it possible to find an integer n, depending only on k (or, at least, on k and some invariant of the group G), such that $G/Z_n(G)$ is finite, for all finitely generated soluble groups G satisfying the condition $(\mathcal{N}_k, \infty)$?

For $k = 2$, it was proved in [1] that a finitely generated soluble group G satisfies the condition $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite.

Unfortunately, it is not possible to extend this result for $k > 2$. For, there exists a finitely generated torsion-free 3-Engel group G which is nilpotent of class exactly 4 (see, for instance, [4]). By a well-known result of H. Heineken (see [3]), all 2-generated

(*) Nella seduta del 18 giugno 1998.

subgroups of a torsion-free 3-Engel group are nilpotent of class at most 3. It follows that G satisfies the condition $(\mathcal{N}_3, \infty)$, and $G/Z_3(G)$ is infinite, otherwise $\gamma_4(G)$ is finite by a theorem of R. Baer (see [7, 14.5.1]), so $\gamma_4(G) = 1$, and the nilpotency class of G is less than 4.

The following result enable us to transfer well-known properties of soluble k-Engel groups to groups satisfying the condition $(\mathcal{N}_k, \infty)$.

THEOREM. *Let G be a torsion-free nilpotent group satisfying the condition $(\mathcal{N}_k, \infty)$. Then every 2-generated subgroup of G has nilpotency class at most k. In particular, G is k-Engel.*

PROOF. If $k = 2$, the result is true by Lemma 2.1 of [1]. So assume $k > 2$. Let a and b in G, and consider the infinite set $\{ab, a^2b, \dots, a^nb, \dots\}$. By the condition $(\mathcal{N}_k, \infty)$, there exist integers $r < s$ such that the subgroup $\langle a^r b, a^s b\rangle$ has nilpotency class at most k. Let $t = s - r$. Then the subgroup $\langle a^t, a^s b\rangle$ has nilpotency class at most k. Let c be the nilpotency class of G, and assume $c > k$. Then $c - 1 \geq k$ and from $[a^s b, {}_{c-1}a^t] = 1$ it follows that $[b, {}_{c-1}a^t] = 1$, so $[b, {}_{c-1}a]^t = 1$ as G has class c (note that it suffices that $\langle a, b\rangle$ has class c). It follows that $[b, {}_{c-1}a] = 1$ because G is torsion-free. Now consider the set of c-tuples

$$S = \{(b, a, x_1, \dots, x_{c-2}) : x_i \in \{a, b\} \;\; \forall\, i \in \{1, \dots, c-2\}\}.$$

For all $w = (b, a, x_1, \dots, x_{c-2}) \in S$, let $\sigma(w)$ be the number of occurrences of b in w. Put $[w] = [b, a, x_1, \dots, x_{c-2}]$. By induction on $\sigma(w)$ we can prove that $[w] = 1$ for all w in S. If $\sigma(w) = 1$ then $[w] = [b, {}_{c-1}a] = 1$. Let $\sigma(w) > 1$, and assume $[v] = 1$ for all v in S with $\sigma(v) < \sigma(w)$. Consider the c-tuple w_1 that we obtain by replacing in w all occurrences of b with $a^s b$ and all occurrences of a with a^t. Since $\langle a^t, a^s b\rangle$ has nilpotency class at most $k \leq c - 1$ we get $[w_1] = 1$. On the other hand, as the nilpotency class of G is at most c, standard commutators computation gives $[w_1] = [w]^n[u]$, where $n = t^{c-\sigma(w)}$ and $[u]$ is a product of certain commutators $[v_j]$ with $\sigma(v_j) < \sigma(w)$ for all j. Then from $[u] = 1$ it follows $[w] = 1$ as G is torsion-free. Therefore $\langle a, b\rangle$ has nilpotency class at most $c - 1$. Iterating this argument $c - k$ times, we get that $\langle a, b\rangle$ has nilpotency class at most k, as required. □

COROLLARY 1. *Let G be a finitely generated soluble group satisfying the condition $(\mathcal{N}_k, \infty)$. Then G is an extension of a finite group by a torsion-free k-Engel group.*

PROOF. The result of Lennox and Wiegold quoted in the Introduction shows that G is finite-by-nilpotent, so there exists a finite subgroup $H \triangleleft G$ such that G/H is nilpotent. Let T/H be the torsion subgroup of G/H. Then T/H is finite, and so is T, and G/T is torsion-free. Therefore the result follows from our Theorem. □

The following result gives an affirmative answer to the question posed in the Introduction.

COROLLARY 2. *Let G be a finitely generated soluble group satisfying the condition $(\mathcal{N}_k, \infty)$. If d is the derived length of G, then $G/Z_{kd-1}(G)$ is finite.*

PROOF. By Corollary 1, the group G is an extension of a finite group by a torsion-free k-Engel group. By a result of K. W. Gruenberg (see [6, Theorem 7.36]) a torsion-free

k-Engel group which is soluble of derived length d is nilpotent of class at most k^{d-1}. So G is an extension of a finite group by a nilpotent group of class at most k^{d-1}. Therefore $\gamma_{k^{d-1}+1}(G)$ is finite, and the result follows (see [6, Theorem 4.24]). □

An obvious consequence of Corollary 2 in the characterization of finitely generated metabelian groups satisfying the condition $(\mathcal{N}_k, \infty)$:

COROLLARY 3. *A finitely generated metabelian group G satisfies the condition $(\mathcal{N}_k, \infty)$ if and only if $G/Z_k(G)$ is finite.*

Finally, for $k = 3$ we obtain a bound for n which does not depend on the derived length.

COROLLARY 4. *Let G be a finitely generated soluble group satisfying the condition $(\mathcal{N}_3, \infty)$. Then $G/Z_4(G)$ is finite.*

PROOF. By Corollary 1, the group G is an extension of a finite group by a torsion-free 3-Engel group. By a theorem of H. Heineken (see [3]) torsion-free 3-Engel groups are nilpotent of class at most 4. Therefore G is an extension of a finite group by a nilpotent group of class at most 4, so $\gamma_5(G)$ is finite. It follows that $G/Z_4(G)$ is finite, as required. □

Notice that the bound 4 for n in the previous corollary is the best possible. Indeed, as we already showed, there exists a finitely generated soluble group G satisfying the condition $(\mathcal{N}_3, \infty)$ and such that $G/Z_3(G)$ infinite.

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