Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Philippe Clément, Sylvie Guerre-Delabrière

On the regularity of abstract Cauchy problems and boundary value problems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.4, p. 245–266.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_4_245_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:245-266 (1998)

Analisi matematica. — *On the regularity of abstract Cauchy problems and boundary value problems.* Nota di PHILIPPE CLÉMENT e SYLVIE GUERRE-DELABRIÈRE, presentata (*) dal Corrisp. G. Da Prato.

ABSTRACT. — Maximal regularity (in L_p-sense) for abstract Cauchy problems of order one and boundary value problems of order two is studied. In general, regularity of the first problems implies regularity of the second ones; the converse is shown to hold if the underlying Banach space has the UMD property. A stronger notion of regularity, introduced by Sobolevskii, plays an important role in the proofs.

KEY WORDS: Abstract differential equations; UMD-spaces; Maximal regularity.

RIASSUNTO. — *Sulla regolarità di problemi di Cauchy astratti e di problemi al contorno.* Viene studiata la regolarità massimale (in L_p) di problemi di Cauchy astratti di ordine uno e di problemi al contorno di ordine due. In generale, la regolarità del primo tipo di problemi implica la regolarità del secondo; l'inverso viene dimostrato quando lo spazio di Banach in oggetto ha la proprietà UMD. Una nozione di regolarità forte, introdotta da Sobolevskii, ha un ruolo importante nelle dimostrazioni.

0. INTRODUCTION

The aim of this paper is to compare several notions of regularity associated with the abstract inhomogeneous Cauchy-problem of order one:

$$(P_{1,T}) \quad \begin{cases} u'(t) + Au(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u(0) = 0, \end{cases}$$

and the abstract boundary value problems of order two:

$$(P_{2,DD}) \begin{cases} -u''(t) + A^2 u(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u(0) = u(T) = 0, \end{cases}$$

and

$$(P_{2,NN}) \begin{cases} -u''(t) + A^2 u(t) = f(t), & t \in [0, T], \quad 0 < T < \infty, \\ u'(0) = u'(T) = 0. \end{cases}$$

The function $u : [0, T] \to X$ takes its values in a complex Banach space $(X, |\cdot|)$, u' (resp. u'') denotes the first derivative (resp. the second derivative) of u with respect to t, the operator A (in general unbounded and not necessarily densely defined) is supposed to be the negative generator of an exponentially stable analytic semigroup on X, denoted by $(e^{-tA})_{t>0}$, see [11]. Notice that an analytic semigroup is exponentially

(*) Nella seduta dell'8 maggio 1998.

stable if and only if 0 is in the resolvent set of the generator A, see Remark 7 below. The function f is a given function in $L^p([0\,,\,T];X)$ for some $p \in (1\,,\infty)$.

The problem $(P_{1\,,T})$ (resp. $(P_{2\,,D,D})\,,(P_{2\,,N,N})$) is called *p-regular* if for every $f \in \in L^p([0\,,\,T];X)$, it has one and only one solution $u \in W^{1,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A))$ (resp. $u \in W^{2\,,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A^2)))$ where $D(A)$ (resp. $D(A^2)$) denotes the domain of A (resp. A^2) equipped with the graph norm, see Definition 1.1 below.

It will appear (see Lemma 1.0 below) that if $(P_{1\,,T})$ (resp.$(P_{2\,,DD})$ or $(P_{2\,,NN})$) is p-regular, then (0.1) holds (resp. (0.2)):

$$(0.1)\qquad \begin{cases} \text{There exists } M \geq 1 \text{ such that} \\ \|u'\|_{L^p([0\,,\,T];X)} + \|Au\|_{L^p([0\,,\,T];X)} \leq M\|u' + Au\|_{L^p([0\,,\,T];X)}\,, \\ \text{for every } u \in W^{1\,,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A)) \\ \text{satisfying } u(0) = 0\,, \end{cases}$$

respectively:

$$(0.2)\qquad \begin{cases} \text{There exists } M \geq 1 \text{ such that} \\ \|u''\|_{L^p([0\,,\,T];X)} + \|A^2u\|_{L^p([0\,,\,T];X)} \leq M\| - u'' + A^2u\|_{L^p([0\,,\,T];X)}\,, \\ \text{for every } u \in W^{2\,,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A^2)) \\ \text{satisfying } u(0) = u(T) = 0 \text{ (resp. } u'(0) = u'(T) = 0). \end{cases}$$

Recall that if $(P_{1\,,T})$ is p-regular for some $p \in (1\,,\infty)$ and some $T > 0$, it is also p-regular for all $p \in (1\,,\infty)$ and all $T > 0$, [26, 8, 5], but of course the constant M may depend on p and T.

We shall call problem $(P_{1\,,T})$ (resp. $(P_{2\,,DD})$, $(P_{2\,,NN})$) *λ-p-regular* if problem

$$(P_{1\,,T})_\lambda \begin{cases} u'(t) + \lambda Au(t) = f(t)\,, \quad t \in [0\,,\,T]\,, \quad 0 < T < \infty\,, \\ u(0) = 0\,, \end{cases}$$

(resp. $(P_{2\,,DD})_\lambda$, $(P_{2\,,NN})_\lambda$) is p-regular for all $\lambda > 0$ and inequality (0.1) (resp. (0.2)) holds where A (resp. A^2) is replaced by λA (resp. $\lambda^2 A^2$) with $\lambda > 0$ and M *is independent of* λ, see Definition 1.2 below. We will denote by $(0.1)_\lambda$ and $(0.2)_\lambda$ the corresponding inequalities.

In §1, we first give precise definitions of regularity and λ-regularity in a general setting. These notions will be used in a essential way in §2 and §3. Then, we recall without proofs some results of Da Prato-Grisvard [9] and Sobolevskii [25] concerning the sum of commuting positive operators in Banach spaces, which are used in this paper. Finally, we recall some results concerning the λ-p-regularity of the Cauchy problem of order one.

In §2, under hypothesis (H_A) defined in §1, we compare the p-regularity (resp. the λ-p-regularity) of problems of order one and two in a general setting.

In §3, under the same assumption, we consider the p-regularity (resp. the λ-p-regularity) of periodic problems $(P_{1,P,T})$ and $(P_{2,P,T})$ of order one and two, that is:

$$(P_{1,P,T})\begin{cases} u'(t) + Au(t) = f(t)\,, & t \in [0\,,\,T]\,,\\ u(0) = u(T)\,, \end{cases}$$

and

$$(P_{2,P,T})\begin{cases} -u''(t) + A^2 u(t) = f(t)\,, & t \in [0\,,\,T]\,,\\ u(0) = u(T)\,, \quad u'(0) = u'(T). \end{cases}$$

We study the different notions of p-regularity of these problems and compare them with the p-regularity of $(P_{1,T})$ for the first order and with the p-regularity of $(P_{2,DD})$ and $(P_{2,NN})$ for the second order.

We deduce from this that, if $(P_{1,T})$ is p-regular, then both $(P_{2,DD})$ and $(P_{2,NN})$ are λ-p-regular. Moreover if X has the UMD-property, then the p-regularity of $(P_{2,DD})$ or $(P_{2,NN})$ implies the λ-p-regularity of $(P_{1,T})$.

As a consequence of these results, it appears that if $p \in (1\,,\,\infty)$, the space X has the UMD property and the operator A satisfies (H_A), then all problems defined above are simultaneously p-regular and λ-p-regular.

1. Preliminaries

Let E be a complex Banach space with norm $\|\cdot\|$ and let $\mathcal{A}$ and $\mathcal{B}$ be two closed operators in E (not necessarily densely defined). Consider the problem:

$$(P) \quad \mathcal{A}u + \mathcal{B}u = f \text{ in E.}$$

Definition 1.1. *Problem* (P) *is called* regular *in E, or equivalently, the pair* $(\mathcal{A}\,,\,\mathcal{B})$ *is called* regular, *if for all* $f \in E$, *there exists a unique* $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ *such that* (P) *holds.*

If (P) is regular, it follows from Banach theorem that inequality (1.0) holds:

$$\|u\| + \|\mathcal{A}u\| + \|\mathcal{B}u\| \leq M\|\mathcal{A}u + \mathcal{B}u\| \tag{1.0}$$

for some $M \geq 1$ and for all $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.
It is easy to verify the following lemma:

Lemma 1.0. *Let $\mathcal{A}$ and $\mathcal{B}$ be two closed operators in E (not necessarily densely defined). Then (P) is regular if and only if:*

1) (1.0) *holds*
2) $R(\mathcal{A} + \mathcal{B})$ *is dense in E.*

If moreover $0 \in \rho(\mathcal{A})$ *or* $\rho(\mathcal{B})$ *(where $\rho(.)$ denotes the resolvent set of an operator),* (1.0) *is equivalent to* (1.1):

$$\|\mathcal{A}u\| + \|\mathcal{B}u\| \leq M\|\mathcal{A}u + \mathcal{B}u\| \tag{1.1}$$

for some $M \geq 1$ and for all $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.

REMARK 0. The operator $\mathcal{A}+\mathcal{B}$ is closed if and only if

$$\|u\| + \|\mathcal{A}u\| + \|\mathcal{B}u\| \leq M\big(\|\mathcal{A}u + \mathcal{B}u\| + \|u\|\big)$$

for some $M \geq 1$ and for all $u \in D(\mathcal{A}) \cap D(\mathcal{B})$. In particular, if (P) is regular, $\mathcal{A}+\mathcal{B}$ has to be closed.

A pair of operators $(\mathcal{A}, \mathcal{B})$ such that problem (P) is regular is called *coercive* in [25].

Also, the stronger notion of *coercively positive* pair is introduced in [25], which motivates our Definition 1.2:

DEFINITION 1.2. *Problem* (P) *is called* λ-regular *in* E, *or equivalently, the pair* $(\mathcal{A}, \mathcal{B})$ *is called* λ-regular, *if problem*

$$(P)_\lambda \quad \lambda\mathcal{A}u + \mathcal{B}u = f \text{ in } E$$

is regular for all $\lambda > 0$ *and moreover, inequality* $(1.1)_\lambda$ *holds, for all* $\lambda > 0$:

$$(1.1)_\lambda \qquad \|\lambda\mathcal{A}u\| + \|\mathcal{B}u\| \leq M\|\lambda\mathcal{A}u + \mathcal{B}u\|$$

for some $M \geq 1$, independent of λ *and for all* $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.

REMARK 1. 1) In [2], an example of two operators $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{B}$ bounded, on a Hilbert space H such that $(\mathcal{A}, \mathcal{B})$ is regular but not λ-regular is given.

2) Clearly if $(1.1)_\lambda$ holds, then the inequality

$$\lambda\|\mathcal{A}u\| + \mu\|\mathcal{B}u\| \leq M\|\lambda\mathcal{A}u + \mu\mathcal{B}u\|$$

holds for some $M \geq 1$, for all $\lambda, \mu > 0$ and $u \in D(\mathcal{A}) \cap D(\mathcal{B})$, which shows that the definition of λ-regularity is symmetric in $\mathcal{A}$ and $\mathcal{B}$. It is also clear that this inequality is equivalent to the following ones:

$$\|\mathcal{A}u\| \leq M\|\mathcal{A}u + \lambda\mathcal{B}u\|,$$

for some $M \geq 1$ and all $\lambda > 0$ and $u \in D(\mathcal{A}) \cap D(\mathcal{B})$, and

$$\lambda\|\mathcal{B}u\| \leq M\|\mathcal{A}u + \lambda\mathcal{B}u\|$$

for some $M \geq 1$ and all $\lambda > 0$ and $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.

The following analogue for the λ-regularity of Lemma 1.0 is easily verified:

LEMMA 1.0.λ. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two closed operators in* E (*not necessarily densely defined*). *If* $0 \in \rho(\mathcal{A})$, *then* (P) *is* λ-*regular if and only if*:

1) $(1.1)_\lambda$ *holds for all* $\lambda > 0$,

2) *There exists* $\lambda_0 > 0$ *such that* $R(\lambda_0\mathcal{A} + \mathcal{B})$ *is dense in* E.

Let us recall classical definitions on closed operators:

A closed linear operator $\mathcal{A} : D(\mathcal{A}) \subset E \to E$ (not necessarily densely defined) is called *positive* in $(E, \|\cdot\|)$, [28], if there exists $C > 0$ such that (1.2) holds:

$$(1.2) \qquad \|u\| \leq C\|u + \lambda\mathcal{A}u\|, \text{ for every } \lambda > 0 \text{ and } u \in D(\mathcal{A}),$$

and if $R(\mathcal{I} + \lambda\mathcal{A}) = E$ for some $\lambda > 0$, equivalently for all $\lambda > 0$.

REMARK 2. In [28], an operator $\mathcal{A}$ is called positive if it is positive and satisfies the additional assumption that $0 \in \rho(\mathcal{A})$. In this paper, it is convenient to relax this extra condition.

If $\mathcal{A}$ is positive, injective and densely defined, it is easy to prove that $\mathcal{A}^{-1}$ is also positive.

If X is reflexive and $\mathcal{A}$ is positive, then $\mathcal{A}$ is densely defined [20].

Let $\Sigma_\sigma := \{\lambda \in \mathbb{C}\backslash\{0\}; |\arg \lambda| \leq \sigma\} \cup \{0\}$, for $\sigma \in [0, \pi)$. If $\mathcal{A}$ is positive, there exists $\theta \in [0, \pi)$ such that (1.3) holds, [20, p. 288]:

$$(1.3) \qquad \left[\begin{array}{ll} i) & \sigma(\mathcal{A}) \subseteq \Sigma_\theta \\ ii) & \text{for each } \theta' \in (\theta, \pi]\,, \text{ there exists} \\ & M(\theta') \geq 1 \text{ such that } \|\lambda(\lambda I - \mathcal{A})^{-1}\| \leq M(\theta')\,, \\ & \text{for every } \lambda \in \mathbb{C}\backslash\{0\} \text{ with } |\arg \lambda| \geq \theta'\,, \end{array}\right.$$

where $\sigma(\mathcal{A})$ denotes the spectrum of $\mathcal{A}$.

The number $\omega_{\mathcal{A}} := \inf\{\theta \in [0, \pi); (1.3) \text{ holds}\}$ is called the *spectral angle* of the operator $\mathcal{A}$.

Clearly $\omega_{\mathcal{A}} \in [0, \pi)$.

An operator $\mathcal{A}$ is said to be *of type* (ω, M) [27], if $\mathcal{A}$ is positive, ω is the spectral angle of $\mathcal{A}$ and

$$M := \min\{C \geq 0; (1.2) \text{ holds}\}.$$

Note that M is also the smallest constant in (1.3) *ii*) for $\theta' = \pi$.

In general, we have $M \geq 1$.

If $M = 1$, the operator $\mathcal{A}$ is called *m-accretive*, and then $\omega \leq \pi/2$ [27].

If $\mathcal{A}$ is of type (ω, M) with $\omega < \pi/2$, then $\mathcal{A}^2$ (with $D(\mathcal{A}^2) := \{u \in D(\mathcal{A}); \mathcal{A}u \in \in D(\mathcal{A})\}$) is of type $(2\omega, M')$ for some $M' \geq 1$.

Let $\mathcal{A}$ be an operator of type (ω, M) and densely defined, then the *fractional powers* of $\mathcal{A}$, $\mathcal{A}^\alpha$ with $\alpha \in (0, 1)$, [20], are well-defined and are positive operators of type $(\alpha\omega, M')$. Moreover the domain of $\mathcal{A}$ is dense in the domain of $\mathcal{A}^\alpha$ equipped with the graph norm [27]. If $\mathcal{A}$ is an operator of type (ω, M) with $\omega < \pi/2$ and if $0 \in \rho(\mathcal{A})$, then $(\mathcal{A}^2)^{1/2} = \mathcal{A}$.

Two positive operators $\mathcal{A}$ and $\mathcal{B}$ in E are said to be (*resolvent*) *commuting* if the bounded operators $(\mathcal{I} + \lambda\mathcal{A})^{-1}$ and $(\mathcal{I} + \mu\mathcal{B})^{-1}$ commute for some $\lambda, \mu > 0$, equivalently for all $\lambda, \mu > 0$.

If $\mathcal{A}$ and $\mathcal{B}$ are commuting positive operators then $\mathcal{A} + \mathcal{B}$ (with domain $D(\mathcal{A}) \cap D(\mathcal{B})$) is closable [9].

The following theorem, which is a consequence of a theorem of Da Prato-Grisvard [9], will be essential in the sequel:

THEOREM 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two commuting positive operators in E satisfying*:
i) $D(\mathcal{A}) + D(\mathcal{B})$ *is dense in* E,
ii) $\omega_{\mathcal{A}} + \omega_{\mathcal{B}} < \pi$.

Then the closure of $\mathcal{A}+\mathcal{B}$ *is of type* (ω, M) *with* $\omega \leq \max(\omega_{\mathcal{A}}, \omega_{\mathcal{B}})$.
If moreover
iii) $0 \in \rho(\mathcal{A})$ *(resolvent set of* $\mathcal{A}$*), then*
a) there exists $M > 0$ *such that*

$$\|u\| \leq M\|\mathcal{A}u + \mathcal{B}u\|, \quad \text{for all } u \in D(\mathcal{A}) \cap D(\mathcal{B}), \tag{1.4}$$

and $0 \in \rho(\overline{\mathcal{A}+\mathcal{B}})$.
b) $R(\mathcal{A}+\mathcal{B}) \supseteq D(\mathcal{A}) + D(\mathcal{B})$,
c) $\mathcal{A}+\mathcal{B}$ *is closed if and only if* $R(\mathcal{A}+\mathcal{B}) = E$ *if and only if* (1.1) *holds.*

REMARK 3. 1) Under hypotheses *i)-iii)* of Theorem 1.1, assumption 2) of Lemma 1.0 is always satisfied. Therefore, in order to prove the regularity of problem (P), it is sufficient to verify inequality (1.1).

2) Similarly, under hypotheses *i)-iii)* of Theorem 1.1, assumption 2) of Lemma 1.0.λ is always satisfied. Therefore, in order to prove the λ-regularity of problem (P), it is sufficient to verify inequality $(1.1)_\lambda$.

In this paper, we shall always be in the situation of *i)-iii)* of Theorem 1.1.

REMARK 4. Under assumptions *i) - iii)* of Theorem 1.1, the pair $(\mathcal{A}, \mathcal{B})$ is λ-regular if and only if the closed operator $\mathcal{B}\mathcal{A}^{-1}$ is positive.

A useful consequence of λ-regularity is the following result of Sobolevskii [25]:

THEOREM 1.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two commuting positive operators in E satisfying*:
i′) $D(\mathcal{A})$ *and* $D(\mathcal{B})$ *are dense in E,*
ii′) $\omega_{\mathcal{A}} + \omega_{\mathcal{B}} < \pi$,
iii′) $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$,
iv′) $(\mathcal{A}, \mathcal{B})$ *satisfies* $(1.1)_\lambda$.
Then for every $\alpha \in (0, 1)$, $D(\mathcal{A}) \cap D(\mathcal{B}) \subset D(\mathcal{A}^\alpha \mathcal{B}^{1-\alpha})$ *and there is* $M_\alpha > 0$ *such that* (1.5) *holds*:

$$\|\mathcal{A}^\alpha \mathcal{B}^{1-\alpha} u\| \leq M_\alpha \|\mathcal{A}u + \mathcal{B}u\|, \tag{1.5}$$

for all

$$u \in D(\mathcal{A}) \cap D(\mathcal{B}).$$

REMARK 5. It is also shown in [25] that under the assumptions of Theorem 1.2, the following interpolation inequality holds:

$$\|\mathcal{A}^\alpha \mathcal{B}^{1-\alpha} u\| \leq M'_\alpha \|\mathcal{A}u\|^\alpha \|\mathcal{B}u\|^{1-\alpha}, \tag{1.6}$$

for some $M'_\alpha > 0$ and for all $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.

We mention in Remark 6, some sufficient conditions on $\mathcal{A}$, $\mathcal{B}$ and E of Theorem 1.1 which guarantee the regularity of the pair $(\mathcal{A}, \mathcal{B})$:

REMARK 6. 1) If E has the UMD property, $\mathcal{A}$ and $\mathcal{B}$ are positive, injective, resolvent commuting and satisfy $\|\mathcal{A}^{is}\| \leq Me^{\alpha|s|}$ and $\|\mathcal{B}^{is}\| \leq Me^{\beta|s|}$ for some $M \geq 1$, $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta < \pi$ and for all $s \in \mathbb{R}$, then:

a) in [15], if $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$, then $(\mathcal{A}, \mathcal{B})$ is a regular pair.

b) in [16] and later in [24], inequality (1.1) is proved without using the condition $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$.

c) It follows from the proof of [24] that not only (1.1) holds but also stronger inequality $(1.1)_\lambda$ holds. This implies in particular that if $0 \in \rho(\mathcal{A})$, $(\mathcal{A}, \mathcal{B})$ forms a λ-regular pair.

2) If E is a Hilbert space, $\mathcal{A}$ and $\mathcal{B}$ are positive with $\omega_\mathcal{A} + \omega_\mathcal{B} < \pi$, injective, resolvent commuting and satisfying $\|\mathcal{A}^{is}\| \leq M$ for some $M \geq 1$ and for all $s \in [-1, +1]$, then:

a) in [15], if $0 \in \rho(\mathcal{A}) \cap \rho(\mathcal{B})$, then $(\mathcal{A}, \mathcal{B})$ is a regular pair.

b) in [2], it is shown that under the weaker condition $0 \in \rho(\mathcal{A})$ only, then $(\mathcal{A}, \mathcal{B})$ is a λ-regular pair.

In this paper we use Theorem 1.1 in the following particular case:

Let $(X, |\cdot|)$ be a complex Banach space and $T > 0$.

For $p \in (1, \infty)$, we shall denote by E_p, the Banach space $L^p([0, T]; X)$, equipped with the usual norm, which will be denoted by $\|\cdot\|$.

Let A be a positive operator on X. Define the operator $\mathcal{A}$ on E_p by:

$$D(\mathcal{A}) := L^p([0, T]; D(A))$$

where $D(A)$ is equipped with the graph norm and

$$(\mathcal{A}u)(t) := Au(t), \ \text{a.e. in } [0, T]. \tag{1.7}$$

The operator $\mathcal{A}$ is also a positive operator on E_p with spectral angle $\omega_\mathcal{A} = \omega_A$. Moreover the following holds:

$$((\mathcal{I} + \mathcal{A})^{-1}u)(t) = (I + A)^{-1}(u(t)), \ \text{a.e. in } [0, T], \ \text{and all } u \in E_p.$$

Next, we define the operator $\mathcal{B}_T$ acting in E_p as follows:

$$D(\mathcal{B}_T) := \{u \in W^{1,p}([0, T]; X); u(0) = 0\}$$

and

$$(\mathcal{B}_T u)(t) := u'(t), \ \text{a.e. in } [0, T].$$

We recall that the operator $\mathcal{B}_T$ (see [9]) is *m*-accretive and densely defined on E_p and commute in the sense of the resolvent with the operator $\mathcal{A}$ defined in (1.9).

Since the spectral angle of $\mathcal{B}_T$ is equal to $\pi/2$, Theorem 1.1 applies to the pair $(\mathcal{A}, \mathcal{B}_T)$, provided that the operator A satisfies assumption (H_A):

$$(H_A) \qquad \begin{aligned} &(H_A)1) \qquad \omega_A < \pi/2, \\ &(H_A)2) \qquad 0 \in \rho(A). \end{aligned}$$

REMARK 7. 1) We recall that if $D(A)$ is dense in X, the assumption $(H_A)1)$ is equivalent to saying that $(-A)$ is the generator of a bounded analytic semi-group $(e^{-tA})_{t>0}$ and moreover that if $(H_A)1)$ holds, then $(H_A)2)$ is equivalent to saying that the semi-group $(e^{-tA})_{t>0}$ is exponentially stable, which means that

$$|e^{-tA}| \leq M_A e^{-\omega t}$$

for some $M \geq 1$, $\omega > 0$ and for all $t > 0$.

2) More generally, if $D(A)$ is not dense in X and $(H_A)1)$ holds, then it is shown in [11] that $(-A)$ generates in an appropriate sense a bounded analytic semi-group $(e^{-tA})_{t>0}$ on X which is strongly continuous only on $\overline{D(A)}$ and satisfying (1.8) when $(H_A)2)$ is fulfilled:

$$|e^{-tA}| \leq M_A e^{-\omega t}\ ,\ |tAe^{-tA}| \leq M_A e^{-\omega t}\ ,\ |t^2 A^2 e^{-tA}| \leq M_A e^{-\omega t} \tag{1.8}$$

for some $M_A \geq 1$, $\omega > 0$ and for all $t > 0$.
Moreover, if (H_A) is satisfied, then

$$1 \in \rho(e^{-tA}) \tag{1.9}$$

for all $t > 0$.

It is clear from the definitions that the pair $(\mathcal{A}, \mathcal{B}_T)$ is regular in E_p (resp. λ-regular) if and only if $(P_{1,T})$ is p-regular (resp. λ-p-regular), see §0 for definitions of Cauchy problem $(P_{1,T})$.

The analogy is the same between general inequality (1.1) and inequalities (0.1) for Cauchy problems of §0 (resp. $(1.1)_\lambda$ and inequalities $(0.1)_\lambda$).

In Proposition 1.3, we show that, under the hypothesis (H_A), the p-regularity and the λ-p-regularity of problem $(P_{1,T})$, is equivalent; the main step is a result of Dore-Kato [14, Theorem 2.4] which proves that if the pair $(\mathcal{A}, \mathcal{B}_T)$ is regular in E_p for some $T > 0$, then the natural extension of this pair to $[0, \infty)$ is regular in $L_p([0, \infty); X)$. Recently, Le Merdy, [21], gave an interesting example of a bounded operator A in X for which the Cauchy problem is not regular in $L^p([0, \infty); X)$, hence probleme $(P_{1,T})$ is not λ-p-regular.

PROPOSITION 1.3. *Let $p \in (1, \infty)$ and A satisfy (H_A), then the following statements are equivalent* :

1. *Problem $(P_{1,T})$ is p-regular*
2. *Problem $(P_{1,T})$ is λ-p-regular.*

PROOF. 1⇒2
If $(P_{1,T})$ is p-regular and (H_A) holds, it follows from [14] that the problem on $[0, \infty)$ is also p-regular (in an obvious sense).

Next, by the change of variables $t \to \lambda t$, the problem on $[0, \infty)$ is clearly λ-p-regular.

Let $u \in W^{1,p}([0\,,\,T];X) \cap (L^p([0\,,\,T];D(A))$ such that $u(0) = 0$ be given and set

$$\begin{aligned} f(t) &= u'(t) + \lambda A u(t) \qquad \text{for } t \in [0\,,\,T] \\ &= 0 \qquad\qquad\qquad\quad\ \ \text{for } t > T. \end{aligned}$$

Clearly, $f \in L^p([0\,,\,\infty);X)$. From the λ-p-regularity of the problem on $[0\,,\,\infty)$, there exists one and only one $\widetilde{u} \in W^{1,p}([0\,,\,\infty);X) \cap (L^p([0\,,\,\infty);D(A))$ with $\widetilde{u}(0) = 0$, satisfying $\widetilde{u}'(t) + \lambda A\widetilde{u}(t) = f$ on $[0\,,\,\infty)$. Moreover there exists $C > 0$ with, for all $\lambda > 0$:

$$\|\widetilde{u}'\| + \lambda\|\mathcal{A}\widetilde{u}\| \le C\|f\|.$$

Since $\widetilde{u}(t) = u(t)$ on $[0\,,\,T]$, by uniqueness of the solution of $(P_{1\,,\,T})$, we get by restriction, since $\|.\|_{L^p([0\,,\,T];X)} \le \|.\|_{L^p([0\,,\,\infty);X)}$:

$$\|u'\| + \lambda\|\mathcal{A}u\| \le C\|f\|$$

and thus $(P_{1\,,\,T})$ is λ-p-regular, which proves 2.

$2 \Rightarrow 1$ is obvious. □

REMARK 8. For the sake of completeness, we recall without proof that if $(P_{1\,,\,T})$ is p-regular, the solution u is given by:

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\, ds = \int_0^t e^{-sA} f(t-s)\, ds$$

for $t \ge 0$.

2. First order versus second order problems

In this Section we are concerned with the regularity of pairs $(\mathcal{A}\,,\,\mathcal{B})$ and $(\mathcal{A}^2\,,\,-\mathcal{B}^2)$, in a general setting, as it was done in Theorem 1.1. This will give us a technical result for comparing problems of order one and of order two.

PROPOSITION 2.1. *Let E be a Banach space and $\mathcal{A}$ be a positive operator in E satisfying*:
i) $\omega_{\mathcal{A}} < \pi/2$,
ii) $0 \in \rho(\mathcal{A})$.
Let $\mathcal{B}$ be the generator of a bounded strongly continuous group on E. Suppose that $\mathcal{A}$ and $\mathcal{B}$ as well as $\mathcal{A}$ and $-\mathcal{B}$ are resolvent commuting. Then, the two following assumptions are equivalent:
a) The pairs $(\mathcal{A}\,,\,\mathcal{B})$ and $(\mathcal{A}\,,\,-\mathcal{B})$ are regular (resp. λ-regular).
b) The pair $(\mathcal{A}^2\,,\,-\mathcal{B}^2)$ is regular (resp. λ-regular) and satisfies (2.1) (*resp.* $(2.1)_\lambda$) :

$$D(\mathcal{A}^2) \cap D(\mathcal{B}^2) \subset D(\mathcal{B}\mathcal{A}) \text{ and } \|\mathcal{B}\mathcal{A}u\| \le M\|\mathcal{A}^2u - \mathcal{B}^2u\| \tag{2.1}$$

for some $M > 0$ and for all $u \in D(\mathcal{A}^2) \cap D(\mathcal{B}^2)$,

$$D(\mathcal{A}^2) \cap D(\mathcal{B}^2) \subset D(\mathcal{B}\mathcal{A}) \text{ and } \lambda\|\mathcal{B}\mathcal{A}u\| \le M\|\lambda^2\mathcal{A}^2u - \mathcal{B}^2u\| \tag*{$(2.1)_\lambda$}$$

for some $M > 0$, independent of λ, *for all $u \in D(\mathcal{A}^2) \cap D(\mathcal{B}^2)$ and all $\lambda > 0$.*

We shall need the following:

LEMMA 2.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two positive operators in a Banach space* E*, resolvent commuting, with* $0 \in \rho(\mathcal{A})$*. Then:*

$$D(\mathcal{BA}) \subset D(\mathcal{AB}) \text{ and } \mathcal{BA}u = \mathcal{AB}u \tag{2.2}$$

for all $u \in D(\mathcal{BA})$.

PROOF OF LEMMA 2.2. We shall identify operators in E with their graphs in $E \times E$. From $\mathcal{A}^{-1}(\mathcal{I} + \mathcal{B})^{-1} = (\mathcal{I} + \mathcal{B})^{-1}\mathcal{A}^{-1}$, we obtain $\big((\mathcal{I} + \mathcal{B})\mathcal{A}\big)^{-1} = \big(\mathcal{A}(\mathcal{I} + \mathcal{B})\big)^{-1}$, hence $(\mathcal{I} + \mathcal{B})\mathcal{A} = \mathcal{A}(\mathcal{I} + \mathcal{B})$. Then for any $x \in D(\mathcal{BA})$, [that is $x \in D(\mathcal{A})$ with $\mathcal{A}x \in D(\mathcal{B})$], we have $x \in D(\mathcal{B})$ together with $x + \mathcal{B}x \in D(\mathcal{A})$ and $\mathcal{A}x + \mathcal{BA}x = \mathcal{A}x + \mathcal{AB}x$. This implies (2.2). □

PROOF OF PROPOSITION 2.1. *a)* $\Rightarrow$ *b)*:

In order to prove the regularity of the pair $(\mathcal{A}^2, -\mathcal{B}^2)$, we shall use Theorem 1.1 with $\mathcal{A}$ replaced by $\mathcal{A}^2$ and $\mathcal{B}$ replaced by $-\mathcal{B}^2$.

Observe that the operator $\mathcal{A}^2$ is positive with $\omega_{\mathcal{A}^2} = 2\omega_{\mathcal{A}} < \pi$ and $0 \in \rho(\mathcal{A}^2)$.

It is known (see *e.g.* [22]) that under the assumption on $\mathcal{B}$, the operator $-\mathcal{B}^2$ is positive with $\omega_{-\mathcal{B}^2} = 0$.

Moreover, $\mathcal{A}^2$ and $-\mathcal{B}^2$ are resolvent commuting: indeed, it is enough to check the equality: $(\mathcal{A}^2)^{-1}(\mathcal{I} - \mathcal{B}^2)^{-1} = (\mathcal{I} - \mathcal{B}^2)^{-1}(\mathcal{A}^2)^{-1}$ which follows from the assumption that $\mathcal{A}$ and $\mathcal{B}$ as well as $\mathcal{A}$ and $-\mathcal{B}$ are resolvent commuting and the observation that $(\mathcal{A}^2)^{-1}(\mathcal{I} - \mathcal{B}^2)^{-1} = \mathcal{A}^{-1}\mathcal{A}^{-1}(\mathcal{I} - \mathcal{B})^{-1}(\mathcal{I} + \mathcal{B})^{-1}$.

Moreover, since $\mathcal{B}$ is in particular the generator of a strongly continuous semigroup on E, $D(\mathcal{B}^2)$ is dense in E, see [23].

In view of Theorem 1.1, the regularity of the pair $(\mathcal{A}^2, -\mathcal{B}^2)$ is a consequence of the surjectivity of $\mathcal{A}^2 - \mathcal{B}^2$. In order to prove it, we need the following regularity property of the solution to the equations $\mathcal{A}x + \mathcal{B}x = y \in E$ and $\mathcal{A}x - \mathcal{B}x = y \in E$. Without loss of generality, we only consider the first equation. We claim that if $y \in D(\mathcal{A}) \cap D(\mathcal{B})$, then the unique solution x satisfies $x \in D(\mathcal{A}^2) \cap D(\mathcal{BA}) \cap D(\mathcal{B}^2)$. Indeed if $y \in D(\mathcal{B})$ and $z = (\mathcal{I} + \mathcal{B})y$, let $\hat{x} \in D(\mathcal{A}) \cap D(\mathcal{B})$ be the unique solution to $\mathcal{A}x + \mathcal{B}x = z$. We obtain:

$$y = (\mathcal{I} + \mathcal{B})^{-1}z = (\mathcal{I} + \mathcal{B})^{-1}\mathcal{A}\hat{x} + (\mathcal{I} + \mathcal{B})^{-1}\mathcal{B}\hat{x} = \mathcal{A}(\mathcal{I} + \mathcal{B})^{-1}\hat{x} + \mathcal{B}(\mathcal{I} + \mathcal{B})^{-1}\hat{x}$$

using the resolvent commutativity of $\mathcal{A}$ and $\mathcal{B}$.

By uniqueness, we get $x = (\mathcal{I} + \mathcal{B})^{-1}\hat{x}$, hence $x \in D(\mathcal{AB}) \cap D(\mathcal{B}^2)$. Similarly, if $y \in D(\mathcal{A})$, we have $x \in D(\mathcal{A}^2) \cap D(\mathcal{BA})$, which establishes the claim.

We are in a position to prove the surjectivity of $\mathcal{A}^2 - \mathcal{B}^2$: let $f \in E$ and let $v \in D(\mathcal{A}) \cap D(\mathcal{B})$ be such that $\mathcal{A}v + \mathcal{B}v = f$. Let $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ satisfy $\mathcal{A}u - \mathcal{B}u = v$. We have $u \in D(\mathcal{A}^2) \cap D(\mathcal{AB}) \cap D(\mathcal{BA}) \cap D(\mathcal{B}^2)$, hence by Lemma 2.2,

$$f = (\mathcal{A} + \mathcal{B})v = (\mathcal{A} + \mathcal{B})(\mathcal{A} - \mathcal{B})u = \mathcal{A}^2u - \mathcal{AB}u + \mathcal{BA}u - \mathcal{B}^2u = \mathcal{A}^2u - \mathcal{B}^2u$$

which shows that $\mathcal{A}^2 - \mathcal{B}^2$ is surjective.

Next we prove (2.1) and $(2.1)_\lambda$. We only consider the case of $(2.1)_\lambda$. Let $u \in D(\mathcal{A}^2) \cap D(\mathcal{B}^2)$ and let $f := \mathcal{A}^2 u - \mathcal{B}^2 u$. From what precedes and the injectivity of $\mathcal{A}^2 - \mathcal{B}^2$, we have $u \in D(\mathcal{B}\mathcal{A})$. For any $\lambda > 0$,

$$\lambda \|\mathcal{B}\mathcal{A}u\| = \|(\mathcal{B})(\lambda\mathcal{A}u)\| \leq M_1 \|(\lambda\mathcal{A} + \mathcal{B})\lambda\mathcal{A}u\|$$

in view of the λ-regularity of the pair $(\mathcal{A}, \mathcal{B})$.

By using (2.2), we have $(\lambda\mathcal{A} + \mathcal{B})\lambda\mathcal{A}u = \lambda\mathcal{A}(\lambda\mathcal{A} + \mathcal{B})u$ and by the λ-regularity of the pair $(\mathcal{A}, -\mathcal{B})$, we obtain

$$\|(\lambda\mathcal{A} + \mathcal{B})\lambda\mathcal{A}u\| = \|\lambda\mathcal{A}(\lambda\mathcal{A} + \mathcal{B})u\| \leq M_2 \|(\lambda\mathcal{A} - \mathcal{B})(\lambda\mathcal{A} + \mathcal{B})u\| = M_1 M_2 \|\lambda^2 \mathcal{A}^2 u - \mathcal{B}^2 u\|.$$

b) $\Rightarrow$ *a)*:

Since the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}, -\mathcal{B})$ satisfy the hypotheses *i)-iii)* of Theorem 1.1, it suffices to prove (1.1) (resp. $(1.1)_\lambda$) for these two pairs.

Let $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ and set $f := (\mathcal{A} + \mathcal{B})u$. By hypothesis, there exists $w \in D(\mathcal{A}^2) \cap D(\mathcal{B}^2)$ such that

$$f = \mathcal{A}^2 w - \mathcal{B}^2 w.$$

By using (2.1) and (2.2), we can write:

$$f = (\mathcal{A} + \mathcal{B})(\mathcal{A} - \mathcal{B})w.$$

By the injectivity of $(\mathcal{A} + \mathcal{B})$, we get $u = (\mathcal{A} - \mathcal{B})w$.

By the regularity of the pair $(\mathcal{A}^2, -\mathcal{B}^2)$, Lemma 2.2 and (2.1), we get $M, M' > 0$ such that:

$$\|\mathcal{A}u\| \leq \|\mathcal{A}^2 w\| + \|\mathcal{A}\mathcal{B}w\| \leq (M + M')\|f\|,$$

which proves the regularity of the pair $(\mathcal{A}, \mathcal{B})$. We proceed similarly for the regularity of the pair $(\mathcal{A}, -\mathcal{B})$ and the λ-regularity. $\square$

3. The periodic case and the conclusion

In this Section, we are concerned with the regularity and the p-regularity of problems $(P_{1,T})$ on one side and $(P_{2,DD})$ and $(P_{2,NN})$ on the other side.

In order to prove their equivalence, we introduce periodic problems of order one and two, $(P_{1,P,T})$ and $(P_{2,P,T})$, which are defined in the Introduction.

As in §0, we define the p-regularity of $(P_{1,P,T})$ and $(P_{2,P,T})$ as follows: $(P_{1,P,T})$ (resp. $(P_{2,P,T})$) is *p-regular* if for all $f \in L^p([0, T]; X)$ there exists a unique function $u \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A))$ (resp. $u \in W^{2,p}([0, T]; X) \cap L^p([0, T]; D(A^2))$) with $u(0) = u(T)$ (resp. $u(0) = u(T)$ and $u'(0) = u'(T)$) such that $(P_{1,P,T})$ (resp. $(P_{2,P,T})$) is satisfied.

As above, it appears that if $(P_{1,P,T})$ (resp.$(P_{2,P,T})$) is p-regular, then (3.1) holds (resp. (3.2)):

$$(3.1) \qquad \begin{cases} \text{There exists } M \geq 1 \text{ such that} \\ \|u'\| + \|Au\| \leq M\|u' + Au\|\,, \\ \text{for every } u \in W^{1,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A)) \\ \text{satisfying } u(0) = u(T)\,, \end{cases}$$

respectively:

$$(3.2) \qquad \begin{cases} \text{There exists } M \geq 1 \text{ such that} \\ \|u''\| + \|A^2u\| \leq M\|-u'' + A^2u\|\,, \\ \text{for every } u \in W^{2,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A^2)) \\ \text{satisfying } u(0) = u(T) \text{ and } u'(0) = u'(T). \end{cases}$$

As in §0, we shall call problem $(P_{1,P,T})$ (resp. $(P_{2,P,T})$) *λ-p-regular* if problem

$$(P_{1,P,T})_\lambda \begin{cases} u'(t) + \lambda Au(t) = f(t)\,, & t \in [0\,,\,T]\,,\quad 0 < T < \infty\,, \\ u(0) = u(T)\,, \end{cases}$$

(resp. $(P_{2,P,T})_\lambda$), is p-regular for all $\lambda > 0$ and inequality (3.1) (resp. (3.2)) holds where A (resp. A^2) is replaced by λA (resp. $\lambda^2 A^2$) with $\lambda > 0$ and *M is independent of λ*. We will denote by $(3.1)_\lambda$-$(3.2)_\lambda$ the corresponding inequalities.

Let us state our results:

Theorem 3.1. *Let $p \in (1\,,\,\infty)$ and A satisfy (H_A) on X. Then:*

1) If $(P_{1,T})$ is p-regular (respectively λ-p-regular), then $(P_{1,P,T})$ is also p-regular (respectively λ-p-regular).

2) If $(P_{2,P,T})$ is p-regular, then $(P_{2,P,T})$ is λ-p-regular.

Theorem 3.2. *Let $p \in (1\,,\,\infty)$ and A satisfy (H_A) on X. Then, if X has the UMD property:*

1) $(P_{2,DD})$ is p-regular if and only if $(P_{2,NN})$ is also p-regular.

2) If $(P_{2,P,T})$ is p-regular, then (3.3) *holds:*

$$(3.3) \qquad \|Au'\| \leq M\|-u'' + A^2u\|$$

for some $M > 0$ and for all $u \in L^p(\mathbb{R};D(A^2)) \cap W^{1,p}(\mathbb{R};D(A)) \cap W^{2,p}(\mathbb{R};X)$ satisfying $u(0) = u(T)$, $u'(0) = u'(T)$.

In order to finish to compare the regularity of these problems, we need the following proposition:

Proposition 3.3. *Let $p \in (1\,,\,\infty)$ and A satisfy (H_A) on X.*

1) $(P_{1,P,T})$ is p-regular if and only if $(P_{2,P,T})$ is p-regular and verifies (3.3).

2) $(P_{2,P,T})$ is p-regular (resp. λ-p-regular) if and only if $(P_{2,DD})$ and $(P_{2,NN})$ are p-regular (resp. λ-p-regular).

3) If $(P_{1,P,T})$ is p-regular, then $(P_{1,T})$ is p-regular.

All together, these results imply the following Corollary, which closes the circle of implications comparing the regularity of Cauchy problems and boundary value problems defined in §0:

Corollary 3.4. *Let* $p \in (1, \infty)$ *and* A *satisfy* (H_A) *on* X.

1) *If* $(P_{1,T})$ *is p-regular, then* $(P_{2,DD})$ *and* $(P_{2,NN})$ *are* λ*-p-regular.*

2) *If* X *has the UMD property, if* $(P_{2,DD})$ *or* $(P_{2,NN})$ *is p-regular, then* $(P_{1,T})$ *is* λ*-p-regular.*

As in §1, we set $E_p = L^p([0, T]; X)$ and define the operator $\mathcal{A}$ by (1.7). We set

$$D(\mathcal{B}_P) := \{u \in W^{1,p}([0, T]; X); u(0) = u(T)\} \tag{3.4}$$

and

$$(\mathcal{B}_P u)(t) := u'(t), \; t \in [0, T], \text{ for } u \in D(\mathcal{B}_P). \tag{3.5}$$

The operators $\mathcal{B}_P$ and $-\mathcal{B}_P$ are m-accretive in E, as generators of translation groups. Obviously, $\mathcal{A}$ and $\mathcal{B}_P$ are resolvent commuting. So by Theorem 1.1, if A satisfies (H_A), $(P_{1,P,T})$ is λ-p-regular if and only if the pair $(\mathcal{A}, \mathcal{B}_P)$ verifies $(1.1)_\lambda$.

Similarly if A satisfies (H_A), since $-\mathcal{B}_P^2$ is positive with $\omega_{-\mathcal{B}^2} = 0$, $\omega_{\mathcal{A}^2} < \pi$ and $\mathcal{A}^2$ and $-\mathcal{B}_P^2$ are resolvent commuting, in order to prove that $(P_{2,P,T})$ is λ-p-regular, it is sufficient to verify $(1.1)_\lambda$, with $\mathcal{A}$ replaced by $\mathcal{A}^2$ and $\mathcal{B}$ by $-\mathcal{B}_P^2$.

For the sequel, it is convenient to identify periodic functions defined on $\mathbb{R}$ and functions defined on $[0, T]$, taking the same value on 0 and T.

Moreover, replacing u and f by their periodic extensions, we get easily that $(P_{1,P,T})$ is p-regular if and only if the extended problem of order one on $\mathbb{R}$ is p-regular (in an obvious sense) and $(P_{2,P,T})$ is p-regular if and only if the extende problem of order two on $\mathbb{R}$ is p-regular.

We will use these two formulations in the proofs, when needed.

Proof of Theorem 3.1.

1) $(P_{1,T})$ p-regular (resp. λ-p-regular) $\Rightarrow$ $(P_{1,P,T})$ p-regular (resp. λ-p-regular).

We already know from Proposition 1.3 that problem $(P_{1,T})$ is p-regular if and only if it is λ-p-regular. So it is sufficient to establish: $(P_{1,T})$ λ-p-regular implies $(P_{1,P,T})$ λ-p-regular.

Suppose $\lambda > 0$, $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ and set $f_\lambda = \lambda \mathcal{A} u + \mathcal{B} u$. Since u is a solution to $(P_{1,P,T})_\lambda$, then u satisfies:

$$u(t) = e^{-\lambda t A} u(0) + \int_0^t e^{-\lambda(t-s)A} f_\lambda(s)\, ds \tag{3.6}$$

where $\{e^{-tA}\}_{t>0}$ is the semigroup constructed in [11]. Since, $u(0) = u(T)$ and $1 \in \rho(e^{-tA})$ for all $t > 0$ by Remark 7. 2), property (1.9), we have:

$$u(0) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-(T-s)A} f_\lambda(s) ds. \tag{3.7}$$

Defining $u_1(t)$ and $u_2(t)$ by

$$u_1(t) = \int_0^t e^{-\lambda(t-s)A} f_\lambda(s)\, ds, \tag{3.8}$$

and

$$u_2(t) = e^{-\lambda tA}(I - e^{-\lambda TA})^{-1} \int_0^T e^{-\lambda(T-s)A} f_\lambda(s)\, ds, \tag{3.9}$$

we get by (3.6) and (3.7) that

$$u(t) = u_1(t) + u_2(t) = \int_0^t e^{-\lambda(t-s)A} f_\lambda(s)\, ds + e^{-\lambda tA}(I - e^{-\lambda TA})^{-1} \int_0^T e^{-\lambda(T-s)A} f_\lambda(s)\, ds. \tag{3.10}$$

From the λ-p-regularity of $(P_{1,T})$, we have

$$u_1 \in W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A)), \tag{3.11}$$

and there exists $M \geq 1$ independent of $\lambda > 0$, such that

$$\lambda \|Au_1\| \leq M \|f_\lambda\|. \tag{3.12}$$

Concerning u_2 observe that

$$\int_0^T e^{-\lambda(T-s)} f_\lambda(s)\, ds = u_1(T)$$

lies in the trace space associated with $W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A))$, see [9, 11].

The same holds for $(I - e^{-\lambda TA})^{-1} u_1(T)$ since:

$$(I - e^{-\lambda TA})^{-1} u_1(T) = \int_0^T e^{-\lambda(T-s)A} (I - e^{-\lambda TA})^{-1} f_\lambda(s)\, ds. \tag{3.13}$$

It follows that, since $u_2(t) = e^{-\lambda tA}(I - e^{-\lambda TA})^{-1} u_1(T)$, then by [13, 11], u_2 belongs to $W^{1,p}([0, T]; X) \cap L^p([0, T]; D(A))$.

It remains to prove (3.12) with u_1 replaced by u_2.

By definition,

$$u_2(t) = e^{-\lambda tA} \int_0^T e^{-\lambda(T-s)A} (I - e^{-\lambda TA})^{-1} f_\lambda(s)\, ds = \int_0^T e^{-\lambda(t+T-s)A} (I - e^{-\lambda TA})^{-1} f_\lambda(s)\, ds =$$

$$= \int_0^T e^{-\lambda(t+s)A} (I - e^{-\lambda TA})^{-1} f_\lambda(T-s)\, ds.$$

Thus,

$$\int_0^T |\lambda A u_2(t)|^p\, dt = \int_0^T \left| \int_0^T \lambda A e^{-\lambda(t+s)A} (I - e^{-\lambda TA})^{-1} f_\lambda(T-s)\, ds \right|^p dt \leq$$

$$\leq |(I - e^{-\lambda TA})^{-1}|^p \int_0^T \left[\int_0^T M_A e^{-\lambda\omega(t+s)} \frac{1}{t+s} |f_\lambda(T-s)|\, ds \right]^p dt$$

by (1.8). Thus,

$$\int_0^T |\lambda A u_2(t)|^p \, dt \leq |(I - e^{-\lambda TA})^{-1}|^p M_A^p \int_0^T \left[\int_0^T \frac{1}{t+s} |f_\lambda(T-s)| \, ds \right]^p dt.$$

Since the kernel $\dfrac{1}{t+s}$ defines a bounded operator on $L^p([0\,,T];\mathbb{R})$, [14], there exists M, depending only on p and T, such that:

$$\int_0^T |\lambda A u_2(t)|^p \, dt \leq M_A^p M^p |(I - e^{-\lambda TA})^{-1}|^p \int_0^T |f_\lambda(t)|^p \, dt.$$

Hence,

$$\|\lambda A u_2\| \leq M_A M |(I - e^{-\lambda TA})^{-1}| \|f_\lambda\|.$$

Then we prove that

$$\sup_{0<\lambda_0\leq\lambda} |(I - e^{-\lambda TA})^{-1}| < \infty \text{ for every } \lambda_0 > 0. \tag{3.14}$$

Since $|e^{-tA}| \leq M_A e^{-\omega t}$, for some $\omega > 0$ and $M_A \geq 1$ by (1.8), we have $|e^{-\lambda TA}| \leq$ $\leq 1/2$, for all $\lambda \in [\lambda_1\,,\infty)$, for some $\lambda_1 > 0$.

It follows that $|(I - e^{-\lambda TA})^{-1}| \leq 2$ for $\lambda \in [\lambda_1\,,\infty)$.

Observe that $|(I - e^{-\lambda TA})^{-1}|$ is a continuous function of $\lambda \in (0\,,\infty)$, since $t \to e^{-tA}$ is analytic from $(0\,,\infty)$ to $\mathcal{L}(X)$ and $(I - e^{-tA})$ is invertible for all $t > 0$ by (1.9) in Remark 7.2).

Therefore, $|(I - e^{-\lambda TA})^{-1}|$ is bounded on intervals $[\lambda_0\,,\lambda_1]$, with $0 < \lambda_0 < \lambda_1$, which implies (3.12) for u_2.

We have shown that given $\lambda_0 > 0$, there exists $M = M(\lambda_0) \geq 1$ such that for every $u \in W^{1,p}([0\,,T];X) \cap L^p([0\,,T];D(A))$ satisfying $u(0) = u(T)$, we have

$$\lambda \|Au\| \leq M \|u' + \lambda A u\|\,, \tag{3.15}$$

for every $\lambda \in [\lambda_0\,,\infty)$.

Finally we remove the restriction on λ_0.

For $u \in D(\mathcal{A}) \cap D(\mathcal{B}_p)$ and $n \in \mathbb{N}$, let v be the T/n-periodic function with values in X which coincides with $u(t/n)$ on $[0\,,T/n]$. Then

$$v \in D(\mathcal{A}) \cap D(\mathcal{B}_p)$$

and for $\lambda \in [\lambda_0\,,\infty)$, we have:

$$\lambda^p \|Av\|^p_{L^p([0\,,T];X)} \leq M_1^p \|v' + \lambda A v\|^p_{L^p([0\,,T];X)}.$$

Observe that $\|Av\|^p_{L^p([0\,,T];X)} = n\|Av\|^p_{L^p([0\,,T/n];X)}$ as well as

$$\|v' + \lambda A v\|^p_{L^p([0\,,T];X)} = n\|v' + \lambda A v\|_{L^p([0\,,T/n];X)}.$$

It follows that $\lambda \|Av\|_{L^p([0\,,T/n];X)} \leq M_1 \|v' + \lambda A v\|_{L^p([0\,,T/n];X)}$, which implies by rescaling:

$$(\lambda/n)\|Au\|_{L^p([0\,,T];X)} \leq M_1 \|u' + (\lambda/n) A u\|_{L^p([0\,,T];X)}. \tag{3.16}$$

It follows that (3.15) holds for all $\lambda > 0$. This completes the proof of the λ-p-regularity of problem $(P_{1,P,T})$.

2) If $(P_{2,P,T})$ is p-regular, then $(P_{2,P,T})$ is λ-p-regular.

In this part, we will consider periodic functions extended to $\mathbb{R}$, as mentioned in the beginning of the proof.

Let us denote by $L^p_T(Y)$ the space of T-periodic functions defined on $\mathbb{R}$ with values in a Banach space Y, p-integrable on $[0\,,\,T]$ and by $W^{m,p}_T(Y)$ the corresponding Sobolev spaces for $m = 1\,,2\,,\ldots$

Under the hypothesis that $(P_{2,P,T})$ is regular, it is sufficient to prove that for some $M > 0$ independant of λ, for all $u \in W^{2,p}_T(X) \cap L^p_T(D(A^2))$ and for all $\lambda > 0$,

$$\lambda^2 \|A^2 u\| \le M \| - u'' + \lambda^2 A^2 u\|$$

where $\|.\| := \|.\|_{L^p_T(X)}$.

Choose a sequence of mollifiers $(\varphi_n)_{n\in\mathbb{N}}$, with compact supports in $\mathbb{R}$ such that, for $u \in W^{m,p}_T(Y)$:

$$\varphi_n * u \in C^\infty_T(Y) \text{ and } \varphi_n * u \to u \text{ in } W^{m,p}_T(Y).$$

Let $u \in W^{2,p}_T(X) \cap L^p_T(D(A^2))$ and define $u_n := \varphi_n * u \in C^\infty_T(D(A^2))$.

Set $v_n := u'_n + \lambda A u_n \in C^\infty_T(D(A))$ and $f_n := -v'_n + \lambda A v_n \in C^\infty_T(X)$.

Then, since $u_n \in C^\infty_T(D(A^2))$, we have $(Au_n)' = Au'_n$ and:

$$-u''_n + \lambda^2 A^2 u_n = f_n.$$

We recall a result of [11]:

For all $g \in W^{1,p}_T(X)$, if A verifies (H_A), there exists a unique solution of the problem

$$z'(t) + \lambda A z(t) = g(t)\ ,\quad t \in \mathbb{R}$$

in $W^{1,p}_T(X) \cap L^p_T(D(A))$. Let us denote by $S_{\lambda,T}g$ this solution.

We claim that

$$S_{\lambda,T}g(t) = z(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma. \tag{3.17}$$

To prove this claim, we write

$$\begin{aligned}
(I - e^{-\lambda TA}) z(t) &= \int_0^T e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma = \\
&= \int_0^t e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma + \int_t^{t+T} e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma - \int_T^{t+T} e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma = \\
&= \int_0^t e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma + \int_0^T e^{-(t+\sigma)\lambda A} g(-\sigma)\, d\sigma - \int_0^t e^{-(\sigma+T)\lambda A} g(t-\sigma)\, d\sigma = \\
&= (I - e^{-\lambda TA}) \int_0^t e^{-\sigma\lambda A} g(t-\sigma)\, d\sigma + e^{-\lambda tA} \int_0^T e^{-\sigma\lambda A} g(T-\sigma)\, d\sigma.
\end{aligned}$$

This last expression is the one which was already proved in (3.10).

Thus, if $\mathcal{P}$ denotes as before the operator on $\mathbb{R}$ such that $(\mathcal{P}z)(t) = z(-t)$, we have $(\mathcal{P}z)'(t) = -(\mathcal{P}z')(t)$ and applying this to v_n, we get:

$$(\mathcal{P}v_n)(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma\lambda A}(\mathcal{P}f_n)(t-\sigma)\, d\sigma$$

which is the same as

$$v_n(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-s\lambda A} f_n(s+t)\, ds. \tag{3.18}$$

From the definition of u_n, we have:

$$u_n(t) = (I - e^{-\lambda TA})^{-1} \int_0^T e^{-\sigma\lambda A} v_n(t-\sigma)\, d\sigma.$$

Replacing v_n by its value, we get:

$$u_n(t) = (I - e^{-\lambda TA})^{-2} \int_0^T e^{-\sigma\lambda A} \int_0^T e^{-s\lambda A} f_n(s+t-\sigma)\, ds\, d\sigma.$$

Set $D_\lambda = (I - e^{-\lambda TA})^{-2}$.
From the proof of the first part of Theorem 3.1, D_λ is uniformly bounded, say by C, for $\lambda \geq 1$. Then:

$$\|\lambda^2 \mathcal{A}^2 u_n\|^p = \int_0^T |\lambda^2 A^2 u_n(t)|^p\, dt = \int_0^T |\lambda^2 A^2 \int_0^T e^{-\sigma\lambda A} \int_0^T e^{-s\lambda A} D_\lambda f_n(s+t-\sigma)\, ds\, d\sigma|^p\, dt.$$

Let us make the change of variables: $\sigma\lambda \to \sigma$, $s\lambda \to s$, $t\lambda \to t$:

$$\|\lambda^2 \mathcal{A}^2 u_n\|^p = \frac{1}{\lambda} \int_0^{\lambda T} |A^2 \int_0^{\lambda T} e^{-\sigma A} \int_0^{\lambda T} e^{-sA} D_\lambda f_n(\frac{s+t-\sigma}{\lambda})\, ds\, d\sigma|^p\, dt\,.$$

To complete the proof, let $\lambda = m \in \mathbb{N}^\star$ and divide the triple integral above in four parts:

$$I_1 = \frac{1}{m} \int_0^{mT} |A^2 \int_0^T e^{-\sigma A} \int_0^T e^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right) ds\, d\sigma|^p\, dt$$

$$I_2 = \frac{1}{m} \int_0^{mT} |A^2 \int_0^T e^{-\sigma A} \int_T^{mT} e^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right) ds\, d\sigma|^p\, dt$$

$$I_3 = \frac{1}{m} \int_0^{mT} |A^2 \int_T^{mT} e^{-\sigma A} \int_0^T e^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right) ds\, d\sigma|^p\, dt$$

$$I_4 = \frac{1}{m} \int_0^{mT} |A^2 \int_T^{mT} e^{-\sigma A} \int_T^{mT} e^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right) ds\, d\sigma|^p\, dt.$$

For I_1, by periodicity and using the λ-regularity of $(P_{2,P,T})$, we get:

$$I_1 = \frac{1}{m}\sum_{k=0}^{m-1}\int_{kT}^{(k+1)T} |A^2 \int_0^T e^{-\sigma A}\int_0^T e^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right)\, ds\, d\sigma|^p\, dt =$$

$$= \frac{1}{m}\sum_{k=0}^{m-1}\int_0^T |A^2 \int_0^T e^{-\sigma A}\int_0^T e^{-sA} D_m f_n\left(\frac{s+t+kT-\sigma}{m}\right)\, ds\, d\sigma|^p\, dt \le$$

$$\le M^p \frac{1}{m}\sum_{k=0}^{m-1}\int_0^T |D_m f_n\left(\frac{t+kT}{m}\right)|^p\, dt \le$$

$$\le M^p |D_m|^p \sum_{k=0}^{m-1}\int_{\frac{kT}{m}}^{\frac{(k+1)T}{m}} |f_n(t)|^p\, dt = M^p |D_m|^p \|f_n\|^p.$$

For I_4, using (1.8), we define a kernel on $\mathbb{R}$ by setting $K(s) = 0$ for $0 \le s \le T$ and $K(s) = \frac{M_A}{T} e^{-\omega s}$ for $s > T$ and we get by using Young's inequality for convolutions:

$$I_4 = \frac{1}{m}\int_0^{mT} |\int_T^{mT} Ae^{-\sigma A}\int_T^{mT} Ae^{-sA} D_m f_n\left(\frac{s+t-\sigma}{m}\right)\, ds\, d\sigma|^p\, dt \le$$

$$\le \frac{1}{m}|D_m|^p \int_0^{mT} [\int_0^{mT} K(\sigma)\int_0^{mT} K(s)|f_n\left(\frac{s+t-\sigma}{m}\right)|\, ds\, d\sigma]^p\, dt =$$

$$= |D_m|^p m^{2p}\int_0^T \left[\int_0^T K(m\sigma)\int_0^T K(ms)|f_n(s+t-\sigma)|\, ds\, d\sigma\right]^p dt \le$$

$$\le |D_m|^p \left(\int_0^T K(s)\, ds\right)^{2p}\int_0^T |f_n(t)|^p\, dt \le \left(\frac{M_A}{\omega T}\right)^{2p} |D_m|^p \|f_n\|^p.$$

For I_2 (and in the same way for I_3) we also use (1.8):

$$I_2 = \frac{1}{m}\int_0^{mT} |\int_0^T\int_T^{mT} A^2 e^{-(\sigma+s)A} D_m f_n\left(\frac{s+t-\sigma}{m}\right)\, ds\, d\sigma|^p\, dt \le$$

$$\le |D_m|^p \frac{1}{m}\int_0^{mT}\left(\int_0^T\int_T^{mT}\frac{M_A^2}{(s+\sigma)^2} e^{-(\sigma+s)\omega}|f_n\left(\frac{s+t-\sigma}{m}\right)|\, ds\, d\sigma\right)^p dt \le$$

$$\le |D_m|^p \frac{1}{m}\int_0^{mT}\left[\int_0^{mT}\int_T^{mT}\left(\frac{M_A}{T}\right)^2 e^{-\omega(s+\sigma)}|f_n\left(\frac{s+t-\sigma}{m}\right)|\, ds\, d\sigma\right]^p dt =$$

$$= |D_m|^p \frac{1}{m}\int_0^{mT} (\int_0^{mT} K_1(s)\int_0^{mT} K(s)|f_n\left(\frac{s+t-\sigma}{m}\right)|\, ds\, d\sigma)^p\, dt$$

where $K_1(\sigma) = \dfrac{M_A}{T} e^{-\omega\sigma}$, for $\sigma \geq 0$

$$= |D_m|^p m^{2p} \int_0^T \left(\int_0^T K_1(m\sigma) \int_0^T K(ms)|f_n(s + t - \sigma)| \, ds \, d\sigma \right)^p dt \leq$$

$$\leq |D_m|^p m^{2p} \left(\int_0^T K_1(m\sigma) \, d\sigma \right)^p \left(\int_0^T K(ms) \, ds \right)^p \|f_n\|^p \leq$$

$$\leq |D_m|^p \left(\int_0^{mT} K_1(s) \, ds \right)^p \left(\int_0^{mT} K(s) \, ds \right)^p \|f_n\|^p \leq |D_m|^p \frac{M_A^{2p}}{(\omega T)^{2p}} \|f_n\|^p.$$

Putting together these four majorations, we see that $\|m^2 \mathcal{A}^2 u_n\|^p$ is uniformly bounded by a constant times $\|f_n\|^p$ if $m \geq 1$.

To remove this last restriction, we conclude like in the first part of the proof of Theorem 3.1 and we get inequality $(1.1)_\lambda$ for u_n, first for positive rational numbers of the form m^2/q, $m, q \in \mathbb{N}^*$, hence for all positive real numbers λ.

Taking limits when $n \to \infty$, this inequality remains true for $u \in W_T^{2,p}(X) \cap \cap Ł_T^p(D(A^2))$, which is sufficient to prove the λ-p-regularity of $(P_{2,P,T})$ as announced at the beginning. □

Proof of Theorem 3.2.

Recall that in this part, we suppose that X has the UMD property.

1. $(P_{2,DD})$ p-regular $\iff$ $(P_{2,NN})$ p-regular

Note that, because of the boundary conditions, we can extend functions on $[0, T]$ to $[-T, T]$ by oddness if we are working with $(P_{2,DD})$ and by evenness if we are working with $(P_{2,NN})$. It is clear that like for the periodic case, we can extend these functions to $\mathbb{R}$ by periodicity.

Under the assumptions of Theorem 3.2, $(P_{2,DD})$ (resp. $(P_{2,NN})$) is p-regular if and only if there exists $M > 0$ such that, for all $u \in W_T^{2,p}(X) \cap L_T^p(D(A^2))$, odd (resp. even), the following inequality holds:

$$\|A^2 u\| \leq M \|A^2 u - u''\| . \tag{3.19}$$

It is enough to prove that if this property is true for odd functions, then it holds for even functions and conversely. Let us prove the direct part, the converse being similar. Suppose that $(P_{2,DD})$ is p-regular. Then inequality (3.19) is true for odd functions.

Let u be an even function of $W_T^{2,p}(X) \cap L_T^p(D(A^2))$. Then if u_0 is the average of u on $[-T, T]$, we can work with $\widetilde{u} = u - u_0$ and without loss of generality, we can suppose that $u_0 = 0$.

It is sufficient to show (3.19) when u belongs to a dense subset of even functions belonging to $W_T^{2,p}(X) \cap Ł_T^p(D(A^2))$, namely the even trigonometrical polynomials with values in $D(A^2)$:

$$P_n(t) := \sum_{k=1}^{n} \cos \frac{2\pi kt}{T} a_k\,, \text{ with } a_k \in D(A^2).$$

If $\mathcal{H}$ denotes the Hilbert transform, acting on $L^p_T(\mathbb{R};X)$, we get that

$$(\mathcal{H}P_n)(t) := \sum_{k=1}^{n} \sin \frac{2\pi kt}{T} a_k.$$

Since $\mathcal{H}P_n$ is odd, (3.19) holds and we have:

$$\|\mathcal{A}^2\mathcal{H}P_n\| \leq M\|\mathcal{A}^2\mathcal{H}P_n - (\mathcal{H}P_n)''\|\,.$$

We have that $\mathcal{H}^2 = -I$, $\mathcal{H}$ commutes with $\mathcal{A}^2$ and with the second derivative. Since X has the UMD property, there exists $C >$ such that:

$$\|\mathcal{A}^2 P_n\| \leq MC\|\mathcal{A}^2 P_n - P_n''\|\,.$$

This proves the p-regularity of $(P_{2\,,NN})$.

2. $(P_{2\,,P,\,T})$ p-regular $\Rightarrow$ (3.3) holds

We want to apply Sobolevskii's result written in Theorem 1.2. We already know that if $(P_{2\,,P,\,T})$ is p-regular then it is also λ-p-regular by Theorem 3.1 2). After substracting averages, which is possible since $0 \in \rho(A)$, it is convenient to work with spaces of periodic functions with null average. Set:

$$W^{m,p}_{T,*}(Y) := \left\{ u \in W^{m,p}_T(Y) \ / \ \frac{1}{T}\int_{-T/2}^{T/2} u(t)\,dt = 0 \right\}$$

for $m = 0\,,1\,,2\,,\ldots.$ On $W^{m,p}_{T,*}(X)$, the second derivative $-\mathcal{B}^2_P$ is invertible. Since $\mathcal{A}^2$ is also invertible by hypothesis, we are in position to apply Theorem 1.2 to $\mathcal{A}^2_*$ and $-\mathcal{B}^2_*$, which denotes the restrictions of these operators to $L^p_{T,*}(D(A^2))$ and $W^{2,p}_{T,*}(X)$. From the Banach valued version of Marcinkiewicz's theorem instead of Mihlin's theorem, see [1, 31], we obtain:

$$D(\mathcal{B}^2_*) \cap D(\mathcal{A}^2_*) \subset D\big[\big(\mathcal{I} - \mathcal{B}^2_*\big)^{1/2}\big(\mathcal{A}^2\big)^{1/2}\big]$$

and

$$\|\big[\big(\mathcal{I} - \mathcal{B}^2_*\big)^{1/2}\big(\mathcal{A}^2\big)^{1/2}\big]u\| \leq M\|\mathcal{A}^2_* u - u''\|$$

for some $M > 0$ and for all $u \in W^{2,p}_{T,*}(X) \cap L^p_{T,*}(D(A^2))$.

Observe that $\big(\mathcal{A}^2_*\big)^{1/2} = \mathcal{A}_*$ and $\big(\mathcal{I}-\mathcal{B}^2_*\big)^{1/2} = \mathcal{H}B_*$ where $\mathcal{H}$ is the Hilbert transform, [1]. We get then that, on $W^{2,p}_{T,*}(X) \cap L^p_{T,*}(D(A^2))$

$$\|\mathcal{H}A_* u'\| \leq M\|\mathcal{A}^2_* - u''\|.$$

Since X has the UMD property, $\mathcal{H}$ is bounded on $L^p_T(\mathbb{R};X)$ and this inequality implies (3.3). $\square$

PROOF OF PROPOSITION 3.3.

1. $(P_{1,P,T})$ p-regular $\iff$ $(P_{2,P,T})$ p-regular and (3.3)

Since $\mathcal{B}_p$ is the generator of the translation group on $L^p([0\,,\,T];X)$, 1) follows from Proposition 2.1.

2. $(P_{2,P,T})$ p-regular $\iff$ $(P_{2,DD})$ and $(P_{2,NN})$ p-regular

It is sufficient to remark that, after extending these problems to $\mathbb{R}$ by periodicity for $(P_{2,P,T})$, oddness then periodicity for $(P_{2,DD})$ and evenness then periodicity for $(P_{2,NN})$, the solution of $(P_{2,DD})$ is the odd part of the solution of $(P_{2,P,T})$ and the solution of $(P_{2,NN})$ its even part. Conversely, by decomposing functions into odd parts and even parts, the solution of $(P_{2,P,T})$ is the sum of the solution of $(P_{2,DD})$ and the solution of $(P_{2,NN})$.

3. $(P_{1,P,T})$ p-regular $\Rightarrow$ $(P_{1,T})$ p-regular

It is enough to show that problem $(P_{1,T})$ has a solution for every $f \in L^p([0\,,\,T];X)$. For such function f, let u_1 be the solution to $(P_{1,P,T})$. Then there exists exactly one function $u_2 \in W^{1,p}([0\,,\,T];X) \cap L^p([0\,,\,T];D(A))$ satisfying $u_2'(t) + Au_2(t) = 0\,,\, u_2(0) = -u_1(0)$, (see [9, appendix on trace spaces] and [11]).

Thus $u = u_1 + u_2$ is a solution to $(P_{1,T})$. □

ACKNOWLEDGEMENTS

This paper was written during a visit of the first author at the Equipe d'Analyse of University Paris 6. He would like to thank the second author for warm hospitality.

REFERENCES

[1] J. BOURGAIN, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*. Ark. Math., 21, 1983, 163-168.

[2] S. BU - PH. CLÉMENT - S. GUERRE-DELABRIÈRE, *Regularity of pairs of positive operators*. Ill. J. of Maths., 1998, to appear.

[3] R. BURKHOLDER, *A geometrical characterization of Banach spaces in which martingale differences are unconditional*. Ann. Prob., 9, 1981, 997-1011.

[4] P. BUTZER - H. BERENS, *Semi-groups of operators and approximation*. Springer-Verlag, NY Inc., 1967.

[5] P. CANNARSA - V. VESPRI, *On maximal L_p-regularity for the abstract Cauchy Problems*. Boll. U.M.I., 1986, 165-175.

[6] PH. CLÉMENT - P. GRISVARD, *Somme d'opérateurs et régularité L^p dans les problèmes aux limites*. CRAS Paris, t. 314, 1992, 821-824.

[7] PH. CLÉMENT - S. GUERRE-DELABRIÈRE, *On the regularity of abstract Cauchy problems of order one and boundary value problems of order two*. Publ. Math. Univ. P. et M. Curie (Paris 6), 119, 1997, 1-44.

[8] T. COULHON - D. LAMBERTON, *Régularité L_p pour les équations d'évolution*. Séminaire d'analyse fonctionnelle Paris VI-VII, 1984-85, 155-165.

[9] G. DA PRATO - P. GRISVARD, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*. J. Math. pures et appl., 54, 1975, 305-387.

[10] G. DA PRATO - P. GRISVARD, *Equations d'évolution abstraites non linéaires de type parabolique*. Ann. Math. Pura Appl., 120, 1979, 329-396.

[11] G. DA PRATO - E. SINESTRARI, *Differential operators with non dense domain*. Ann. Scuola Norm. Sup. Pisa cl. Sci., 4, 14, 1987, 185-344.
[12] L. DE SIMON, *Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*. Rend. Sem. Mat. Univ. Padova, 1964, 205-223.
[13] G. DI BLASIO, *Linear parabolic evolution equation in L_p-spaces*. Annali Mat., 138, 1984, 55-104.
[14] G. DORE, *L_p-regularity of abstract differential equations*. Lecture notes in math., 1540, Springer-Verlag, 1993, 25-38.
[15] G. DORE - A. VENNI, *On the closedness of the sum of two closed operators*. Math. Zeitschr., 196, 1987, 189-201.
[16] Y. GIGA - H. SOHR, *Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes euqtions in exterior domains*. Hokkaido University Preprint, series in Mathematics, series 60, Nov. 1989, 38 pp.
[17] P. GRISVARD, *Equations différentielles abstraites*. Ann. Sc. Ecole Normale Sup., 2, 1969, 311-395.
[18] S. GUERRE-DELABRIÈRE, *L_p-regularity of the Cauchy problem and the geometry of Banach spaces*. Ill. J. of Maths., 39, 1995, 556-566.
[19] M. HIEBER - J. PRÜSS, *Heat kernel and maximal L^p-L^q estimates for parabolic evolution equations*. Comm. PDE, 22, 1997, 1647-1669.
[20] H. KOMATSU, *Fractional powers of operators*. Pacific J. of Math., 19, 1966, 285-346.
[21] C. LE MERDY, *Counterexamples on L_p-Maximal regularity*. Preprint
[22] R. NAGEL (ed.), *One-parameter semi groups of positive operators*. Lecture notes in math., 1184, Springer-Verlag, Berlin-New York 1986.
[23] A. PAZY, *Semi groups of linear operators and applications to partial differential equations*. Springer-Verlag, New York 1983.
[24] J. PRÜSS - H. SOHR, *On operators with bounded imaginary powers in Banach spaces*. Math. Z., 203, 1990, 429-452.
[25] P.E. SOBOLEVSKII, *Fractional powers of coercively positive sums of operators*. Soviet Math. Dokl., 16, 1975.
[26] P.E. SOBOLEVSKII, *Coerciveness inequalities for abstract parabolic equations*. Dokl. Acad. Nauk, 157, 1964, 52-55.
[27] H. TANABE, *Equations of Evolution*. Pitman, London, San Francisco, Melbourne 1979.
[28] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*. North Holland, 1978.
[29] W. VON WAHL, *The equation $u' + Au = f$ in a Hilbert space and L^p-estimates for parabolic equations*. J. London Math. Soc., 25, 1982, 483-497.
[30] K. YOSIDA, *Functional Analysis*. Springer-Verlag, New York 1971.
[31] F. ZIMMERMANN, *On vector-valued Fourier multiplier theorems*. Studia Math., 93, 1989, 201-222.

Pervenuta il 7 novembre 1997,
in forma definitiva il 24 aprile 1998.

Ph. Clément:
Department of Technical Mathematics and Informatics
Delft University of Technology - P. O. Box 5031
2600 GA DELFT (Paesi Bassi)
ph.p.j.e.clement@twi.tudelft.nl

S. Guerre-Delabrière:
Equipe d'Analyse, Case 186 - Université Paris 6
4, Place Jussieu - 75252 PARIS Cedex 05 (Francia)
syg@ccr.jussieu.fr