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Some remarks on groups in which elements with the same p-power commute

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Teoria dei gruppi. — *Some remarks on groups in which elements with the same p-power commute.* Nota di Patrizia Longobardi e Mercede Maj, presentata (*) dal Socio G. Zappa.

Abstract. — In this paper we characterize certain classes of groups G in which, from $x^p = y^p$ ($x, y \in G$, p a fixed prime), it follows that $xy = yx$. Our results extend results previously obtained by other authors, in the finite case.

Key words: p-powers; p-elements; Locally nilpotent groups.

Riassunto. — *Alcune osservazioni sui gruppi in cui sono permutabili elementi con la stessa potenza p-ma.* In questa *Nota* si caratterizzano alcune classi di gruppi G tali che da $x^p = y^p$($x, y \in G$, p primo fissato), segue $xy = yx$. In particolare si estendono risultati precedentemente ottenuti da altri autori, nel caso finito.

1. Introduction

Let p be a prime. We will denote by C_p the class of all groups G which satisfy the following property:

$$x^p = y^p\,,\ x, y \in G \qquad \text{implies} \qquad xy = yx.$$

Groups in the class C_2 have been studied by L. Brailovsky and G.A. Freiman in [3] and by L. Brailovsky and M. Herzog in [4]. Finite groups in C_p, with $p \neq 2$, have been investigated by M. Bianchi, A. Gillio and L. Verardi in [1]. They proved that a finite p-group G, with p odd, is in C_p if and only if the elements of order p form a subgroup $\Omega(G) \leq Z(G)$. Finite p-groups with this property constitute an interesting class, studied by many authors [2, 7-9]. A classical result due to J. Thompson (see, for instance, [6, III, 12.2]) ensures that if G is a finite p-group, with $p > 2$, and any element of G of order p is central, then $d(G) < d(Z(G))$, where $d(H)$ denotes the minimal number of generators of a finite group H. More recently, D. Bubboloni and G. Corsi Tani [5] have studied the relationship between this class and the class of regular p-groups.

In [1] it is also proved that a finite group G is in C_p if and only if G possesses a normal Sylow p-subgroup $P \in C_p$. In this paper we extend the results of [1] to not necessarily finite groups, by proving the following theorems.

Theorem 1. *Let G be a p-group, with p odd. G is in the class C_p if and only if G is hypercentral of length $\leq \omega$ and every element of G of order p is contained in $Z(G)$.*

Theorem 2. *Let G be a p-group, with p odd. G is in the class C_p if and only if G is hypercentral of length $\leq \omega$ and, for any positive integer i, every element of G of order p^i is contained in the i-centre $\zeta_i(G)$.*

(*) Nella seduta del 12 febbraio 1999.

THEOREM 3. *Let G be a group in the class C_p, with p odd. Then for any integer i the p-elements of G of order at most p^i form a normal subgroup P_i of class $\leq i$.*

From Theorem 3 it easily follows

COROLLARY 4. *A torsion group G is in the class C_p, with p odd, if and only if the p-elements of G form a normal subgroup P in the class C_p.*

We notice that there exist infinite non-hypercentral p-groups with every element of order p in the centre, hence in Theorem 1 the condition G hypercentral is essential. For example, A. Yu. Ol'shanskii constructed in [10] an infinite torsion-free group H with $Z(H)$ infinite cyclic and $H/Z(H)$ isomorphic to the infinite Burnside group $B(n, p)$ of exponent p. Now, if we write $Z(H) = \langle z \rangle$, and $G = H/\langle z^p \rangle$, from $a^p \in \langle z^p \rangle$, we get $a^p = (z^p)^\alpha$, $(a^{-1}z^\alpha)^p = 1$, and $a^{-1}z^\alpha = 1$, therefore $a \in Z(H)$, and every element of G of order p is in the center of G.

We also remark that the result proved by Bianchi, Gillio, Verardi for finite p-groups actually holds for any locally nilpotent group. In fact we have:

THEOREM 5. *A locally nilpotent group G is in the class C_p if and only if every element of G of order p is contained in the centre $Z(G)$.*

Now let p be an odd prime and write $\Im_{2,p}$ the class of groups G such that the subgroup generated by two p-elements is a finite p-group. Then, by Theorem 3, the class C_p is a subclass of the class $\Im_{2,p}$. Conversely, if $\mathcal{H}$ is a subclass of $\Im_{2,p}$, closed under subgroups and homomorphic images and G is a group in $\mathcal{H}$ with no p-elements, then it is easy to prove that G is in the class C_p. On the other hand, there exist torsion-free groups that are not in the class C_p. In fact, let H be the group constructed by Ol'shanskii, and mentioned before. If a, b are elements of H non-commuting mod $Z(G)$, then we have $a^p = z^\alpha$ and $b^p = z^\beta$, for some integers α, β and obviously p does not divide α, β, since H is torsion-free, therefore $a^{\beta p} = b^{\alpha p}$, with $[a^\beta, b^\alpha] \neq 1$.

Our final result is the following

THEOREM 6. *Let G be a group such that every finitely generated subgroup of G has finite abelian subgroup rank. Then G is in the class C_p if and only if the p-elements of G form a normal subgroup P in the class C_p such that G/P is in the class C_p and every nilpotent subgroup of G is in the class C_p.*

2. PRELIMINARY RESULTS

Throughout this Section p is an odd prime. In order to prove the results stated in the Introduction we shall require the following lemmas.

LEMMA 1. *Let $G \in C_p$ be a locally nilpotent group. Then the p-elements of G of order at most p form a subgroup $\Omega(G) \leq Z(G)$ such that $G/\Omega(G) \in C_p$.*

Proof. We will show that any element of G of order p is in $Z(G)$. Let $a \in G$ be of order p, $g \in G$ and $a \in \zeta_2(\langle a, g\rangle)$; we claim that $a \in Z(\langle a, g\rangle)$. In fact, from $a \in \zeta_2(\langle a, g\rangle)$, we get $[a, g] \in Z(\langle a, g\rangle)$, and $1 = [a^p, g] = [a, g]^p$, therefore $(ga)^p = g^p a^p [a, g]^{p(p-1)/2} = g^p$, and $[ga, g] = 1 = [a, g]$. Now let $b \in G$ be of order p, $x \in G$, and assume $[b, x] \neq 1$. Then $\langle b, x\rangle$ is nilpotent of class $i > 1$. Hence $[b,_{i-2} x] \in \zeta_2(\langle b, x\rangle)$ has order p, and $[b,_{i-2} x] \in Z(\langle b, x\rangle)$, by the previous remark. Thus $\langle b, x\rangle$ is nilpotent of class $i-1$, a contradiction. Therefore the p-elements of G of order p form a subgroup $\Omega(G)$ contained in $Z(G)$. Now let $x^p\Omega(G) = y^p\Omega(G)$, $x, y \in$ $\in G$, then we have $x^p = y^p c \in \Omega(G) \leq Z(G)$ and $(x^y)^p = (x^p)^y = (y^p c)^y = y^p c = x^p$. Hence $[x^y, x] = 1$, from which $[[x, y], x] = 1$. Thus $1 = [x^p, y] = [x, y]^p$, and $[x, y] \in \Omega(G)$, as required. □

Lemma 2. *Let G be a group in the class C_p. If a, b are elements in G such that $a^{p^n} = b^{p^n}$, then $\langle a, b\rangle$ is nilpotent of class $\leq n$.*

Proof. We argue by induction on n. If $n = 1$, the result is true since $G \in C_p$. Now assume $n > 1$ and $a^{p^n} = b^{p^n}$, then $(a^{p^{n-1}})^p = (b^{p^{n-1}})^p$, and $a^{p^{n-1}} b^{p^{n-1}} = b^{p^{n-1}} a^{p^{n-1}}$, since $G \in C_p$. Write $c = a^{p^{n-1}} (b^{-1})^{p^{n-1}}$. Then $|c| = p$ if $c \neq 1$ and $a^{p^{n-1}} = b^{p^{n-1}} c$. From $(a^{b^{p^{n-1}}})^{p^{n-1}} = a^{p^{n-1}}$, we get by induction that the subgroup $\langle a, a^{b^{p^{n-1}}}\rangle$ is nilpotent of class $\leq n-1$. Hence $\langle [a, b^{p^{n-1}}], a\rangle$ is nilpotent of class $\leq n-1$, and similarly $\langle [a^{p^{n-1}}, b], b\rangle$ is nilpotent of class $\leq n-1$. Hence $1 = [a^{p^{n-1}}, b,_{n-1} b] = [c, b,_{n-1} b] =$ $= [c,_n b] = 1$. From $\langle c\rangle^G$ abelian, we easily get $\langle c, b\rangle$ nilpotent of class $\leq n-1$. But $|c| = p$, then $[b, c] = 1$, by Lemma 1. Arguing similarly on a, we get that $\langle c, a\rangle$ is nilpotent and $[a, c] = 1$. Therefore $c \in Z(\langle a, b\rangle)$ and $\langle a, b\rangle/\langle c\rangle \in C_p$, by Lemma 1. Moreover, $a^{p^{n-1}}\langle c\rangle = b^{p^{n-1}}\langle c\rangle$, hence, by induction, $\langle a, b\rangle/\langle c\rangle$ is nilpotent of class $\leq n-1$, and $\langle a, b\rangle$ is nilpotent of class $\leq n$, as required. □

3. Proofs

Proof of Theorem 1. Assume $G \in C_p$ and let $a, b \in G$, with $|a| = p$. By Lemma 2, $\langle a, b\rangle$ is nilpotent and Lemma 1 applies. Therefore $\Omega(G) \leq Z(G)$. Moreover, arguing as in the proof of Lemma 1, we obtain that $G/Z(G) \in C_p$. Conversely, let G be hypercentral, and assume $\Omega(G) \leq Z(G)$. Then, for any $x, y \in G$, $\langle x, y\rangle$ is a finite p-group with every element of order p in the centre and the result follows from [1, Theorem 1]. □

Proof of Theorem 2. If $G \in C_p$, then $\Omega(G) \leq Z(G)$, and $G/\Omega(G) \in C_p$, by Lemma 1. Assume, by induction, $\Omega_i(G/\Omega(G)) = \Omega_{i+1}(G)/\Omega(G) \leq \zeta_i(G/\Omega(G))$, then we get easily $\Omega_{i+1}(G) \leq \zeta_{i+1}(G)$, and the result follows. The converse follows from Theorem 1. □

Proof of Theorem 3. The result is true if $i = 1$, by Lemma 1. Assume $i > 1$ and argue by induction on i. Let $a, b \in G$, with $|a| = |b| = p^i$ and write $H = \langle a, b\rangle$.

Then $a^{p^i} = b^{p^i} = 1$, and H is a nilpotent p-group of class $\leq i$ by Lemma 2. Thus $a^{p^{i-1}}, b^{p^{i-1}} \in \Omega(H) \leq Z(H)$ and $H/\Omega(H) \in C_p$, by Lemma 1. Hence, by induction, $H/\Omega(H)$ has exponent p^{i-1}, and H has exponent p^i. □

Proof of Theorem 5. If $G \in C_p$, then the result is true by Lemma 1. Conversely, assume that G is locally nilpotent and $\Omega(G) \leq Z(G)$, we show that $G \in C_p$. Assume that there exist $a, b \in G$, with $a^p = b^p$ and $[a, b] \neq 1$, and choose a, b such that the nilpotent class $n > 1$ of $\langle a, b\rangle$ is minimal. Then $a^p = (a^b)^p$, and $\langle a, a^b\rangle$ is nilpotent of class $< n$, hence $[a, a^b] = 1$, by minimality of n. Similarly, $[b, b^a] = 1$, and $n = 2$. Therefore, from $a^p = b^p$, we get $(a^{-1}b)^p[a, b]^{p(p-1)/2} = 1$, and $(a^{-1}b)^p = 1$. Then $a^{-1}b \in Z(G)$, and $\langle a, b\rangle$ is abelian, a contradiction. □

Proof of Theorem 6. Assume $G \in C_p$. Then, by Theorem 3, the p-elements of G form a normal subgroup P. We show that $G/P \in C_p$. For, let $a, b \in G$, and assume $a^pP = b^pP$. Then $a^p = b^pc$, with $c \in P$. Write $|c| = p^i$, and $H = \langle a, b\rangle$. Then $N = \langle c\rangle^H$ is a nilpotent p-group, of class $\leq i$, and exponent p^i, by Theorem 2. Moreover, N has finite abelian subgroup rank, hence N is finite (see, for example, [11, Corollary 2, p. 38]). Then $H/C_H(N)$ is finite. Write $|H/C_H(N)| = m = p^h k$, where p does not divide k. Then $(a^p)^m = (b^p)^m d$, with $d \in N$ and $(a^m)^{p^{i+1}} = (b^m)^{p^{i+1}}$. Hence $(a^k)^{p^{i+1+h}} = (b^k)^{p^{i+1+h}}$ and $\langle a^k, b^k\rangle$ is nilpotent, by Lemma 2. Then, by Lemma 1, $\langle a^k, b^k\rangle P/P \in C_p$, and from $a^pP = b^pP$ we get $[a^k, b^k] \in P$, and $[a, b] \in P$ since p and k are coprime.

Conversely, let $a^p = b^p$, $a, b \in G$. Then $[a, b] \in P$ and $\langle a, b\rangle'$ is a finite p-group, by Theorem 2. Then $\langle a, b\rangle/Z(\langle a, b\rangle)$ is a finite p-group, and $\langle a, b\rangle$ is nilpotent. Hence $\langle a, b\rangle \in C_p$, and $[a, b] = 1$. □

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