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On the Cauchy problem for a class of parabolic equations with variable density

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Equazioni a derivate parziali. — *On the Cauchy problem for a class of parabolic equations with variable density.* Nota (*) di Shoshana Kamin, Robert Kersner e Alberto Tesei, presentata dal Corrisp. A. Tesei.

Abstract. — The well-posedness of the Cauchy problem for a class of parabolic equations with variable density is investigated. Necessary and sufficient conditions for existence and uniqueness in the class of bounded solutions are proved. If these conditions fail, sufficient conditions are given to ensure well-posedness in the class of bounded solutions which satisfy suitable constraints at infinity.

Key words: Cauchy problem; Well-posedness; Conditions at infinity.

Riassunto. — *Sul problema di Cauchy per una classe di equazioni paraboliche con densità variabile.* Si studia la buona posizione del problema di Cauchy per una classe di equazioni paraboliche con densità variabile. Si ricavano condizioni necessarie e sufficienti per l'esistenza e l'unicità nella classe delle soluzioni limitate. Se tali condizioni non sono verificate, si danno condizioni sufficienti a garantire la buona posizione del problema nella classe delle soluzioni limitate che all'infinito soddisfano opportune restrizioni.

1. Introduction

In this paper we investigate existence and uniqueness of nonnegative solutions to the following Cauchy problem:

$$(P) \qquad \begin{cases} \rho u_t = \{a[G(u)]_x\}_x & \text{in } \mathbb{R} \times (0\,,\,T] =: S \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}\ . \end{cases}$$

Here ρ (referred to as density) and a are positive functions of the space variable, u_0 is nonnegative; concerning G, a typical choice is $G(u) = u^m\,,\ m \geq 1$. Precise assumptions will be made in the following (see Section 2).

The motivation of our study comes from the problem:

$$(1.1) \qquad \begin{cases} \rho\, u_t = \Delta[G(u)] & \text{in } \mathbb{R}^N \times (0\,,\,T] \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}\ , \end{cases}$$

which arises in situations of physical interest (see [16]). If ρ and u_0 are radially symmetric, radially symmetric solutions of (1.1) satisfy the equation:

$$(1.2) \qquad \rho\, u_t = \frac{1}{r^{N-1}}\,\{r^{N-1}[G(u)]_r\}_r$$

(where $r := |x|$), which is a particular case of the differential equation in (P).

(*) Pervenuta in forma definitiva all'Accademia il 16 luglio 1998.

An interesting feature of problem (1.1) is that it can be ill-posed in the set of bounded solutions, depending on the behaviour of ρ as $|x| \to \infty$ and on the space dimension N. In fact, it turns out to be well-posed in the class of bounded solutions for any smooth, positive ρ if $N \leq 2$. On the other hand, if $N \geq 3$ and $\rho \to 0$ sufficiently fast as $|x| \to \infty$ (depending on N), some *conditions at infinity* are needed to restore well-posedness (see [6-8, 16]). We expect that a similar ill-posedness arises for problem (P), depending on the behaviour of both ρ and a as $|x| \to \infty$.

It is well known that the Cauchy problem for linear parabolic equations is ill-posed, if the coefficients of the equation do not satisfy suitable growth conditions as $|x| \to \infty$ (*e.g.*, see [15]). After the pioneering papers by Holmgren [14], Täcklind [25], Tikhonov [26] and Widder [27] concerning the classes of well-posedness for the heat equation, the same question was raised for general parabolic equations and systems by Petrowsky (see [20]). Several results obtained in this direction are accounted for in [17] (see also [1, 4, 5, 11, 15, 19, 21, 23, 24, 28]). Results concerning existence, uniqueness and smoothing properties of solutions to parabolic equations with unbounded coefficients in $\mathbb{R}^N$ have been recently obtained in [18].

As it is known, probability methods are successfully applied to investigate linear partial differential equations. For the present case, interesting results concerning problem (P) with $G(u) = u$ have been obtained in the framework of the probabilistic theory of diffusion processes (see [9]; see also [10, 12, 13, 22]).

In this paper we prove necessary and sufficient conditions for the well-posedness of problem (P) in the class of bounded solutions (see Corollary 2.1). When these conditions are not satisfied, the lack of *uniqueness* in this class is established by proving *existence* in some more restricted class of solutions to (P). Such a class is defined as consisting of bounded solutions of (P), which satisfy some additional conditions, namely

$$\int_0^T G(u(x,t))\,dt \to 0 \qquad \text{as } |x| \to \infty \ .$$

A constraint like the one above can be regarded as a *condition at infinity*, which is needed to restore the well-posedness of the Cauchy problem (P).

The conditions we give for the well-posedness of (P) (either in the class of bounded solutions or in some more restricted class) depend on the behaviour as $|x| \to \infty$ of solutions both to the ordinary differential equation

$$(ay')' = -\rho \qquad \text{in } \mathbb{R} \tag{1.3}$$

and to the associated homogeneous differential equation, namely:

$$(a\,y')' = 0 \qquad \text{in } \mathbb{R}\,. \tag{1.4}$$

Our conditions generalize those given in [9] for (P) in the linear case $G(u) = u$ (see Remark 2.1); they also suggest possible extensions of the present results to space dimension $N > 1$ (in this respect, see Section 6 for a discussion of the case with radial symmetry).

2. Mathematical framework and results

The following assumption will be made throughout the paper:

$$(H_0)\qquad \begin{cases} (i) & \rho \in C(\mathbb{R})\,,\ a \in C^1(\mathbb{R})\,,\ \rho > 0\,,\ a > 0\ ; \\ (ii) & G \in C^1([0\,,\infty)) \cap C^{2+\sigma}((0\,,\infty))\ , \\ & G(0) = 0\,,\ G'(s) > 0 \text{ for any } s > 0\ , \\ & G' \text{ increasing in } (0\,,\delta) \text{ if } G'(0) = 0\ ; \\ (iii) & u_0 \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})\,,\ u_0 \geq 0\ . \end{cases}$$

By a weak solution of problem (P) we mean a function $u = u(x, t)$ continuous and nonnegative in S such that:

$$\begin{aligned} \int_0^\tau \int_{x_1}^{x_2} \{\rho\, u\, \psi_t + G(u)\,[a\psi_x]_x\}\, dx\, dt &= \int_{x_1}^{x_2} \rho[u(x, \tau)\psi(x, \tau) - u_0(x)\psi(x, 0)]\, dx + \\ &+ \int_0^\tau [a(x_2)\,G(u(x_2\,, t))\psi_x(x_2\,, t) - a(x_1)\,G(u(x_1\,, t))\psi_x(x_1\,, t)]\, dt \end{aligned} \tag{2.1}$$

for any bounded rectangle $V := (x_1\,, x_2) \times (0\,, \tau) \subseteq S$ and any $\psi \in C^{2,1}(\overline{V})\,, \psi \geq 0$ such that $\psi(x_1\,, t) = \psi(x_2\,, t) = 0$ for any $t \in [0\,, \tau]$. Weak supersolutions (subsolutions) of (P) are defined replacing "$=$" by "$\leq$" ("$\geq$", respectively) in (2.1).

In the following we only consider weak solutions and super-, subsolutions.

Denote by $\mathcal{B}$ the set of bounded solutions of problem (P). Set also:

$$\mathcal{B}\,\mathcal{Z} \ := \left\{ u \in \mathcal{B} \mid \int_0^T G(u(x, t))\, dt \underset{|x|\to\infty}{\longrightarrow} 0 \right\} .$$

Concerning existence of solutions of problem (P), the following result will be proved.

Theorem 2.1. (i) *There exists a solution* $u \in \mathcal{B}$ *of problem* (P).

(ii) *Let any solution of equation* (1.3) *be bounded in* $\mathbb{R}$. *Then there exists a solution* $u \in \mathcal{B}\,\mathcal{Z}$ *of problem* (P).

Concerning uniqueness in the above classes, the following result holds.

Theorem 2.2. (i) *Let there exist a solution* y *of equation* (1.3) *such that* $|y| \to \infty$ *as* $|x| \to \infty$. *Then there exists at most one solution* $u \in \mathcal{B}$ *of problem* (P).

(ii) *There exists at most one solution* $u \in \mathcal{B}\,\mathcal{Z}$ *of problem* (P).

Similar results hold if we allow a different behaviour of solutions of (P) as $x \to \infty$ or as $x \to -\infty$. In fact, define:

$$\mathcal{B}\,\mathcal{Z}_+ := \left\{ u \in \mathcal{B} \mid \int_0^T G(u(x, t))\, dt \underset{x\to\infty}{\longrightarrow} 0 \right\} ,$$

$$\mathcal{B}\,\mathcal{Z}_- := \left\{ u \in \mathcal{B} \mid \int_0^T G(u(x, t))dt \underset{x\to-\infty}{\longrightarrow} 0 \right\} .$$

The following existence result can be proved.

THEOREM 2.3. (*i*) *Suppose that for any solution* y *of equation* (1.3) *there holds*:

$$\lim_{x\to\infty} |y| < \infty.$$

Then there exists a solution $u \in \mathcal{B}\mathcal{Z}_+$ *of problem* (P).

(*ii*) *Suppose that for any solution* y *of equation* (1.3) *there holds*:

$$\lim_{x\to-\infty} |y| < \infty.$$

Then there exists a solution $u \in \mathcal{B}\mathcal{Z}_-$ *of problem* (P).

Concerning uniqueness in the classes $\mathcal{B}\mathcal{Z}_+$, $\mathcal{B}\mathcal{Z}_-$, the following result holds.

THEOREM 2.4. (*i*) *Suppose that for any solution* y *of equation* (1.3) *there holds*:

$$\lim_{x\to\infty} |y| < \infty.$$

Moreover, let there exist a solution of (1.3) *such that*

$$\lim_{x\to-\infty} |y| = \infty.$$

Then there exists at most one solution $u \in \mathcal{B}\mathcal{Z}_+$ *of problem* (P).

(*ii*) *Suppose that for any solution* y *of equation* (1.3) *there holds*:

$$\lim_{x\to-\infty} |y| < \infty.$$

Moreover, let there exist a solution of (1.3) *such that*

$$\lim_{x\to\infty} |y| = \infty.$$

Then there exists at most one solution $u \in \mathcal{B}\mathcal{Z}_-$ *of problem* (P).

In connection with the above statements, let us notice that any solution of (1.3) has a limit if either $x \to \infty$ or $x \to -\infty$. Also observe that the existence of solutions of problem (P) in any class $\mathcal{B}\mathcal{Z}$, $\mathcal{B}\mathcal{Z}_+$, $\mathcal{B}\mathcal{Z}_-$ (for any initial data satisfying (H)-(iii)) implies nonuniqueness in the class $\mathcal{B}$. Using this remark the following result can be proved.

THEOREM 2.5. *Let there exist a solution of equation* (1.3) *bounded from below. Then the solution of problem* (P) *in the class* $\mathcal{B}$ *is not unique.*

COROLLARY 2.1. *The following statements are equivalent*:

(*i*) *there exists a solution* y *of equation* (1.3) *such that* $|y| \to \infty$ *as* $|x| \to \infty$;

(*ii*) *there exists only one solution* $u \in \mathcal{B}$ *of problem* (P).

REMARK 2.1. Since the solutions of equation (1.3) are explicitly known, assumptions concerning their behaviour at $|x| = \infty$ may be formulated in terms of the following integrals:

$$I_\rho^+ := \int_0^\infty [a(\xi)]^{-1} \int_0^\xi \rho(\eta)\, d\eta\, d\xi \ ,$$
$$I_\rho^- := \int_{-\infty}^0 [a(\xi)]^{-1} \int_\xi^0 \rho(\eta)\, d\eta\, d\xi.$$

For instance, since

$$y_+(x) := -\int_0^x [a(\xi)]^{-1} \int_0^\xi \rho(\eta)\, d\eta\, d\xi$$

is a solution of (1.3), the integral I_ρ^+ is bounded if any solution of (1.3) converges to a finite limit as $x \to \infty$; similarly for I_ρ^-. There is a similar connection between the integrals:

$$I_a^+ := \int_0^\infty [a(\xi)]^{-1}\, d\xi \ ,$$
$$I_a^- := \int_{-\infty}^0 [a(\xi)]^{-1}\, d\xi$$

and the solutions of the homogeneous differential equation (1.4).

As it is easily seen, there holds:

$$\left\{ \begin{array}{l} I_\rho^+ < \infty \ \Rightarrow \ I_a^+ < \infty \ , \\ I_\rho^- < \infty \ \Rightarrow \ I_a^- < \infty \ . \end{array} \right. \tag{2.2}$$

3. PROOF OF EXISTENCE RESULTS

Let us prove Theorem 2.1.

PROOF OF THEOREM 2.1. (*i*) For any $R > 0$ consider the problem:

$$(P)_R \qquad \left\{ \begin{array}{ll} \rho u_t = \{a[G(u)]_x\}_x & \text{in } (-R, R) \times (0\,, T] =: Q_R \\ u = 0 & \text{in } \{-R, R\} \times (0\,, T] \\ u = u_0 & \text{in } (-R, R) \times \{0\} \ , \end{array} \right.$$

where $u_0 \geq 0$.

By a solution of $(P)_R$ we mean any continuous, nonnegative function u in Q_R such that equality (2.1) holds in any rectangle V with $\overline{V} \subseteq \overline{Q}_R$ and moreover $u(-R, t) = u(R, t) = 0$ for any $t \in (0\,, T]$. Subsolutions and supersolutions of $(P)_R$ are similarly defined. Existence, uniqueness and comparison results for $(P)_R$ can be proved as in [2, 3].

Denote by u_R the unique solution of problem $(P)_R$. By comparison we have

$$0 \leq u_R \leq \|u_0\|_\infty =: M \qquad \text{in } Q_R \ . \tag{3.1}$$

Let $R \to \infty$. By usual arguments a sequence $\{u_{R_n}\}$ exists, which converges uniformly in any bounded subset of S to a solution of problem (P), namely

$$u := \lim_{R_n \to \infty} u_{R_n} \ . \tag{3.2}$$

Since $u_R \geq 0$ for any R, by construction there holds

$$0 \leq u \leq M \qquad \text{in } S \ .$$

This proves the claim.

(ii) Consider the solution $u \in \mathcal{B}$ of problem (P) constructed in (i) above. Let us prove that $u \in \mathcal{B}\mathcal{Z}$ if $I_\rho^- < \infty$, $I_\rho^+ < \infty$.

Set

$$z_0(x) := C_0 \int_x^\infty [a(\xi)]^{-1} \int_1^\xi \rho(\eta)\, d\eta\, d\xi \qquad (x \geq 1) \ , \tag{3.3}$$

where

$$C_0 := \max\left\{ M \, , \frac{G(M) \ T}{\int_1^\infty [a(\xi)]^{-1} \int_1^\xi \rho(\eta)\, d\eta\, d\xi} \right\}.$$

Since by assumption $I_\rho^+ < \infty$, the function z_0 tends to zero as $x \to \infty$. Moreover, it is a supersolution of the problem:

$$\begin{cases} (a\, v')' = -M\rho & \text{in } (1\, , R) \\ v(1) = G(M)\ T \ , \quad v(R) = 0 \, , \end{cases} \tag{3.4}$$

where $R > 1$.

Set

$$v_R(x\, , t) := \int_0^t G(u_R(x\, , \tau))\, d\tau \qquad ((x\, , t) \in Q_R)$$

(u_R being the solution of problem $(P)_R$) and

$$v(x\, , t) := \int_0^t G(u(x\, , \tau))\, d\tau \qquad ((x\, , t) \in S).$$

Since the convergence (3.2) is uniform in any compact domain, we have

$$\lim_{R_n \to \infty} v_{R_n} = v \quad \text{in } S \ . \tag{3.5}$$

By inequality (3.1) there holds

$$0 \leq v_R \leq G(M)\ T \quad \text{in } Q_R \ .$$

From the first equation in $(P)_R$ we obtain:

$$[a\, (v_R)_x]_x = \rho u_R - \rho u_0 \geq -M\rho \quad \text{in } Q_R \tag{3.6}$$

due to (3.1), the above inequality being satisfied in the weak sense.

Hence the function v_R is a subsolution of problem (3.4) for any fixed $t \in (0\, , T)$. It follows that

$$0 \leq v_R(x\, , t) \leq z_0(x) \quad \text{for any } (x\, , t) \in (1\, , R) \times (0\, , T) \ . \tag{3.7}$$

Passing to the limit as $R \to \infty$ in (3.7) we obtain:

$$0 \leq v(x, t) \leq z_0(x) \quad \text{for any } (x, t) \in (1, \infty) \times (0, T) .$$

Since the function z_0 tends to zero as $x \to \infty$, we conclude that

$$\lim_{x \to \infty} \int_0^T G(u(x, t))\, dt = 0 .$$

Due to the assumption $I_\rho^- < \infty$, it is similarly proved that

$$\lim_{x \to -\infty} \int_0^T G(u(x, t))\, dt = 0 ;$$

hence the conclusion follows. □

REMARK 3.1. Observe that the solution constructed in the proof of Theorem 2.1 is minimal among all nonnegative, nontrivial solutions of problem (P).

The proof of Theorem 2.3 is similar to *(ii)* above, thus we omit it.

4. Proof of uniqueness results

Let u be any solution of problem (P) in the class $\mathcal{B}$; denote by $\underline{u}$ the minimal solution of (P) in this class (see Remark 3.1). Then

$$u - \underline{u} \geq 0 \quad \text{in } S ;$$

the conclusion of Theorem 2.2 will follow if we prove that

$$u = \underline{u} \quad \text{in } S .$$

For this purpose it suffices to prove the equality

$$\int_0^T \int_{\mathbb{R}} [G(u) - G(\underline{u})] F \, dx\, dt = 0 \tag{4.1}$$

for any test function $F = F(x) \in C_0^\infty(\mathbb{R})$, $F \geq 0$, as it is easily seen.

Without loss of generality we can assume $\operatorname{supp} F \subseteq (-1, 1)$. Consider the solution of the problem:

$$\begin{cases} (a\psi')' = -F & \text{in } (-R, R) \\ \psi(-R) = \psi(R) = 0 , \end{cases} \tag{4.2}$$

where $R > 1$.

From (P) we obtain:

$$\begin{cases} \rho(u - \underline{u})_t = \{a[G(u) - G(\underline{u})]_x\}_x & \text{in } S \\ u - \underline{u} = 0 & \text{in } \mathbb{R} \times \{0\} . \end{cases}$$

Using equality (2.1) with $x_1 = -R$, $x_2 = R$, and ψ given by (4.2), we obtain:

$$\int_{-R}^{R} \rho[u(x,\tau) - \underline{u}(x,\tau)]\psi\, dx + \int_0^\tau \int_{-R}^{R} [G(u) - G(\underline{u})]\, F\, dx\, dt = \tag{4.3}$$
$$= -\int_0^\tau [a\{G(u) - G(\underline{u})\}\psi']_{-R}^{R}\, dt\ .$$

Since $F \geq 0$, $\psi \geq 0$ (see Lemma 4.1 below) and $u \geq \underline{u}$, from the above equality with $\tau = T$ we obtain:

$$\int_0^T \int_{\mathbb{R}} [G(u) - G(\underline{u})]\, F\, dx\, dt \leq \varliminf_{R\to\infty} \left| \int_0^T [a\{G(u) - G(\underline{u})\}\psi']_{-R}^{R}\, dt \right| \ .$$

Hence the equality (4.1) will follow, if we prove that

$$\varliminf_{R\to\infty} \left| \int_0^T \left[a\{G(u) - G(\underline{u})\}\psi'\right]_{-R}^{R} dt \right| = 0\ . \tag{4.4}$$

To this purpose we need some properties of the solution of problem (4.2). According to Remark 2.1, they are related to properties of the integrals I_ρ^+, I_ρ^- and I_a^+, I_a^-.

Set

$$H(x) := \int_{-R}^{x} F(\xi)\, d\xi \qquad (x \in (-R, R))\,,$$

$$H_0 := \int_{-1}^{1} F(\xi)\, d\xi\ .$$

Since supp $F \subseteq (-1\,, 1)$ and $R > 1$, we have:

$$\begin{cases} (i) & H = 0 & \text{in } (-R, -1)\ , \\ (ii) & 0 \leq H \leq H_0 & \text{in } [-1\,, 1]\ , \\ (iii) & H = H_0 & \text{in } (1\,, R)\ . \end{cases} \tag{4.5}$$

The solution of problem (4.2) is

$$\psi(x) \equiv \psi_R(x) := \int_{-R}^{x} \{c_R - H(\xi)\}[a(\xi)]^{-1}\, d\xi \qquad (x \in (-R, R))\,, \tag{4.6}$$

where

$$c_R := \frac{\int_{-R}^{R} H(\xi)\, [a(\xi)]^{-1}\, d\xi}{\int_{-R}^{R} [a(\xi)]^{-1}\, d\xi}\ . \tag{4.7}$$

We always set $\psi \equiv \psi_R$ in the sequel. Let us note the following result.

Lemma 4.1. *Let ψ be the function* (4.6). *Then*:

(*i*) $$0 \leq \psi \leq H_0 \int_{-R}^{R} [a(\xi)]^{-1}\, d\xi \qquad in\ [-R, R]\,;$$

(*ii*) *for any* $R > 0$

$$\max\{a(-R)|\psi'(-R)|\,,\, a(R)|\psi'(R)|\} \leq H_0\,. \tag{4.8}$$

Proof. (*i*) Since $F \geq 0$, the nonnegativity of ψ follows by the maximum principle. By (4.5) and (4.7) we also have

$$|c_R - H| \leq H_0 \qquad \text{in } [-R, R]\ ; \tag{4.9}$$

in particular, the second inequality in (*i*) follows.

(*ii*) Integrating the first equation in (4.2) we have:

$$a(R)\psi'(R) - a(-R)\psi'(-R) = -H_0\,. \tag{4.10}$$

Since $\psi'(R) \leq 0\,, \psi'(-R) \geq 0$, the conclusion follows. □

Observe that by (4.5) and (4.7) we have:

$$c_R = \frac{\int_{-1}^{1} H(\xi)[a(\xi)]^{-1}\,d\xi + H_0\int_1^R [a(\xi)]^{-1}\,d\xi}{\int_{-R}^{1}[a(\xi)]^{-1}\,d\xi + \int_1^R [a(\xi)]^{-1}\,d\xi}\ . \tag{4.11}$$

It follows from (4.11) that the limit

$$\overline{c} := \lim_{R\to\infty} c_R \tag{4.12}$$

exists; moreover, there holds:

$$\begin{cases} (i) & I_a^- < \infty\,,\ I_a^+ < \infty \Rightarrow \overline{c} \in (0\,, H_0); \\ (ii) & I_a^- = \infty\,,\ I_a^+ < \infty \Rightarrow \overline{c} = 0; \\ (iii) & I_a^- < \infty\,,\ I_a^+ = \infty \Rightarrow \overline{c} = H_0\,. \end{cases} \tag{4.13}$$

Lemma 4.2. *Let* ψ *be the function* (4.6).

(*i*) *Let* $I_a^+ < \infty$, $I_\rho^+ = \infty$. *Then for any fixed* $R_1 > 0$

$$\int_{R_1}^{R} \rho\,\psi\,dx \to \infty \quad \text{as } R \to \infty\,. \tag{4.14}$$

(*ii*) *Let* $I_a^- < \infty$, $I_\rho^- = \infty$. *Then for any fixed* $R_1 > 0$

$$\int_{-R}^{-R_1} \rho\,\psi\,dx \to \infty \quad \text{as } R \to \infty\,. \tag{4.15}$$

Proof. By (4.5)-(4.6) we have:

$$\psi(x) = \begin{cases} c_R\int_{-R}^{x}[a(\xi)]^{-1}\,d\xi & \text{if } x \in (-R, -1) \\ (H_0 - c_R)\int_x^R [a(\xi)]^{-1}\,d\xi & \text{if } x \in (1\,, R)\ . \end{cases} \tag{4.16}$$

It suffices to consider $R_1 \geq 1$. From the second equality in (4.16) we obtain:

$$\int_{R_1}^{R} \rho(x)\psi(x)\,dx = (H_0 - c_R)\int_{R_1}^{R}\rho(x)\int_x^R [a(\xi)]^{-1}\,d\xi\,dx =$$

$$= (H_0 - c_R)\int_{R_1}^{R}[a(\xi)]^{-1}\int_{R_1}^{\xi}\rho(x)\,dx\,d\xi \to \infty \quad \text{as } R \to \infty\ .$$

In fact,

$$H_0 - c_R \to H_0 - \overline{c} > 0 \qquad \text{as } R \to \infty$$

by (4.13) (*i*)-(*ii*). Moreover,

$$\int_{R_1}^{R} [a(\xi)]^{-1} \int_{R_1}^{\xi} \rho(x)\, dx\, d\xi \to \infty \qquad \text{as } R \to \infty$$

since $I_a^+ < \infty$ and $I_\rho^+ = \infty$; this proves (4.16). The proof of (4.15) is similar, using the first equality in (4.16) and (4.13). Then the conclusion follows. □

If any solution of (1.3) is bounded, the same holds for (1.4). Hence to prove Theorem 2.2 only the following cases must be considered:

(α) there exists a solution y of (1.3) such that $|y| \to \infty$ as $|x| \to \infty$ and any solution of (1.4) is bounded;

(β) there exists a solution y of (1.3) such that $|y| \to \infty$ as $|x| \to \infty$ and there exist unbounded solutions of (1.4);

(γ) any solution of (1.3) (hence of (1.4)) is bounded.

The proof of Theorem 2.2 will be given investigating each of the above cases (α)-(γ). Concerning (α) we can prove the following result.

Proposition 4.1. *Let there exists a solution y of* (1.3) *such that $|y| \to \infty$ as $|x| \to \infty$; moreover, let any solution of* (1.4) *be bounded. Then there exists at most one solution $u \in \mathcal{B}$ of problem* (P).

Remark 4.1. Under the conditions of Proposition 4.1 both integrals I_ρ^-, I_ρ^+ are infinite, while both I_a^- and I_a^+ are finite.

Proof of Proposition 4.1. Define for any $x \in \mathbb{R}$

$$W(x) := \int_0^T \{G(u(x,\tau)) - G(\underline{u}(x,\tau))\}\, d\tau \;; \tag{4.17}$$

observe that

$$W(x) \le L \int_0^T [u(x,\tau) - \underline{u}(x,\tau)]\, d\tau \;, \tag{4.18}$$

where

$$L := \max_{u \in [0\,,M]} G'(u) \;.$$

Let us first prove that

$$\varlimsup_{x\to\infty} W(x) = 0 \;, \qquad \varlimsup_{x\to-\infty} W(x) = 0 \;. \tag{4.19}$$

By absurd, suppose that

$$\varlimsup_{x\to\infty} W(x) = \gamma > 0 \,.$$

Then there exists $R_1 > 1$ such that

$$W(x) \geq \gamma/2 \qquad \text{for any } x > R_1 \ . \tag{4.20}$$

Let (4.20) hold. Observe that by (4.3)

$$\begin{aligned} &\int_0^T \int_{-R}^R \rho[u(x,\tau) - \underline{u}(x,\tau)]\psi \, dx \, d\tau \leq \\ &\leq \left| \int_0^T \int_0^\tau \left[a\{G(u) - G(\underline{u})\}\psi'\right]_{-R}^R dt \, d\tau \right| \leq 2H_0 G(M) \, T^2 \end{aligned} \tag{4.21}$$

for any $R > 1$; here use of inequality (4.8) has been made. On the other hand, from (4.18) and (4.20) we obtain:

$$\int_0^T \int_{-R}^R \rho[u(x,\tau) - \underline{u}(x,\tau)]\psi \, dx \, d\tau \geq \frac{1}{L} \int_{R_1}^R \rho\psi W \, dx \geq \frac{\gamma}{2L} \int_{R_1}^R \rho\psi \, dx \tag{4.22}$$

($R > R_1$). Then by (4.21)-(4.22) there holds:

$$\int_{R_1}^R \rho\psi \, dx \leq \frac{4}{\gamma} H_0 G(M) L T^2$$

for any $R > R_1$, in contrast with Lemma 4.2(i). The second equality in (4.19) is proved similarly; hence the claim follows.

According to (4.19), there exist two sequences $R'_n \to \infty$, $R''_n \to \infty$ such that

$$\lim_{R''_n \to \infty} W(R''_n) = 0 = \lim_{R'_n \to \infty} W(-R'_n) \ . \tag{4.23}$$

Let $\psi \equiv \psi_n$ be the solution of the problem

$$\begin{cases} (a\psi')' = -F & \text{in } (-R'_n, R''_n) \\ \psi(-R'_n) = \psi(R''_n) = 0 \ . \end{cases}$$

A result analogous to Lemma 4.1(ii) holds for ψ defined above; hence we have

$$\left| \int_0^T \left[a\{G(u) - G(\underline{u})\}\psi'\right]_{-R'_n}^{R''_n} dt \right| \leq H_0\{W(-R'_n) + W(R''_n)\}.$$

Repeating the argument used for (4.4) and using (4.23) we obtain the equality (4.1); then the conclusion follows. □

Let us now turn to case (β).

PROPOSITION 4.2. *Let there exists a solution y of* (1.3) *such that $|y| \to \infty$ as $|x| \to \infty$; moreover, let there exist unbounded solutions of* (1.4). *Then there exists at most one solution $u \in \mathcal{B}$ of problem* (P).

REMARK 4.2. Under the conditions of Proposition 4.2 either

(a) $I_a^- = \infty$, $I_a^+ < \infty$, or

(b) $I_a^- < \infty$, $I_a^+ = \infty$ or

(c) $I_a^- = \infty$, $I_a^+ = \infty$.

In all cases both integrals I_ρ^-, I_ρ^+ are infinite.

To prove the above proposition we show that equality (4.4) holds (with u, $\underline{u}$, ψ as above). For this purpose we need some additional properties of the function ψ.

Lemma 4.3. (i) *Let* $I_a^- = \infty$, $I_a^+ < \infty$. *Then*

$$\lim_{R\to\infty} a(-R)\psi'(-R) = 0 .$$

(ii) *Let* $I_a^- < \infty$, $I_a^+ = \infty$. *Then*

$$\lim_{R\to\infty} a(R)\psi'(R) = 0 .$$

Proof. (i) By (4.5)-(i) and (4.6) we have

$$a(-R)\psi'(-R) = c_R \qquad \text{for any } R > 1 .$$

Then the claim follows by (4.13)-(ii).

(ii) By (4.5)-(iii) and (4.10)

$$a(R)\psi'(R) = c_R - H_0 \qquad \text{for any } R > 1 .$$

Hence the conclusion by (4.13)-(iii). □

Lemma 4.4. *Let* $I_a^- = \infty$, $I_a^+ = \infty$. *Then for any* $R_1 > 1$

(4.24) $$\psi \to \infty \quad \textit{uniformly in } [-R_1 , -1] \cup [1 , R_1] \textit{ as } R \to \infty .$$

Proof. By the second equality in (4.16) and (4.11) we have:

$$\psi(R_1) = \frac{H_0 \int_{-R}^{1} [a(\xi)]^{-1}\, d\xi - \int_{-1}^{1} H(\xi)[a(\xi)]^{-1}\, d\xi}{\int_{-R}^{1} [a(\xi)]^{-1}\, d\xi + \int_{1}^{R} [a(\xi)]^{-1}\, d\xi} \int_{R_1}^{R} [a(\xi)]^{-1}\, d\xi \ ,$$

whence

$$\psi(R_1) \to \infty \quad \text{as } R \to \infty .$$

Since ψ is decreasing in $(1 , R_1)$, the divergence of ψ in this interval follows immediately. The proof for the interval $(-R_1 , -1)$ is similar, thus the conclusion follows. □

Proof of Proposition 4.2. (a) Let $I_a^- = \infty$ and $I_a^+ < \infty$, $I_\rho^+ = \infty$. From (4.18) we obtain:

(4.25)
$$\begin{aligned} &\varlimsup_{R\to\infty} \left| \int_0^T \left[a\{G(u) - G(\underline{u})\}\psi' \right]_{-R}^{R} dt \right| \le \\ &\le \varlimsup_{R\to\infty} \{ W(R)a(R) \left|\psi'(R)\right| + W(-R)a(-R) \left|\psi'(-R)\right| \} \le \\ &\le H_0 \varlimsup_{R\to\infty} W(R) + 2G(M)\, T \lim_{R\to\infty} a(-R)|\psi'(-R)| = H_0 \varlimsup_{R\to\infty} W(R) \end{aligned}$$

due to inequality (4.8) and Lemma 4.3(*i*). Since $I_a^+ < \infty$ and $I_\rho^+ = \infty$, we may use Lemma 4.2(*i*) as in the proof of Proposition 4.1 (see (4.19)) to prove that

$$\varliminf_{R\to\infty} W(R) = 0 \ .$$

Then by (4.25) the claim follows.

(*b*) Let $I_a^- < \infty$, $I_\rho^- = \infty$ and $I_a^+ = \infty$. The proof is the same as in (*a*), using inequality (4.8), Lemma 4.3(*ii*) and the following statement:

$$\varliminf_{R\to\infty} W(-R) = 0 \qquad \text{if } I_a^- < \infty\,, \ I_\rho^- = \infty \ .$$

The details are omitted.

(*c*) Let $I_a^- = \infty$, $I_a^+ = \infty$. The proof in this case is the same as for Proposition 4.1, since Lemma 4.4 applies and the statement (4.24) implies (4.14)-(4.15). Then the conclusion follows as before. □

Concerning case (γ) we have the following result.

Proposition 4.3. *Let any solution of* (1.3) *be bounded. Then there exists only one solution* $u \in \mathcal{BZ}$ *of problem* (*P*).

Proof. Due to Theorem 2.1(*ii*) the class $\mathcal{BZ}$ is nonempty. Let u be any solution of problem (*P*) in the class $\mathcal{BZ}$, $\underline{u}$ the minimal solution of (*P*) in this class (see the proof of Theorem 2.1(*ii*) and Remark 3.1). By inequality (4.8) and the definition of class $\mathcal{BZ}$ there holds:

$$\varliminf_{R\to\infty} \left| \int_0^T \left[a\{G(u) - G(\underline{u})\}\psi'\right]_{-R}^{R} dt \right| \le H_0 \lim_{R\to\infty} \{W(R) + W(-R)\} \ = 0 \ ;$$

then by (4.4) the conclusion follows. □

Now the proof of Theorem 2.2 follows by Propositions 4.1-4.3. The same arguments can be used to prove Theorem 2.4.

Proof of Theorem 2.4. We only prove the statement (*i*), the proof of (*ii*) being similar.

It follows from the assumptions that $I_\rho^- = \infty$, $I_\rho^+ < \infty$. Let us consider the following cases:

(*a*) $I_a^- = \infty$, $I_\rho^+ < \infty$;

(*b*) $I_a^- < \infty$, $I_\rho^- = \infty$ and $I_\rho^+ < \infty$.

(*a*) Let W be defined by (4.17), where u is any solution of (*P*) in the class $\mathcal{BZ}_+$ and $\underline{u}$ denotes the minimal solution of (*P*) in the same class (see Remark 3.1). By definition of class $\mathcal{BZ}_+$ there holds:

$$\lim_{R\to\infty} W(R) = 0 \ . \tag{4.26}$$

From (4.18) we obtain:

$$\varlimsup_{R\to\infty}\left|\int_0^T [a\{G(u)-G(\underline{u})\}\psi']_{-R}^{R}\,dt\right| \le H_0 \lim_{R\to\infty} W(R) + 2G(M)\,T \lim_{R\to\infty} a(-R)|\psi'(-R)| = 0\,,$$

due to (4.26) and Lemma 4.3(i). Then the equality (4.4) follows, whence the result in this case.

(b) We have:

$$\varlimsup_{R\to\infty}\left|\int_0^T [a\{G(u)-G(\underline{u})\}\psi']_{-R}^{R}\,dt\right| \le H_0\{\varlimsup_{R\to\infty} W(-R) + \lim_{R\to\infty} W(R)\} = H_0 \varlimsup_{R\to\infty} W(-R)\,,$$

due to (4.26). As in the proof of (4.19), using Lemma 4.2(ii) it is shown that the right-hand side of the above inequality is zero; then by (4.4) the conclusion follows. □

Let us now prove Theorem 2.5.

Proof of Theorem 2.5. Since no solution of equation (1.3) has a local minimum, either

$$\inf_{x\in\mathbb{R}} y(x) = \lim_{x\to\infty} y(x)\,,$$

or

$$\inf_{x\in\mathbb{R}} y(x) = \lim_{x\to-\infty} y(x).$$

Since y is bounded from below, at least one of the above limits is finite. Suppose for instance:

$$\lim_{x\to\infty} y(x) = l \in \mathbb{R};$$

then $y_1 := C(y-l)$ (with $C>0$ suitably chosen) can be used as in the proof of Theorem 2.1(ii) to prove that there exists a solution of problem (P) in the class $\mathcal{B}\mathcal{Z}_+$. Similarly, if $\lim_{x\to-\infty} y(x)$ is finite, there exists a solution of problem (P) in the class $\mathcal{B}\mathcal{Z}_-$. Hence the conclusion follows. □

Let us finally prove Corollary 2.1.

Proof of Corollary 2.1. (i) $\Rightarrow$ (ii): follows by Theorems 2.1(i) and 2.2(i). (ii) $\Rightarrow$ (i): if I_ρ^+ were finite, any solution of equation (1.3) would be bounded as $x\to\infty$; hence by Theorem 2.3(i) there would exist a solution of (P) in the class $\mathcal{B}\mathcal{Z}_+$. However, this contradicts the uniqueness in the class $\mathcal{B}$. Similarly it is proved that $I_\rho^- = \infty$; hence the conclusion follows. □

5. Sufficient conditions

The following results can be proved.

Theorem 5.1. *Let*

$$(H_1) \qquad \lim_{|x|\to\infty} x\,\frac{a'(x)}{a(x)} < 1\,.$$

Then there exists only one solution $u\in\mathcal{B}$ of problem (P).

THEOREM 5.2. *Let the following assumptions be satisfied*:

$$(H_2) \qquad \lim_{|x|\to\infty} x\,\frac{a'(x)}{a(x)} > 1\,;$$

$$(H_3) \qquad \lim_{|x|\to\infty} \frac{a(x)\,a''(x)}{[a'(x)]^2} > 0\,;$$

$$(H_4) \qquad \frac{\rho}{a'} \in L^1(\mathbb{R})\,.$$

Then there exists only one solution $u \in \mathcal{BZ}$ *of problem* (P).

Similar results hold for the classes $\mathcal{BZ}_+$, $\mathcal{BZ}_-$. To prove Theorems 5.1-5.2, let us observe preliminarily that:

(a) if (H_1) holds, then $I_a^- = \infty$, $I_a^+ = \infty$;

(b) if (H_2) holds, then $I_a^- < \infty$, $I_a^+ < \infty$

(the elementary proofs of these statements are omitted). We also need the following lemma.

LEMMA 5.1. *Let assumptions* (H_2)-(H_4) *be satisfied. Then* $I_\rho^- < \infty$, $I_\rho^+ < \infty$.

PROOF. It is easily seen that

$$I_\rho^+ = \int_0^\infty \rho(\eta) \int_\eta^\infty [a(\xi)]^{-1}\, d\xi\, d\eta\ . \tag{5.1}$$

By statement (b) above

$$\int_x^\infty [a(\xi)]^{-1}\, d\xi \to 0 \qquad \text{as } x \to \infty\ ;$$

moreover, the assumption (H_2) implies

$$\frac{1}{a'(x)} \to 0 \qquad \text{as } x \to \infty\ .$$

Set

$$\gamma := \lim_{x\to\infty} \frac{a(x)\,a''(x)}{[a'(x)]^2};$$

by assumption $(H_3)_+$ and de l'Hôpital's rule we have:

$$\lim_{x\to\infty} \frac{\int_x^\infty [a(\xi)]^{-1}\, d\xi}{\frac{1}{a'(x)}} = \lim_{x\to\infty} \frac{1}{\frac{a(x)\,a''(x)}{[a'(x)]^2}} = \frac{1}{\gamma}\ .$$

Hence there exists $L > 0$ such that

$$\int_x^\infty [a(\xi)]^{-1}\, d\xi \le \frac{2}{\gamma}\ \frac{1}{a'(x)} \quad \text{for any } x > L\ .$$

Then from (5.1) we obtain:

$$I_\rho^+ \leq \int_0^L \rho(\eta) \int_\eta^\infty [a(\xi)]^{-1}\, d\xi\, d\eta + \frac{2}{\gamma} \int_L^\infty \frac{\rho(\eta)}{a'(\eta)}\, d\eta < \infty \quad ,$$

due to assumption (H_4). The proof that $I_\rho^- < \infty$ is similar, thus it is omitted. □

Let us now prove Theorems 5.1 and 5.2.

Proof of Theorem 5.1. Due to the above statement (a) and to (2.2), there exist unbounded solutions of both equations (1.3)-(1.4) as $|x| \to \infty$. Then uniqueness in the class $\mathcal{B}$ follows by Proposition 4.2, while existence in the same class is given by Theorem 2.1(i); hence the conclusion follows. □

Proof of Theorem 5.2. Due to Lemma 5.1 and to (2.2), any solution of (1.3), thus of (1.4) is bounded in $\mathbb{R}$. Then the conclusion follows by Proposition 4.3. □

In connection with the above results, an interesting relation between problem (P) and the first order hyperbolic Cauchy problem:

$$(HP) \qquad \begin{cases} \rho u_t = a'\,(G(u))_x & \text{in } \mathbb{R} \times (0\,,\,T) \\ u = u_0 & \text{in } \mathbb{R} \times \{0\} \end{cases}$$

can be pointed out. If the function $\frac{\rho}{a'}$ is integrable, introducing the new variable

$$y := \int \frac{\rho}{a'}\, dx$$

reduces (HP) to an initial value problem on a bounded domain. The latter in general is not well posed, unless boundary conditions are given, which in the original variable read as conditions at infinity. For instance, if (H_2) holds, then $x\,a'(x) > 0$ for any $|x|$ large enough and conditions at infinity are needed to make (HP) well posed. According to Theorem 5.2 the situation for (P) is analogous, provided that condition (H_3) is satisfied.

6. Radial case

In this section we study nonnegative solutions of the Cauchy problem:

$$(P_N) \qquad \begin{cases} \rho u_t = \operatorname{div}\{a \nabla [G(u)]\} & \text{in } \mathbb{R}^N \times (0\,,\,T] \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\} \end{cases}$$

$(N \geq 1)$, assuming that the functions ρ, a, u_0 are radially symmetric. We suppose that (H_0) (with obvious modifications concerning ρ, a, u_0) is satisfied.

We always consider weak solutions of (P_N) in the following. As before, we denote by $\mathcal{B}$ the set of bounded solutions of problem (P_N). Now we set:

$$\mathcal{B}\,\mathcal{Z} := \left\{ u \in \mathcal{B} \mid \int_0^T \max_{|x|=r} G(u(x\,,\,t))\, dt \underset{r\to\infty}{\longrightarrow} 0 \right\} \quad .$$

Observe that solutions of (P_N) with radial symmetry satisfy the equation:

$$(6.1) \qquad \rho\, u_t = \frac{1}{r^{N-1}} \{a\, r^{N-1} [G(u)]_r\}_r$$

(where $r := |x|$, $x \in \mathbb{R}^N$), which reduces to (1.2) when $a \equiv 1$.

The counterparts of equations (1.3)-(1.4) are

$$(a\, r^{N-1} y')' = -\rho r^{N-1} \qquad \text{in } (0\,,\infty) \ , \tag{6.2}$$

respectively

$$(a\, r^{N-1} y')' = 0 \qquad \text{in } (0\,,\infty) \ . \tag{6.3}$$

As in the case $N = 1$, assumptions concerning the behaviour at $r = \infty$ of solutions to (6.2)-(6.3) can be formulated in terms of the following integrals:

$$I_{a,N} := \int_0^\infty [a(r)]^{-1} r^{-(N-1)}\, dr \ ,$$

respectively

$$I_{\rho,N} := \int_0^\infty [a(r)]^{-1} r^{-(N-1)} \int_0^r \rho(s) s^{N-1}\, ds\, dr \ ;$$

these reduce to I_a^+, I_ρ^+ in the case $N = 1$. Observe that as in (2.2):

$$I_{\rho,N} < \infty \ \Rightarrow \ I_{a,N} < \infty \ .$$

The following results are similar to those valid if $N = 1$.

THEOREM 6.1. (*i*) *There exists a radial solution $u \in \mathcal{B}$ of problem* (P_N).

(*ii*) *Let any solution of equation* (6.2) *be bounded in* $(0\,,\infty)$. *Then there exists a radial solution $u \in \mathcal{B}\,\mathcal{Z}$ of problem* (P_N).

THEOREM 6.2. (*i*) *Let there exists a solution y of equation* (6.2) *such that $|y| \to \infty$ as $r \to \infty$. Then there exists at most one solution $u \in \mathcal{B}$ of problem* (P_N).

(*ii*) *There exists at most one solution $u \in \mathcal{B}\,\mathcal{Z}$ of problem* (P_N).

COROLLARY 6.1. *The following statements are equivalent* :

(*i*) *there exists a solution y of equation* (6.2) *such that $|y| \to \infty$ as $r \to \infty$* ;

(*ii*) *there exists only one solution $u \in \mathcal{B}$ of problem* (P_N).

THEOREM 6.3. *Let there exists a solution of equation* (6.2) *bounded from below. Then the solution of problem* (P_N) *in the class $\mathcal{B}$ is not unique.*

The proofs are also similar to those given in Sections 3 and 4, hence they are omitted.

Let us consider the following assumptions, analogous to those introduced in Section 5:

$$\lim_{r\to\infty} r\,\frac{a'(r)}{a(r)} < 2 - N \ ; \tag{R_1}$$

(R_2)
$$\lim_{r\to\infty} r\,\frac{a'(r)}{a(r)} > 2-N\ ;$$

(R_3)
$$\lim_{r\to\infty} \frac{(a'r)^2+(N-1)a^2-a''ar^2}{[a'r+(N-1)a]^2} < 1\ ;$$

(R_4)
$$\frac{r\rho}{a'r+(N-1)a} \in L^1(0\,,\infty)\ .$$

The following results are proved like Theorems 5.1, 5.2.

THEOREM 6.4. *Let* (R_1) *be satisfied. Then there exists only one solution* $u \in \mathcal{B}$ *of problem* (P_N).

THEOREM 6.5. *Let* (R_2)-(R_4) *be satisfied. Then there exists only one solution* $u \in \mathcal{B}\,\mathcal{Z}$ *of problem* (P_N).

Let us finally discuss some examples on the strength of the above results.

EXAMPLE 6.1. Consider problem (P_N) with $\rho \equiv 1$, $a \equiv 1$. It is easily seen that $I_{\rho\,,N} = \infty$ for any $N \geq 1$; hence there exists a solution of (6.2), which diverges as $r \to \infty$. According to Corollary 6.1 there is a unique solution of problem (P_N) in the class $\mathcal{B}$ for any $N \geq 1$, in agreement with previous results.

EXAMPLE 6.2. Let $a \equiv 1$. In this case both conditions (R_2)-(R_3) reduce to the inequality $N > 2$, while (R_4) reads:

$$\int_0^\infty r\,\rho(r)\,dr < \infty\ . \tag{6.4}$$

If these conditions are satisfied, Theorem 6.5 applies, in agreement with the results in [8]. Observe that conditions (R_2)-(R_3) are not satisfied if $N = 2$; in fact, in this case there is uniqueness in the class $\mathcal{B}$ (see [6] and the following example).

EXAMPLE 6.3. Let
$$\frac{a'(r)}{a(r)} = -\frac{\lambda}{r^\alpha} \qquad (\alpha\,,\lambda > 0)\ .$$

Then the assumption (R_1) is satisfied if either

$$\alpha < 1 \quad \text{and } \lambda > 0$$

or

$$\alpha = 1 \quad \text{and } \lambda > N-2\ .$$

In both cases Theorem 6.4 holds; hence problem (P_N) with a coefficient a satisfying either condition above is well posed in the class $\mathcal{B}$. In particular, this is the case for $N \geq 3$, at variance from the case $a \equiv 1$ if (6.4) holds.

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We thank the referee for bringing the above reference to our attention.

The paper deals with uniqueness of nonnegative solutions of the Cauchy problem for a general second order linear parabolic equation, under suitable conditions concerning the coefficients - possibly depending both on space and on time variable. On the other hand, it does not investigate which conditions at infinity can restore the well-posedness of the Cauchy problem, if uniqueness fails.

In the case of time independent coefficients the paper provides interesting sufficient conditions for nonuniqueness of nonnegative solutions. In the radial case, these conditions give results in agreement with our Example 6.3 (in particular, see Example 5.9 of the above mentioned paper).

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