Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Giuseppe Da Prato, Arnaud Debussche

Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton-Jacobi equation

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.4, p. 267–277.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_4_267_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:267-277 (1998)

Analisi matematica. — *Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton-Jacobi equation.* Nota di
Giuseppe Da Prato e Arnaud Debussche, presentata (*) dal Corrisp. G. Da Prato.

Abstract. — We consider a stochastic Burgers equation. We show that the gradient of the corresponding transition semigroup $P_t\varphi$ does exist for any bounded φ, and can be estimated by a suitable exponential weight. An application to some Hamilton-Jacobi equation arising in Stochastic Control is given.

Key words: Stochastic control problem; Burgers equation; Hamilton-Jacobi equation.

Riassunto. — *Differenziabilità del semigruppo di transizione dell'equazine di Burgers stocastica e applicazione all'equazione di Hamilton-Jacobi corrispondente.* Si considera un'equazione di Burgers stocastica. Si prova che il gradiente del semigruppo di transizione corrispondente $P_t\varphi$ esiste per ogni φ limitata e che può essere stimato con un opportuno peso esponenziale. Viene data un'applicazione ad una equazione di Hamilton-Jacobi che interviene in un problema di controllo stocastico.

1. Introduction

We consider the stochastic Burgers equation

$$(1.1)\qquad \begin{cases} dX = \left(AX + \dfrac{1}{2}\,\dfrac{\partial}{\partial\xi}(X^2)\right)dt + \sqrt{Q}dW,\ \ \xi \in [0\,,1]\,,\ t \ge 0\,, \\ X(t\,,0) = X(t\,,1) = 0\,,\ \ t \ge 0\,, \\ X(0\,,\xi) = x(\xi)\,,\ \ \xi \in [0\,,1]\,, \end{cases}$$

where $x \in L^2(0\,,1)$.

Here W is a cylindrical Wiener process on $L^2(0\,,1)$, adapted to a stochastic basis $(\Omega\,,\mathcal{F}\,,\{\mathcal{F}_t\}_{t\ge 0}\,,\mathbb{P})$. Moreover Q is a symmetric linear operator on $L^2(0\,,1)$, and A is the unbounded operator on $L^2(0\,,1)$ defined by

$$Ax = \partial^2 x/\partial\xi^2\,,\ \ D(A) = H^2(0\,,1) \cap H_0^1(0\,,1).$$

Existence and uniqueness of a solution $X(\cdot\,,x)$ to (1.1), have been proved in [5]. We consider the transition semigroup

$$(1.2)\qquad P_t\varphi(x) = \mathbb{E}[\varphi(X(t\,,x))]\,,\ \ x \in H,\ \ \varphi \in B_b(H)\ (^1)\,,$$

where $H = L^2(0\,,1)$. In [6] it is proved that the semigroup P_t, $t \ge 0$ is Strong Feller, that is if $\varphi \in B_b(H)$ and $t > 0$ then $P_t\varphi$ is continuous. In this paper we show that,

(*) Nella seduta del 18 giugno 1998.
(1) $B_b(H)$ is the set of all bounded Borel real functions on H.

under suitable assumptions on Q, $P_t\varphi$ belongs to $C^1(H)$ (2) and that the following estimate holds

$$|DP_t\varphi(x)| \le C_T t^{-\gamma} e^{\delta|x|^2} \|\varphi\|_0 \,, \quad t \in [0\,,\,T]\,, \ x \in H, \tag{1.3}$$

where $\delta > 0$ is arbitrary and $C_T > 0$, $\gamma < 1$ are suitable constants depending on δ.

This result allows us to prove existence and uniqueness of a regular solution of the Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{d}{dt}\, u(t,x) &= \dfrac{1}{2}\,\mathrm{Tr}\,[QD^2u(t,x)] + \left(Ax + \dfrac{1}{2}\,(x^2)_\xi\,,\, Du(t,x)\right) + \\ & \qquad - F(x, Du(t,x)) + g(x)\,, \\ u(0\,,x) &= \varphi(x) \end{cases} \tag{1.4}$$

where F is Lipschitz continuous with respect to Du and exponentially decaying with respect to x. Moreover φ and $g \in C_b(H)$ and D denotes derivatives with respect to x.

Hamilton-Jacobi equations in Hilbert spaces of the form

$$\begin{cases} \dfrac{d}{dt}\, u(t,x) &= \dfrac{1}{2}\,\mathrm{Tr}\,[QD^2u(t,x)] + (Ax + f(x)\,,\, Du(t,x)) + \\ & \qquad -F(x, Du(t,x)) + g(x)\,, \\ u(0\,,x) &= \varphi(x) \end{cases}$$

have been studied in [2, 9], under the assumption that f is a Lipschitz continuous mapping from H into H, by doing a fixed point in a space of C^1 functions. These results can be generalized to the case where f has a polynomial growth, see [3].

The more singular equation (1.4) was studied by the authors, [4], using Hopf transform. This requires a special quadratic form of the Hamiltonian.

In this paper we will solve equation (1.4) in a more general situation by using estimate (1.3) and by doing a fixed point in a suitable space of functions having exponential growth.

This result can be applied, using an usual argument of Dynamic Programming to the following optimal control problem,

Minimize:

$$J(x,z) = \mathbb{E}\left(\int_0^T \left[g(Y(s)) + \frac{1}{2}\, h(Y(s))|z(s)|^2\right] ds + \varphi(Y(T))\right), \tag{1.5}$$

over controls z that are adapted to W, and such that $|z(s)| \le R$, where $R > 0$ is fixed

(2) $C_b(H)$ is the Banach space of all continuous and bounded mappings from H into $\mathbb{R}$, endowed with the norm $\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|$. Moreover $C^1(H)$ is the set of all functions in $C_b(H)$ that are Fréchet differentiable.

and subjected to the state equation

$$
(1.6) \qquad \begin{cases} dY = \left(AY + \dfrac{1}{2}\dfrac{\partial}{\partial\xi}(Y^2) + \sqrt{Q}z\right)dt + \sqrt{Q}dW, \quad \xi \in [0\,,1]\,,\ t \geq 0\,, \\ Y(t\,,0) = Y(t\,,1) = 0\,, \quad t \geq 0\,, \\ Y(0\,,\xi) = x(\xi)\,, \quad \xi \in [0\,,1]\,. \end{cases}
$$

Moreover $g\,,h\,,\varphi$ are nonnegative functions on $C_b(H)$, and

$$
(1.7) \qquad h(x) \leq Ce^{-\varepsilon|x|^2}\,, \quad x \in H,
$$

for some $C\,,\varepsilon > 0$.

In this case the related Hamilton-Jacobi equation is equation (1.4) with

$$
(1.8) \qquad F(x\,,z) = \begin{cases} (1/2)\,|z|^2 & \text{if } |z| \leq R, \\ \left(R|z| - R^2/2\right) h(x) & \text{if } |z| \geq R, \end{cases}
$$

see *e.g.* [8].

2. Estimate of the solution

Let $H = L^2(0\,,1)$ be endowed with the usual norm and inner product denoted by $|\cdot|$ and $(\cdot\,,\cdot)$. As usual, $H^k(0\,,1)$, $k \in \mathbb{N}$, is the Sobolev space of all functions in H whose derivatives up to the order k belong to H, and $H_0^1(0\,,1)$ is the subspace of $H^1(0\,,1)$ of all functions whose traces at 0 and 1 vanish.

The operator A is selfadjoint, strictly negative and has a compact inverse. We can define $(-A)^s$ and $D((-A)^s)$ for any $s \in \mathbb{R}$. For $s = \frac{1}{2}$, we have $D((-A)^{1/2}) = H_0^1(0\,,1)$ and its norm and inner product are denoted by

$$
\|x\| = |(-A)^{1/2}x|\,,\ ((x\,,y)) = \left((-A)^{1/2}x\,,(-A)^{1/2}y\right)\,,\ x\,,y \in H_0^1(0\,,1).
$$

The sequence of eigenvalues of A is

$$
\lambda_k = -k^2\pi^2\,,\ k \in \mathbb{N}\,,
$$

it is associated to the orthonormal basis of eigenvectors $\{e_k\}_{k\in\mathbb{N}}$,

$$
e_k = \sqrt{2/\pi}\ \sin k\xi\,,\ k \in \mathbb{N}\,,\ \xi \in [0\,,1].
$$

For any positive integer m we denote by P_m the orthogonal projector on the space spanned by $e_1\,,\dots\,,e_m$.

We also consider a linear operator Q which is assumed to be symmetric, nonnegative and of trace class, a cylindrical Wiener process W on H associated to a stochastic basis $(\Omega\,,\mathcal{F}\,,\{\mathcal{F}_t\}_{t\geq 0}\,,\mathbb{P})$.

For $x \in H_0^1(0\,,1)$ we set

$$
B(x) = \frac{1}{2}\,\frac{\partial}{\partial\xi}(x^2).
$$

For any $x \in H$, equation (1.1) has a unique solution, see [5]. This solution can be constructed as the limit of Galerkin approximations. For $m \in \mathbb{N}$, we define B_m by

$$B_m(x) = \frac{1}{2} P_m \left[\frac{\partial}{\partial \xi} (P_m x)^2 \right], \quad x \in L^2(0,1),$$

and consider the following Galerkin approximation of (1.1):

$$\begin{cases} dX_m = (AX_m + B_m(X_m))dt + \sqrt{Q_m}\, dW, \\ X_m(0) = x_m = P_m x, \end{cases} \tag{2.1}$$

where $Q_m = P_m Q P_m$. Notice the crucial identity

$$(B_m(X_m), X_m) = 0. \tag{2.2}$$

We start by estimating $|X_m(t)|$ and $\|X_m(t)\|$.

Proposition 2.1. *For all* $T > 0$ *there exists* $C_T > 0$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_m(t)|^2 + \int_0^T \|X_m(s)\|^2 ds \right) \le C_T(|x^2| + 1). \tag{2.3}$$

Proof. For any $m \in \mathbb{N}$, we have from Itô's formula and (2.2), that

$$|X_m(t)|^2 + 2\int_0^t \|X_m(s)\|^2 ds = |x^2| + t \operatorname{Tr} Q_m + 2\int_0^t \left(X_m(s), \sqrt{Q_m}\, dW_s \right), \tag{2.4}$$

that yields

$$\mathbb{E}\left(|X_m(t)|^2 + 2\int_0^t \|X_m(s)\|^2 ds \right) = t \operatorname{Tr} Q_m + |x^2|. \tag{2.5}$$

Moreover by (2.4) it follows

$$\sup_{t\in[0,T]} |X_m(t)|^2 \le |x^2| + T \operatorname{Tr} Q_m + 2 \sup_{t\in[0,T]} \int_0^t \left(X_m(s), \sqrt{Q_m}\, dW_s \right).$$

Here we have dropped expectation of the Itô integral term. This fact can be justified approximating $|X_m(t)|^2$ with $|X_m(t)|^2/(1+\delta|X_m(t)|^2)$ and then letting δ tend to 0.

Taking expectation, and using a well known martingale inequality, we find

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_m(t)|^2 \right) \le |x^2| + T \operatorname{Tr} Q_m + 4 \left[\mathbb{E}\left(\int_0^T |\sqrt{Q_m} X_m(s)|^2 ds \right) \right]^{1/2}.$$

By (2.5) and the boundedness of Q there exists $C_{1,T} > 0$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_m(t)|^2 \right) \le C_{1,T}(|x^2| + 1).$$

Now the conclusion follows from (2.5). □

Now we want to estimate exponential moments of $|X_m(t)|$.

PROPOSITION 2.2. *Let* $\varepsilon \le \varepsilon_0 = \pi^2/(2\|Q\|)$ *, then for any* $t \in [0, T]$ *and* $m \in \mathbb{N}$ *,*

$$\mathbb{E}\left(e^{\varepsilon|X_m(t)|^2+\varepsilon\int_0^t \|X_m(s)\|^2 ds}\right) \le e^{\varepsilon|x|^2+\varepsilon t \operatorname{Tr} Q}. \tag{2.6}$$

PROOF. We set

$$Z(t) = |X_m(t)|^2 + \int_0^t \|X_m(s)\|^2 ds,\ t \le [0, T].$$

Then we have

$$dZ(t) = \left(-\|X_m(t)\|^2 + \operatorname{Tr} Q_m\right) dt + 2\left(X_m, \sqrt{Q_m}\, dW_t\right).$$

By Itô's formula applied to $e^{\varepsilon Z(t)}$, it follows that

$$de^{\varepsilon Z(t)} = \varepsilon e^{\varepsilon Z(t)}\left[-\|X_m(t)\|^2 + 2\varepsilon|\sqrt{Q_m}\, X_m|^2 + \operatorname{Tr} Q_m\right] dt + \\ + 2\varepsilon e^{\varepsilon Z(t)}\left(X_m(t), \sqrt{Q_m}\, dW_t\right).$$

Then, integrating and taking expectation we obtain

$$\begin{aligned}\mathbb{E}\left(e^{\varepsilon Z(t)}\right) &= e^{\varepsilon|x|^2} + \varepsilon \operatorname{Tr} Q_m \int_0^t \mathbb{E}\left(e^{\varepsilon Z(s)}\right) ds + \\ &+ \varepsilon\mathbb{E}\left(e^{\varepsilon Z(t)}\int_0^t (-\|X_m(s)\|^2 + 2\varepsilon|\sqrt{Q_m}\, X_m(s)|^2) ds\right).\end{aligned} \tag{2.7}$$

Here we have dropped expectation of the Itô integral term. This fact can be justified approximating $e^{\varepsilon Z(t)}$ with $e^{\varepsilon Z(t)}/(1+\delta e^{\varepsilon Z(t)})$ and then letting δ tend to 0. Since

$$2\varepsilon|\sqrt{Q_m}\, X_m(s)|^2 \le 2\varepsilon\|Q_m\|\, |X_m(s)|^2 \le 2\varepsilon\pi^{-2}\|Q_m\|\, \|X_m(s)\|^2 \le \|X_m(s)\|^2,$$

by (2.7) we have

$$\mathbb{E}\left(e^{\varepsilon Z(t)}\right) \le e^{\varepsilon|x|^2} + \varepsilon \operatorname{Tr} Q_m \int_0^t \mathbb{E}\left(e^{\varepsilon Z(s)}\right) ds,$$

and the conclusion follows from Gronwall's lemma. □

Now, by Proposition 2.2 we obtain, letting m tend to infinity,

PROPOSITION 2.3. *Let* $X(\cdot)$ *be the mild solution to problem* (1.1)*. Let* $\varepsilon \le \varepsilon_0 = \pi^2/(2\|Q\|)$*, then for any* $t \in [0, T]$

$$\mathbb{E}\left(e^{\varepsilon|X(t)|^2+\varepsilon\int_0^t \|X(s)\|^2 ds}\right) \le e^{\varepsilon|x|^2+\varepsilon t \operatorname{Tr} Q}. \tag{2.8}$$

3. ESTIMATES OF DERIVATIVES

In this Section we are going to prove, using (2.8), some estimates on derivatives of the transition semigroup of Burgers equation. Derivation of our estimates are not rigorous but they can be easily justified by considering Galerkin approximations as in the previous Section.

We start with first derivatives. Let $h \in H$, setting

$$\eta^h(t,x) = \eta(t) = D_x X(t,x)h,$$

we have

$$\left\{ \begin{array}{l} d\eta/dt = A\eta + (X\eta)_\xi \\ \\ \eta(0) = h. \end{array} \right. \tag{3.1}$$

Proposition 3.1. *There exists $C_1 > 0$ such that*

$$|\eta(t)|^2 + \int_0^t \|\eta(s)\|^2 ds \le e^{C_1 \int_0^t \|X(s)\|^{4/3} ds} |h|^2. \tag{3.2}$$

Proof. We start from the equality

$$\frac{1}{2}\frac{d}{dt}|\eta(t)|^2 + \|\eta(t)\|^2 = \int_0^1 (X(t)\eta(t))_\xi \eta(t)\, d\xi =$$
$$= -\int_0^1 \eta(t)\eta_\xi(t)X(t)d\xi = \frac{1}{2}\int_0^1 \eta^2(t)X_\xi(t)d\xi.$$

It follows

$$\frac{1}{2}\frac{d}{dt}|\eta(t)|^2 + \|\eta(t)\|^2 \le \frac{1}{2}\|X(t)\||\eta(t)|^2_{L^4} \le$$
$$\le \frac{C}{2}\|X(t)\||\eta(t)|^2_{H^{1/4}} \le \frac{C}{2}\|X(t)\||\eta(t)|^{3/2}\|\eta(t)\|^{1/2},$$

where we have used the embedding $H^{1/4}(0,1) \subset L^4(0,1)$, and a well known interpolatory inequality. Since

$$ab \le (3/4)\, a^{4/3} + (1/4)\, b^4, \quad a, b > 0,$$

setting

$$a = \frac{C}{2}\|X(t)\||\eta(t)|^{3/2}, \quad b = \|\eta(t)\|^{1/2},$$

we find

$$\frac{1}{2}\frac{d}{dt}|\eta(t)|^2 + \|\eta(t)\|^2 \le \frac{3}{4}\left(\frac{C}{2}\right)^{4/3}\|X(t)\|^{4/3}|\eta(t)|^2 + \frac{1}{4}\|\eta(t)\|^2.$$

It follows

$$\frac{d}{dt}|\eta(t)|^2 + \|\eta(t)\|^2 \le C_1\|X(t)\|^{4/3}|\eta(t)|^2,$$

that implies, by a well known comparison result,

$$|\eta(t)|^2 \le e^{C_1\int_0^t \|X(s)\|^{4/3}ds}|h|^2 - \int_0^t e^{C_1\int_s^t \|X(\sigma)\|^{4/3}d\sigma}\|\eta(s)\|^2 ds\, |h|^2,$$

which yields (3.2). □

Proposition 3.2. *For all* $T>0$ *and* $\varepsilon \in (0, \varepsilon_0)$, *there exists* $C_{\varepsilon, T}>0$ *such that*

$$\mathbb{E}\left(|\eta(t)|^2 + \int_0^t \|\eta(s)\|^2 ds\right) \le C_{\varepsilon, T}|h|^2 e^{\varepsilon|x|^2}. \tag{3.3}$$

Proof. For any $\varepsilon \in (0, \varepsilon_0)$ there exists $C_\varepsilon > 0$ such that

$$C_1 \int_0^t \|X(s)\|^{4/3} ds \le \varepsilon \int_0^t \|X(s)\|^2 ds + C_\varepsilon t.$$

Then by (3.2) it follows

$$\mathbb{E}\left(|\eta(t)|^2 + \int_0^t \|\eta(s)\|^2 ds\right) \le \mathbb{E}\left(e^{\varepsilon \int_0^t \|X(s)\|^2 ds + C_\varepsilon t}\right) |h|^2.$$

Now the conclusion follows from Proposition 2.3. □

We now consider second derivatives. Let $h \in H$, setting

$$\zeta^h(t, x) = \zeta(t) = D_x^2 X_m(t, x)(h, h),$$

we have

$$\begin{cases} \dfrac{d\zeta}{dt} = A\zeta + (X\zeta)_\xi + \left(\eta^2\right)_\xi \\ \zeta(0) = 0. \end{cases} \tag{3.4}$$

Proposition 3.3. *There exists* $C_2, C_3 > 0$ *such that*

$$|\zeta(t)|^2 + \int_0^t \|\zeta(s)\|^2 ds \le C_3 e^{C_2 \int_0^t \|X(s)\|^{4/3} ds} |h|^4. \tag{3.5}$$

Proof. We have, arguing as in the proof of Proposition 3.1,

$$\frac{1}{2}\frac{d}{dt}|\zeta(t)|^2 + \|\zeta(t)\|^2 = \frac{1}{2}\int_0^1 \zeta^2(t) X_\xi(t) d\xi - \int_0^1 \zeta_\xi(t)\eta^2(t) d\xi \le$$

$$\le C_1 \|X(t)\|^{4/3}|\zeta(t)|^2 + \frac{1}{2}\|\zeta(t)\|^2 + |\eta(t)|_{L^4}^4.$$

It follows

$$\begin{aligned} |\zeta(t)|^2 + \int_0^t \|\zeta(s)\|^2 ds &\le \int_0^t e^{C_1 \int_s^t \|X(\sigma)\|^{4/3} d\sigma} |\eta(s)|_{L^4}^4\, ds \le \\ &\le e^{C_1 \int_0^t \|X(\sigma)\|^{4/3} d\sigma} \int_0^t |\eta(s)|_{L^4}^4\, ds. \end{aligned} \tag{3.6}$$

On the other hand by Sobolev embedding and interpolation inequality:

$$|\eta(t)|_{L^4}^4 \le C_3 |\eta(t)|^2\, \|\eta(t)\|^2$$

and by (3.2) we have

$$\int_0^t |\eta(s)|^4 ds \le C_3 e^{2C_1 \int_0^t \|X(s)\|^{4/3} ds} |h|^4.$$

Now the conclusion follows by substituting in (3.6). □

Finally the following result is proved as Proposition 3.2

PROPOSITION 3.4. *For all* $T > 0$ *and* $\varepsilon > 0$ *there exists* $C_{\varepsilon,T} > 0$ *such that*

$$\mathbb{E}\left(|\zeta(t)|^2 + \int_0^t \|\zeta(s)\|^2 ds\right) \le C_{\varepsilon,T}|h|^4 e^{\varepsilon|x|^2}. \tag{3.7}$$

4. Regularity of the transition semigroup

Here we assume that

$$|Q^{-1/2}x| \le C|(-A)^{-\beta/2}x|,\ x \in H, \tag{4.1}$$

for some $C > 0$ and $\beta \in (1/2, 1)$. For simplicity we take $Q = (-A)^{-\beta}$. Now we prove differentiability of the transition semigroup by using the Bismut–Elworthy formula, see [1, 7].

PROPOSITION 4.1. *Assume that* (4.1) *holds, and let* $h \in H$ *and* $\varphi \in C_b(H)$. *Then* $P_t\varphi$ *is twice differentiable and*

$$\left(DP_t\varphi(x), h\right) = \frac{1}{t}\,\mathbb{E}\left[\varphi(X(t,x))\int_0^t \left(Q^{-1/2}\eta^h(s), dW(s)\right)\right]. \tag{4.2}$$

Moreover for any $T > 0$ *and* $\varepsilon > 0$ *there exists* $C_{\varepsilon,T} > 0$ *such that*

$$|DP_t\varphi(x)| \le C_{\varepsilon,T}t^{-(1+\beta)/2}\,\|\varphi\|_0 e^{\varepsilon|x|^2},\ t \in [0, T], \tag{4.3}$$

and

$$\|D^2P_t\varphi(x)\| \le C_{\varepsilon,T}t^{-1-\beta}\,\|\varphi\|_0 e^{\varepsilon|x|^2},\ t \in [0, T]. \tag{4.4}$$

PROOF. We first note that by interpolation

$$|Q^{-1/2}\eta| \le C|(-A)^{-\beta/2}\eta| \le C|\eta|^{1-\beta}\|\eta\|^{\beta}.$$

Therefore

$$\mathbb{E}\int_0^t |Q^{-1/2}\eta(s)|^2 ds \le C\mathbb{E}\int_0^t |\eta(s)|^{2(1-\beta)}\|\eta(s)\|^{2\beta} ds \le$$

$$\le C\left[\mathbb{E}\int_0^t |\eta(s)|^2 ds\right]^{1-\beta}\left[\mathbb{E}\int_0^t \|\eta(s)\|^2 ds\right]^{\beta}.$$

By Proposition 3.2 it follows

$$\mathbb{E}\int_0^t |Q^{-1/2}\eta(s)|^2 ds \le C_{\varepsilon,T}t^{1-\beta}|h|^2 e^{\varepsilon|x|^2}. \tag{4.5}$$

Consequently

$$|\left(DP_t\varphi(x), h\right)|^2 \le t^{-2}\,\|\varphi\|_0^2\mathbb{E}\left[\int_0^t |Q^{-1/2}\eta(s)|^2 ds\right] \le C_{\varepsilon,T}^2\,t^{-1-\beta}\,\|\varphi\|_0^2|h|^2 e^{\varepsilon|x|^2},$$

and (4.3) is proved. (4.4) can proved similarly, using Proposition 3.4 and the semigroup property of P_t.

5. Hamilton-Jacobi equation

We are here concerned with the following Hamilton-Jacobi equation

$$(5.1)\begin{cases} \dfrac{d}{dt}\, u(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[QD^2 u(t,x)] + (Ax + B(x)\,,\, Du(t,x)) - F(x, Du(t,x)) + g(x)\,, \\ u(0\,,x) \qquad = \varphi(x) \end{cases}$$

for $x \in H$, and $t > 0$. We have denoted by $B(x) = (1/2)\,(x^2)_\xi$, and F is a continuous and bounded mapping from $H \times H$ into H, such that

$$|F(x, y_1) - F(x, y_2)| \le L(x)|y_1 - y_2|\,,\ y_1\,, y_2 \in H, \tag{5.2}$$

where $L(x)$ is such that

$$L(x) \le k e^{-\gamma|x|^2}\,, \tag{5.3}$$

for some $k > 0$ and $\gamma > 0$. Moreover $\varphi\,, g \in C_b(H)$.

We write problem (5.1) in the mild form

$$u(t,\cdot) = P_t\varphi + \int_0^t P_{t-s}[(F(\cdot\,, Du(s,\cdot))) + g]ds. \tag{5.4}$$

We define $C^1_\gamma(H)$ as the space of all functions of $C_b(H)$ that are differentiable and such that

$$\|D\varphi\|_{0\,,\gamma} = \sup e^{-\gamma|x|^2}|D\varphi(x)| < \infty.$$

Equation (5.4) will be solved by a fixed point argument in the Banach space $Z_{T,\gamma}$ consisting of all mappings $u : [0\,, T] \times H \to \mathbb{R}$ such that $u \in C([0\,,T] \times H)$, for all $t \in (0\,, T]$ $u(t,\cdot) \in C^1_\gamma(H)$, and

$$\|u\|_{Z_{T,\gamma}} = \sup_{t\in[0\,,T]} \|u(t,\cdot)\|_0 + \sup_{t\in(0\,,T]} t^{\frac{1+\beta}{2}}\,\|Du(t,\cdot)\|_{0\,,\gamma} < +\infty.$$

Theorem 5.1. *Assume that F is continuous and satisfies* (5.2), (5.3)*, and that φ and g are in $C_b(H)$. Then there exists a unique mild solution to problem* (5.1) *in $Z_{T,\gamma}$.*

Proof. We shall denote by C any constant. We write equation (5.4) as

$$u = \Gamma(u) + z,$$

where

$$z(t,\cdot) = P_t\varphi + \int_0^t P_{t-s}\, g\, ds,$$

and

$$\Gamma(u)(t,\cdot) = \int_0^t P_{t-s}F(\cdot\,, Du(s,\cdot))ds.$$

Step 1. $z \in Z_{T,\gamma}$.

We clearly have

$$\|P_t\varphi\|_0 \le C\|\varphi\|_0\,,$$

and

$$\left\| \int_0^t P_{t-s} g \, ds \right\|_0 \le T \|g\|_0.$$

Moreover by Proposition 4.1

$$|DP_t\varphi|_{0,\gamma} = \sup e^{-\gamma|x|^2} |DP_t\varphi(x)| \le Ct^{-(1+\beta)/2} \, \|\varphi\|_0 \,,$$

and

$$\left| D \int_0^t P_{t-s} g \, ds \right|_{0,\gamma} \le C \int_0^t (t-s)^{-(1+\beta)/2} ds \, \|g\|_0 \le C\|g\|_0.$$

Step 2. Γ maps $Z_{T,\gamma}$ into $Z_{T,\gamma}$.

Let $u \in Z_{T,\gamma}$ then

$$\|\Gamma(u)(t,\cdot)\|_0 \le \int_0^t \|F(\cdot\,, Du(s,\cdot))\|_0 ds,$$

and by (5.2), (5.3)

$$\begin{aligned}|F(x, Du(s,x))| \le L(x)|Du(s,x)| + |F(x,0)| \le k|Du(s,\cdot)|_{0,\gamma} + \|F(\cdot\,,0)\|_0 \le \\ \le ks^{-(1+\beta)/2}\|u\|_{Z_{T,\gamma}} + \|F(\cdot\,,0)\|_0.\end{aligned}$$

Thus

$$\|\Gamma(u)(t,\cdot)\|_0 \le k \int_0^t s^{-\frac{1+\beta}{2}} ds \, \|u\|_{Z_{T,\gamma}} + T\|F(\cdot\,,0)\|_0.$$

Also by Proposition 4.1

$$\begin{aligned}\|D\Gamma(u)(t,\cdot)\|_{0,\gamma} &\le C \int_0^t (t-s)^{-(1+\beta)/2} \|F(\cdot\,, Du(s,\cdot))\|_0 ds \le \\ &\le C \int_0^t (t-s)^{-(1+\beta)/2} s^{-(1+\beta)/2} \, ds \, \|u\|_{Z_{T,\gamma}} + C \int_0^t (t-s)^{-(1+\beta)/2} \|F(\cdot\,,0)\|_0 ds \le \\ &\le Ct^{-(1+\beta)/2}\|u\|_{Z_{T,\gamma}} + C\|F(\cdot\,,0)\|_0.\end{aligned}$$

Step 3. Γ is a contraction.

We argue as in Step 2, for $u, v \in Z_{T,\gamma}$

$$\|\Gamma(u)(t,\cdot) - \Gamma(v)(t,\cdot)\|_0 \le \int_0^t \|F(\cdot\,, Du(s,\cdot)) - F(\cdot\,, Dv(s,\cdot))\|_0 ds,$$

and, by (5.2), (5.3)

$$\|F(\cdot\,, Du(s,\cdot)) - F(\cdot\,, Dv(s,\cdot))\| \le k|Du(s,\cdot) - Dv(s,\cdot)|_{0,\gamma}\,,$$

so that

$$\|\Gamma(u)(t,\cdot) - \Gamma(v)(t,\cdot)\|_0 \le k \int_0^t s^{-(1+\beta)/2} ds \, \|u - \|_{Z_{T,\gamma}} \le \frac{2k}{1-\beta} \, T^{(1-\beta)/2} \, \|u - \|_{Z_{T,\gamma}}.$$

Moreover by Proposition 4.1

$$\|D\Gamma(u)(t,\cdot)-D\Gamma(v)(t,\cdot)\|_{0,\gamma}\leq C\int_0^t(t-s)^{-(1+\beta)/2}\|F(\cdot\,,Du(s,\cdot))-F(\cdot\,,Dv(s,\cdot))\|_0 ds\leq$$

$$\leq C\int_0^t (t-s)^{-(1+\beta)/2}s^{-(1+\beta)/2}\,ds\,\|u-v\|_{Z_{T,\gamma}}\leq Ct^{(1-\beta)/2}\|u\|_{Z_{T,\gamma}}.$$

We deduce

$$\|\Gamma(u)-\Gamma(v)\|_{Z_{T,\gamma}}\leq CT^{\frac{1-\beta}{2}}\|u-\|_{Z_{T,\gamma}}.$$

Thus there exists $T_0>0$ such that Γ is a contraction on $Z_{T_0,\gamma}$ and we have existence and uniqueness of a mild solution of (5.1) in $Z_{T_0,\gamma}$. This solution can be continued to $Z_{T,\gamma}$ in a standard way. □

References

[1] J. M. Bismut, *Large deviations and the Malliavin Calculus*. Birkhäuser, 1984.
[2] P. Cannarsa - G. Da Prato, *Some results on nonlinear optimal control problems and Hamilton-Jacobi equations in infinite dimensions*. J. Funct. Anal., 90, 1990, 27-47.
[3] S. Cerrai, *Optimal control problems for reaction-diffusion equations by a Dynamic Programming approach*. Scuola Normale Superiore, Pisa 1998, preprint.
[4] G. Da Prato - A. Debussche, *Control of the stochastic Burgers model of turbulence*. Siam Journal on Control and Optimization, to appear.
[5] G. Da Prato - A. Debussche - R. Temam, *Stochastic Burgers equation*. NoDEA, 1994, 389-402.
[6] G. Da Prato - J. Zabczyk, *Ergodicity for Infinite Dimensional Systems*. London Mathematical Society Lecture Notes, 229, 1996.
[7] K. D. Elworthy, *Stochastic flows on Riemannian manifolds*. In: M. A. Pinsky - V. Vihstutz (eds.), *Diffusion processes and related problems in analysis*. Birkhäuser, 1992, vol. II, 33-72.
[8] W. H. Fleming - R. W. Rishel, *Deterministic and Stochastic Optimal Control*. Springer-Verlag, 1975.
[9] F. Gozzi, *Regularity of solutions of a second order Hamilton-Jacobi equation and application to a control problem*. Commun. in partial differential equations, 20, 1995, 775-826.

Pervenuta il 19 marzo 1998,
in forma definitiva il 2 giugno 1998.

G. Da Prato:
Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa
daprato@sns.it

A. Debussche:
CNRS et Université de Paris Sud
91405 Orsay (Francia)