Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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On a problem in effective knot theory

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **9** (1998), n.4, p. 299–306.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1998_9_9_4_299_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1998.

Rend. Mat. Acc. Lincei
s. 9, v. 9:299-306 (1998)

Topologia. — *On a problem in effective knot theory.* Nota (*) di Stefano Galatolo, presentata dal Corrisp. E. Arbarello.

Abstract. — The following problem is investigated: «Find an *elementary* function $F(n) : \mathbf{Z} \to \mathbf{Z}$ such that if Γ is a knot diagram with n crossings and the corresponding knot is trivial, then there is a sequence of Reidemeister moves that proves triviality such that at each step we have less than $F(n)$ crossings». The problem is shown to be equivalent to a problem posed by D. Welsh in [7] and solved by geometrical techniques (normal surfaces).

Key words: Knots; Complexities; Normal surfaces.

Riassunto. — *Su un problema di teoria effettiva dei nodi.* Viene analizzato il seguente problema: «Trovare una funzione *elementare* $F(n) : \mathbf{Z} \to \mathbf{Z}$ tale che se Γ è un diagramma del nodo banale con n incroci allora esiste una successione di mosse di Reidemeister che portano il diagramma nel diagramma banale tale che ad ogni passo si abbiano non più di $F(n)$ incroci». Il Problema è dimostrato essere equivalente ad un problema posto da D. Welsh in [7] e risolto con tecniche geometriche (superfici normali).

1. Equivalent problems

Working on knots and Reidemeister moves, two problems arise naturally:

Problem 1 [7]. *Find a function $f(n) : \mathbf{Z} \to \mathbf{Z}$ such that if Γ is a diagram with n crossings and representing the trivial knot then there is a sequence of no more than $f(n)$ Reidemeister moves which demonstrates the equivalence.*

Problem 2. *Find a function $F(n) : \mathbf{Z} \to \mathbf{Z}$ such that if Γ is a knot diagram with n crossings and the corresponding knot is trivial then there is a sequence of Reidemeister moves that proves triviality such that at each step we have less than $F(n)$ crossings.*

Let us clarify the meaning of «*find* a function»: the existence of a function is a more or less trivial fact. The existence of a recursive function is a consequence of Haken's algorithm to solve knots; the goal is to find an elementary function, so that we can estimate the complexity of the problem of solving knots and the complexity of the resulting isotopies. Our problems are not trivial, the following fig. 1 [1] shows a diagram such that each sequence of moves leading the diagram to the trivial one, temporarily increases the number of crossings.

The diagram of the figure is analogous to a local minimum of the functional «number of crossings» (in a space of diagrams). From this point of view the function f is the height of the lowest critical point we need to visit in order to go from a given point to the absolute minimum (the circle) of this functional. It is easy to prove:

(*) Pervenuta in forma definitiva all'Accademia il 24 giugno 1998.

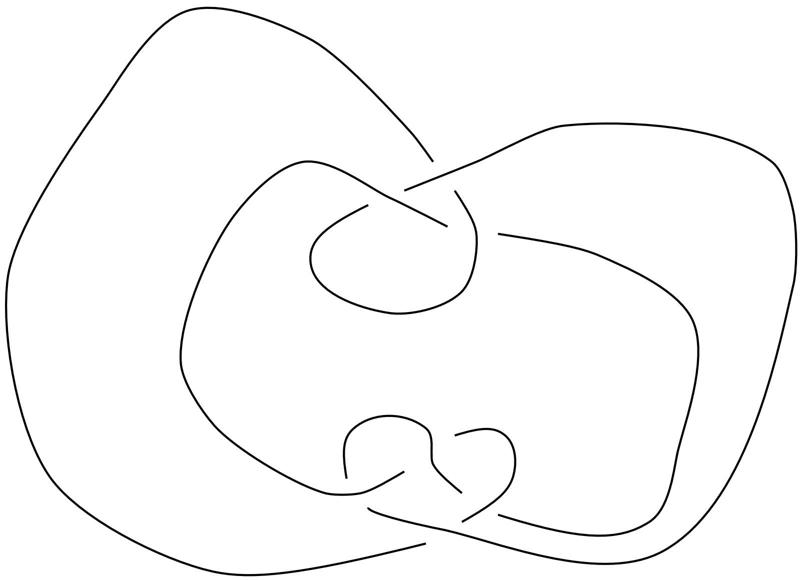

Fig. 1.

Theorem 1.1. *Problem* 1 *is equivalent to Problem* 2 ([1]).

To solve Problems 1 and 2 we will use a similar problem posed in another framework of knot complexity.

2. Polygonal knots

For a polygonal knot γ in $\mathbf{R}^3$ we define the *number of edges* of γ as the smallest number m such that γ is made of m segments.

Definition 2.1. *A n-rigid isotopy is a* (*locally flat*) *isotopy* $\Gamma_t : S^1 \times I \to \mathbf{R}^3$ *with* $\Gamma_0 = \gamma$ *such that at each instant* t *of the isotopy,* Γ_t *has no more than n edges.*

Analogous to Problem 2 in this context is:

Problem 3. *Find a function* $G(n) : \mathbf{Z} \to \mathbf{Z}$ *such that if* γ *is a knot in* $\mathbf{R}^3$ *with n edges and* γ *is isotopically trivial then there is a* $G(n)$*-rigid isotopy leading* γ *to a triangle.*

Theorem 2.2. *Problem 2 can be reduced to Problem 3: if we find* $G(n)$ *solving Problem 3 then,* $F(n) = G(6n)(G(6n) - 3)/2$ *is a solution of Problem 2.*

Proof (sketch). Suppose we have a knot diagram $\mathcal{D}$ with n crossings, the graph $\mathcal{G}$ underlying this diagram may have loops or multiple edges. Adding 2 new vertices

([1]) For example we can have these estimates: if we have $f(n)$ then $F(n) < 2f(n) + n$, if we have $F(n)$ then $f(n) < 2^{2F(n)(\log 2F(n)+1)}$.

for each edge we obtain a graph without loops or multiple edges. Since the graph is four-valent we will add $4n$ vertex. Now by Fary's lemma [2] we know that there exists an homeomorphism of the plane leading G to a graph with straight edges ($6n$ edges). This is the projection of a polygonal knot γ in R^3 constructed by lifting the appropriate vertex for each crossing. This knot is trivialized by a $G(6n)$-rigid isotopy. Perturbing this isotopy by a family of rotations if necessary, we can suppose that the projection of the isotopy is a sequence of Reidemeister moves. Since the projection of a knot with m edges has no more than $m(m-3)/2$ crossings, during this isotopy the projection of γ will have no more than $G(6n)(G(6n)-3)/2$ crossings. □

To find the solution to Problem 3 we perform a geometrical construction and then we translate our problem into a problem about 3-manifolds and surfaces.

3. Normal surfaces

Given a P.L. trivial knot γ in $\mathbf{R}^3$ with n edges we will construct an associated 3-manifold M_γ to which we will apply normal surface theory to find a disc spanning the knot and then estimate the number of faces of the disc.

Roughly speaking M_γ is constructed as follows:

a) put the knot in the interior of a cube;

b) triangulate the cube in such a way that the knot is contained in the 1-skeleton of the triangulation;

c) consider the second derivative of the triangulation and remove the star around the knot.

What we get is a triangulated 3-manifold homeomorphic to a 3-disc minus a regular neighborhood of γ, which has 2 boundary components: a 2-sphere and a torus T.

Notation. In the sequel M_γ will always denote either the specific simplicial complex given in the above construction or the underlying manifold, it being clear from the context which object is to be understood.

Proposition 3.1. *If the knot has n edges, then the construction can be performed to give a triangulated manifold with a triangulation made of less than* $10^6 n^2$ *3-simplices* (2).

Proof (sketch). The proof is elementary, but technical and quite long. We choose a square face Q of the cube with vertices q_1, q_2, q_3, q_4, call π the orthogonal projection of $\mathbf{R}^3$ on the plane containing Q. We can suppose that γ is in general position with respect to π, then $\pi(\gamma)$ is a polygonal curve in Q with double points $p_1, \dots, p_k$, call $y_1, \dots, y_n$ the vertices of γ and $g_1 = \pi(y_1), \dots, g_n = \pi(y_n)$ their projection in Q. Let us also call $q_1', \dots, q_4'$ the middle points of the four edges of the cube containing $q_1, \dots, q_4$ but not contained in Q. The first step is to triangulate Q in a way such that

1) the 0-skeleton of the triangulation is the set $\{q_1, \dots, q_4, p_1, \dots, p_k, g_1, \dots, g_n\}$

(2) This estimate is not supposed to be sharp, the main purpose of this paper is to show the general construction leading to the solution of the problems introduced before.

2) the curve $\pi(\gamma)$ is contained in the 1-skeleton of the triangulation.

The key step to prove that such a triangulation is possible is to observe that, if α is a simple plane P.L. curve (*i.e.* having no double points) with vertices $v_1, \dots, v_n$ then the interior of α can be triangulated with $n-2$ triangles such that the 0-skeleton of the triangulation is the set $\{v_1, \dots, v_n\}$.

From the above observation it is also possible to estimate the number of triangles in the triangulation of Q: there exists a triangulation of Q satisfying 1) and 2) such that the number of triangles is less than $18n^2$. From the triangulation of Q we decompose the cube in a set of triangular prisms such that the base of each prism is a triangle of the triangulation. Each prism will be triangulated appropriately, first we decompose each prism P in three parts: the first part is a prism with its base on Q, the second is the convex hull of $P \cap (\gamma \cup \{q'_1, \dots, q'_4\})$, the third is a prism with its base on the face of the cube which is opposite to Q. The convex hull of $P \cap (\gamma \cup \{q'_1, \dots, q'_4\})$ can be: either a prism or a square pyramid or a 3-simplex or a triangle. All these pieces can be triangulated coherently and such that the knot is contained in the 1-skeleton of the triangulation, as is suggested in the following figure.

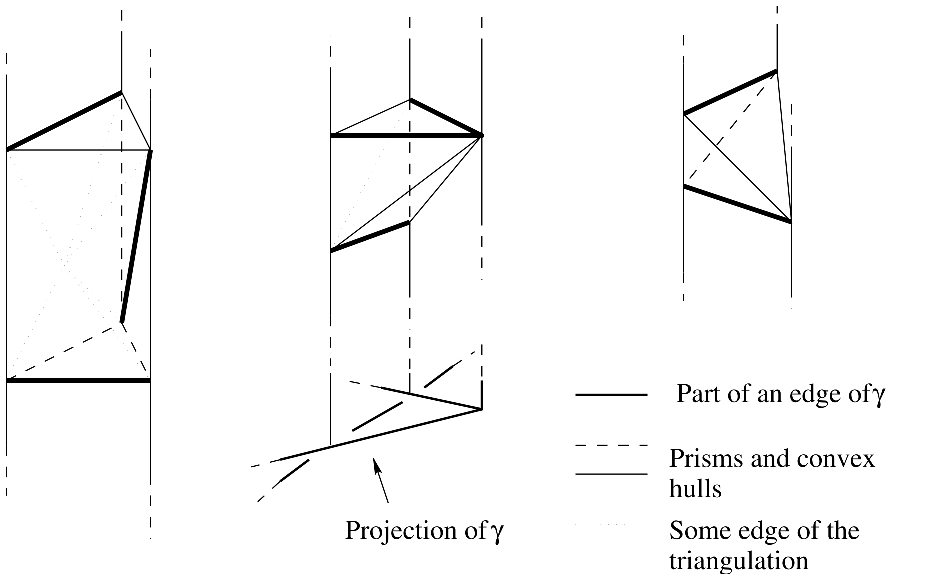

Fig. 2.

The prisms are decomposed into the union of eight 3-simplices, the pyramids into two 3-simplices so that in total we can have no more than $432n^2$ 3-simplices. If we make the second derivative we have no more than $248832n^2$ 3-simplices. $\square$

If the knot is trivial there exists a P.L. disc D properly embedded in M_γ with $\partial D \subset T$ and ∂D is not boundary of a disc contained in T. Such a disc will be called an *admissible disc*. We will show how to evaluate the number of faces for a suitable triangulation of such a disc in terms of the number of the 3-simplices of M_γ. After the construction we have this problem:

PROBLEM 4. *Find a function $D(m)$ such that if m is the number of 3-simplices of M_γ, then there exists an admissible disc consisting of $D(m)$ triangular plane faces.*

Let us explain why the solution to this problem can provide a solution to Problem 3: suppose we have found an admissible disc in M_γ. Then ∂D is a curve on T with l edges (and the number l is not greater than the number $2D(m)$). Following a retraction of the star of γ on γ we see that γ is l-rigid isotopic to ∂D. At this point we construct the desired isotopy (to transform the unknot to a triangle) collapsing the triangles of the disc (and adding a new edge to the knot for each triangle). We can solve the last problem using normal surface theory.

DEFINITION 3.2. *Given a triangulated 3-manifold W we say that a properly embedded P.L. surface S in W is normal iff*:

1) S is in general position with respect to the triangulation;

2) the intersection $S \cap \Delta$ of the surface with each 3-simplex Δ of the triangulation is a collection D_i of discs;

3) each D_i meets the 1-skeleton of the triangulation in 3 or 4 points belonging to distinct edges. In this case we call D_i a combinatorial triangle or combinatorial square respectively.

The discs D_i will be called *combinatorial faces.* They are «like» plane *triangles* or *squares* but until now they are simply P.L. discs and we cannot know how many plane faces they may have.

Before continuing our construction let us make a review of normal surface theory. Note that for each 3-simplex of the triangulation there are 4 possible classes of combinatorial triangles and 3 possible classes of combinatorial squares: for each 3-simplex we have 7 possible classes. We can try to indicate a normal surface in W by the list of its combinatorial faces.

The intersections D_i of the surface with a given 3-simplex can be indicated by a list of 7 natural numbers, that is a point of $\mathbf{Z}^7$, so that if the number of 3-simplexes of the triangulation is m, a surface can be represented by a point of $\mathbf{Z}^{7m}$. More precisely, to a normal surface S can be associated a combinatorial object which is the list of the classes of its combinatorial faces in this way: to S corresponds $(x_{ij})_{i=1,\dots,7,j=1,\dots,m} \in \mathbf{Z}^{7m}$, where x_{ij} is the number of discs of $S \cap \Delta_j$ belonging to the class i (of the 7 possible classes of combinatorial triangles and squares). In this way a point of $\mathbf{Z}^{7m}$ can be associated to a surface. We will consider the norm l^1 on $\mathbf{Z}^{7m}$: $||x|| = \sum |x_{ij}|$. Notice that all the coordinates of a point associated to a surface are positive and that the number of combinatorial faces of a surface is the norm of the corresponding point in $\mathbf{Z}^{7m}$.

For the sake of simplicity from now on all the results will be stated for M_γ. Nevertheless we would like to remark that they are true also for a wider class of manifolds.

The following lemma allows us to estimate the number of triangular faces by using the number of combinatorial faces.

Lemma 3.3. *If S is a normal surface in M_γ then S is isotopic to a normal surface S' such that S and S' induce the same point in $\mathbf{Z}^{7m}$. Furthermore for each 3-simplex Δ we have a correspondence between the sets $D_i = S \cap \Delta$ and $D_i' = S' \cap \Delta$ such that:*

if D_i is a combinatorial triangle then D_i' is a plane triangle, and

if D_i is a combinatorial square then D_i' is a combinatorial square made of two plane triangles with a common edge.

Lemma 3.4. *Let δ be a 1-simplex of the 1-skeleton of M_γ. If the surface S intersects δ in $x_1, x_2, \ldots, x_n$ and ϕ is an ambient isotopy in δ moving $x_1, x_2, \ldots, x_n$ to $x_1', x_2', \ldots, x_n'$, then this isotopy can be extended to an ambient isotopy in M with support in a neighborhood of δ moving S to a surface S' which intersects δ in $x_1', x_2', \ldots, x_n'$.*

Combining Lemmas 3.3 and 3.4 we have:

Theorem 3.5. *If two normal surfaces S and S' induce the same point of $\mathbf{Z}^{7m}$ then S is isotopic to S'.*

Not all the points of $\mathbf{Z}^{7m}$ can be realized by an embedded surface. There are some conditions to be satisfied:

1) $S \cap \Delta_j$ can contain only one class of combinatorial squares (otherwise the surface would have self intersections).

2) Given two adjacent 3-simplices Δ_i, Δ_j the classes of Δ_i must fit well with those of Δ_j in the following sense: let f be the common face of Δ_i and Δ_j. In f there are at most 3 classes of lines allowed on the boundary of the discs D_i. The condition is, for each class l_f of lines the number of the discs in Δ_i with boundary of class l_f must be equal to the number of discs in Δ_j with boundary of class l_f.

Looking at this condition in $\mathbf{Z}^{7m}$ we have that each class of lines will give us a linear equation on the $x_{i,j}$.

Theorem 3.6. *Each point of $\mathbf{Z}^{7m}$ satisfying conditions 1) and 2) above, can be realized by an embedded surface.*

If $x, y \in \mathbf{R}^{7m}$ are realizable by normal surfaces, the sum $x + y \in R^{7m}$ is not always realizable: it could happen that $x + y$ does not satisfy condition 1, conversely by Theorem 3.6 if x, y are realizable and $x + y$ satisfy condition 1 then $x + y$ is realizable by an (embedded) normal surface. If x and y are realized by surfaces S_x, S_y then if $x + y$ is realizable (by S_{x+y}) the surface S_{x+y} can be constructed by a desingularization of $S_x \cup S_y$. More precisely if $S_x \cap S_y = \emptyset$ then $S_x \cup S_y = S_{x+y}$, else $S_x \cup S_y$ is (assuming general position) a singular surface with singularities on lines $\cup_i \{l_i\} = S_x \cap S_y$. For each line there are two possible desingularizations cutting the surface on l_i and gluing the resulting surfaces as in the following figure.

One of the two desingularizations preserves the combinatorial structure of $S_x \cup S_y$ (the point in R^{7m}). Desingularizing $S_x \cup S_y$ in the appropriate way we obtain S_{x+y}.

Now all the ingredients are ready. We first observe the following two things: condition 2) can be transformed into a linear condition. Points of $\mathbf{N}^{7m}$ satisfying condition

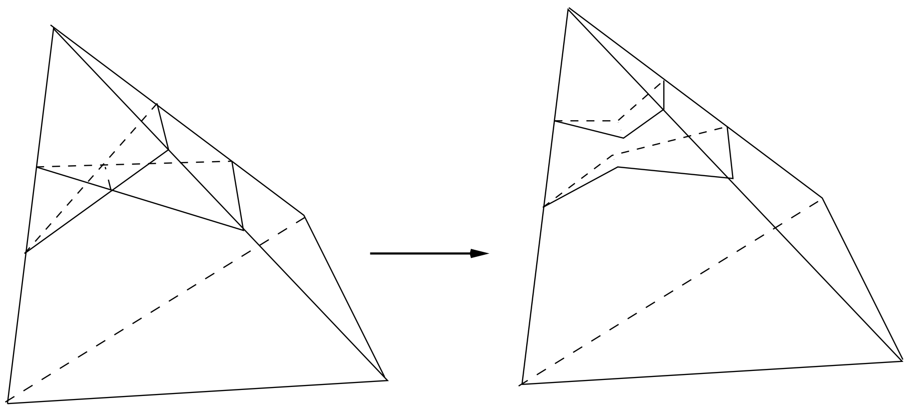

Fig. 3.

2) form a convex unbounded set, that is *generated* by a *finite* set, in the sense that these points are linear combinations with *positive* integer coefficients of a finite set of points. By elementary calculations using Cramer's formula it is possible to estimate the norm of the generators in terms of the number of simplexes as follows:

PROPOSITION 3.7. *There exists a set B of generators of the space of solutions to condition 2) such that* $\forall x \in B, ||x|| < 2^{225m^2}$.

Finally we can conclude by applying the following theorems:

THEOREM 3.8. *If there exists an admissible disc D in M_γ then there exists an admissible normal disc D^*.*

The proof uses techniques similar to the ones used to move an incompressible surface into normal position (see [4, pp. 29, 30; 3, pp. 48-50]).

Since our knot is isotopically trivial an admissible disc exists. Hence we now know that we can find such a disc in the space of normal surfaces. Furthermore the following theorem states that an admissible normal disc must belong to any set of generators.

THEOREM 3.9. *If in M_γ there exists an admissible normal disc D^* then there is an admissible normal disc D' which is irreducible* (i.e. *D' can not be written as a sum $D' = S_1 + S_2$ with S_1 and S_2 non empty normal surfaces*) *and must therefore be in every set of generators.*

This theorem can be proved using essentially the same arguments as in the proof of Lemma 4.1 in [6].

From Theorems 3.8 and 3.9 we know that an admissible disc must be in any set of generators, we use Proposition 3.7 to estimate the number of its combinatorial faces and by Lemma 3.3 we can estimate the number of its plane triangular faces, solving Problem 4.

Finally we summarize:

THEOREM 3.10. *A solution to Problem 4 is* $D(m) = 2^{225m^2+1}$.

Theorem 3.11. *A solution to Problem 3 is*:

$$G(n) = n + 2^{10^{15} n^4}$$

These formulas are not useful in practice because of their huge exponents. It is worthy of note that in a higher dimension situations are even worse, as the following result [5] states: we can consider P.L. embeddings $S^n \to \mathbf{R}^{n+2}$ with $n > 2$. The complexity of an embedding will be the number of linear faces of the image of S^n. We can imagine defining a notion of n-rigid isotopy in the same way as we did before, and posing in $\mathbf{R}^{n+2}$ a problem analogous to Problem 3. Nabutowsky and Weinberger prove that in this case a function $F(n)$, that is solution to Problem 3 in $\mathbf{R}^{n+2}$, $n > 2$, cannot be majorized by any recursive function.

References

[1] J. S. Birman, *New points of view in knot theory*. Bulletin of the AMS (New Series), vol. 28, n. 2, 1993, 253-285.

[2] I. Fáry, *On straight line representation of planar-graphs*. Acta Sci. Math. (Szeged), 1948, 11.

[3] G. Hemion, *The Classification of Knots and 3-Dimensional Spaces*. Oxford University Press, 1992.

[4] J. Hempel, *3-Manifolds*. Princeton University Press and University of Tokyo Press, 1976.

[5] A. Nabutowsky - S. Weinberger, *Algorithmic unsolvability of the triviality problem for multidimensional Knots*. Comment. Math. Helv., 71, n. 3, 1996, 426-434.

[6] W. Jaco - U. Oertel, *An algorithm to decide if a manifold is a Haken manifold*. Topology, vol. 23, No. 2, 1984, 195-209.

[7] D. J. A. Welsh, *The complexity of Knots*. Annals of Discrete Mathematics, 55, 1993, 159-172.

Pervenuta l'8 ottobre 1997,
in forma definitiva il 24 giugno 1998.

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