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Alessandra Lunardi

## On optimal $L^p$ regularity in evolution equations

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ABSTRACT. — Using interpolation techniques we prove an optimal regularity theorem for the convolution $u(t) = \int_0^t T(t-s)f(s)ds$, where $T(t)$ is a strongly continuous semigroup in general Banach space. In the case of abstract parabolic problems – that is, when $T(t)$ is an analytic semigroup – it lets us recover in a unified way previous regularity results. It may be applied also to some non analytic semigroups, such as the realization of the Ornstein-Uhlenbeck semigroup in $L^p(\mathrm{R}^n)$, $1 < p < \infty$, in which case it yields new optimal regularity results in fractional Sobolev spaces.

KEY WORDS: Abstract evolution equations; Optimal regularity; Interpolation.

RIASSUNTO. — *Regolarità ottimale $L^p$ in equazioni di evoluzione.* Usando tecniche di interpolazione si dimostra un teorema di regolarità ottimale per la convoluzione $u(t) = \int_0^t T(t-s)f(s)ds$, dove $T(t)$ è un semigruppo fortemente continuo in uno spazio di Banach qualunque. Nel caso dei problemi parabolici astratti – cioè quando $T(t)$ è un semigruppo analitico – esso permette di ritrovare in modo unificato risultati di regolarità già noti. Il teorema può essere applicato anche nel caso di alcuni semigruppi non analitici, come ad esempio la realizzazione del semigruppo di Ornstein-Uhlenbeck in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, per il quale dà nuovi risultati di regolarità ottimale in spazi di Sobolev frazionari.

## 1. INTRODUCTION

Let $T(t)$ be a strongly continuous semigroup in a Banach space $X$, with generator $A : D(A) \mapsto X$; let $T > 0$ and let $f : (0, T) \mapsto X$. The initial value problem

$$(1.1)\qquad \begin{cases} u'(t) = Au(t) + f(t), & 0 < t < T, \\ u(0) = 0 \end{cases}$$

has been the object of deep investigations for many years. In particular, much effort has been devoted to optimal $L^p$ regularity, $1 < p < \infty$. It is well known that if $f \in L^p(0, T; X)$ the solution $u$ does not necessarily belong to $L^p(0, T; D(A))$, even if $T(t)$ is an analytic semigroup. Several counterexamples may be found in [15]. Sufficient conditions in order that $u \in L^p(0, T; D(A))$ whenever $f \in L^p(0, T; X)$ were given by Da Prato and Grisvard in [5], and by Dore and Venni in [10].

Our point of view is slightly different, although related. We do not demand that $u \in L^p(0, T; D(A))$ but we look for a (as small as possible) subspace $Y$ such that $u \in L^p(0, T; Y)$. Our main result is the following.

THEOREM 1.1. *Let $Y_i$, $i = 0, 1, 2$ be Banach spaces such that $Y_2 \subset Y_1 \subset Y_0$, and let $T(t)$ be a semigroup in $Y_0$ such that for every $x \in Y_0$ the function $t \mapsto T(t)x$ is measurable*

(*) Nella seduta dell'11 dicembre 1998.

*with values in* $Y_i$, $i = 1, 2$, *and*

$$(1.2)\qquad \begin{cases} t^{\gamma_i}\|T(t)\|_{L(Y_0, Y_i)} \le c_i(t), & t > 0, \ i = 1, 2, \\ c_i \in L^p_*(0, T), & i = 1, 2, \end{cases}$$

*with* $0 \le \gamma_1 < 1 < \gamma_2$, $1 \le p < \infty$, $T > 0$.

*Then for every* $f \in L^p(0, T; Y_0)$ *the function*

$$u(t) = \int_0^t T(t-s)f(s)\,ds$$

*belongs to* $L^p(0, T; (Y_1, Y_2)_{\gamma, p})$, *with* $\gamma = (1-\gamma_1)/(\gamma_2-\gamma_1)$, *and there is* $C > 0$, *independent of* $f$, *such that*

$$\|u\|_{L^p(0, T; (Y_1, Y_2)_{\gamma, p})} \le C\|f\|_{L^p(0, T; Y_0)}.$$

Let us illustrate some consequences of the theorem.

1) Let $T(t)$ be an analytic semigroup in a Banach space $X$ with generator $A$, let $0 < \theta < \alpha < 1$, $1 \le p < \infty$, and choose $Y_0 = D_A(\theta, p) = (X, D(A))_{\theta, p}$, $Y_1 = D_A(\alpha, p)$, $Y_2 = D_A(\alpha + 1, p) = \{x \in D(A) : Ax \in D_A(\alpha, p)\}$ (definitions and properties of such spaces are in §3). We shall prove that (1.2) holds with $\gamma_1 = \alpha - \theta$, $\gamma_2 = 1 + \alpha - \theta$. Theorem 1.1 implies that for each $f \in L^p(0, T; D_A(\theta, p))$ the solution $u$ of (1.1) belongs to $L^p(0, T; (D_A(\alpha, p), D_A(\alpha + 1, p))_{1-(\alpha-\theta), p})$. On the other hand, the space $(D_A(\alpha, p), D_A(\alpha + 1, p))_{1-(\alpha-\theta), p}$ coincides with $D_A(\theta + 1, p)$, with equivalence of the respective norms: this is well known and can be seen, for instance, using Theorems 1.14.5 and 1.10.2 (reiteration theorem) of [23]. It follows that $u \in L^p(0, T; D_A(\theta + 1, p))$. So we recover an old result of Da Prato and Grisvard [5] (see also [7]).

2) Let $T(t)$ be an analytic semigroup of negative type with generator $A$ in a Banach space $X$, and set $Y_0 = D_A(0, p) =$ completion of $\{x \in X : t \mapsto t\|AT(t)x\| \in L^p_*(0, \infty)\}$ in the norm $|x| = (\int_0^\infty (t\|AT(t)x\|)^p dt/t)^{1/p}$, $1 \le p < \infty$. The semigroup $T(t)$ has a natural extension to $Y_0$. Choosing $Y_1 = D_A(1/2, p)$, $Y_2 = D_A(3/2, p)$, (1.2) holds with $\gamma_1 = 1/2$, $\gamma_2 = 3/2$. Theorem 1.1 implies that for each $f \in L^p(0, T; D_A(0, p))$ the solution $u$ of (1.1) belongs to $L^p(0, T; (D_A(1/2, p), D_A(3/2, p))_{1/2, p}) = L^p(0, T; D_A(1, p))$ (the equality $(D_A(1/2, p), D_A(3/2, p))_{1/2, p} = D_A(1, p)$ follows again from Theorems 1.14.5 and 1.10.2 of [23]). So we recover a result of Di Blasio [8].

3) Other – and probably more interesting – applications are provided by non analytic semigroups, such as the Ornstein-Uhlenbeck semigroup generated by the realization $A$ of the differential operator

$$\mathcal{A}u = \mathrm{Tr}(QD^2u) + \langle Bx, Du\rangle$$

(where $Q$ is symmetric and positive definite, and $B \ne 0$) in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. See [22]. We choose here $Y_0 = W^{\theta, p}(\mathbb{R}^n)$, $Y_1 = W^{\alpha, p}(\mathbb{R}^n)$, $Y_2 = W^{\alpha+2, p}(\mathbb{R}^n)$, with $0 < \theta < \alpha < 1$, and we prove that (1.2) holds with $\gamma_1 = (\alpha-\theta)/2$, $\gamma_2 = 1 + (\alpha-\theta)/2$.

Theorem 1.1 implies that for each $f \in L^p(0, T; W^{\theta,p}(\mathbb{R}^n))$ the solution $u$ of (1.1) belongs to $L^p(0, T; (W^{\alpha,p}(\mathbb{R}^n), W^{2+\alpha,p}(\mathbb{R}^n))_{1-(\alpha-\theta)/2,p}) = L^p(0, T; W^{2+\theta,p}(\mathbb{R}^n))$. Consequently, $t \mapsto u_t(t,\cdot) - \langle B\cdot, Du(t,\cdot)\rangle$ belongs to $L^p(0, T; W^{\theta,p}(\mathbb{R}^n))$, but in general $u_t$ and $\langle B\cdot, Du(t,\cdot)\rangle$ do not.

The study of elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^n$ grew considerably in these years: see the recent papers [2, 3, 6, 18-22]. Presumably, the developments of the theory will allow to apply Theorem 1.1 to a wider class of parabolic problems with unbounded coefficients in the near future.

In a recent paper [17] a stationary version of Theorem 1.1 has been proved: we have shown that if (1.2) holds, then the domain of the part of $A$ in $Y_0$ is continuously embedded in $(Y_1, Y_2)_{\gamma,p}$. Moreover we have obtained a result similar to the one of Theorem 1.1 in the case $p = \infty$, which allowed us to prove several optimal Hölder regularity results for parabolic problems [4, 17, 19, 20]. The technique used here is similar.

## 2. Proof of Theorem 1.1

We recall that for $0 < \theta < 1$, $1 \le p < \infty$, $(Y_1, Y_2)_{\theta,p}$ is the subspace of $Y_1$ consisting of all $y$ such that $\xi \mapsto \xi^{-\theta} K(\xi, y) \in L^p_*(0,1)$, where

$$K(\xi, y) = \inf_{y=a+b,\ a\in Y_1,\ b\in Y_2} \|a\|_{Y_1} + \xi\|b\|_{Y_2}.$$

$(Y_1, Y_2)_{\theta,p}$ is endowed with the norm

$$\|y\|_{\theta,p} = \|\xi \mapsto \xi^{-\theta} K(\xi, y)\|_{L^p_*(0,1)} = \left(\int_0^1 K(\xi, y)^p \frac{d\xi}{\xi^{1+\theta p}}\right)^{1/p}.$$

We shall use the Hardy-Young inequalities (see [14, pp. 245-246]), which hold for every nonnegative measurable function $\varphi : (0, +\infty) \mapsto \mathbb{R}$, and $\alpha > 0$, $1 \le p < \infty$.

$$(2.1)\qquad \begin{cases} (i) & \displaystyle\int_0^\infty \sigma^{-\alpha p}\left(\int_0^\sigma \varphi(s)\frac{ds}{s}\right)^p \frac{d\sigma}{\sigma} \le \frac{1}{\alpha^p}\int_0^\infty s^{-\alpha p}\varphi(s)^p \frac{ds}{s}, \\ (ii) & \displaystyle\int_0^\infty \sigma^{\alpha p}\left(\int_\sigma^\infty \varphi(s)\frac{ds}{s}\right)^p \frac{d\sigma}{\sigma} \le \frac{1}{\alpha^p}\int_0^\infty s^{\alpha p}\varphi(s)^p \frac{ds}{s}. \end{cases}$$

Let $u : [0, T] \mapsto Y_0$ be given by the variation of constants formula,

$$u(t) = \int_0^t T(s) f(t-s)\, ds, \quad 0 < t < T.$$

For every $t \in (0, T)$ and $\xi \in (0, 1)$ split $u(t)$ as

$$u(t) = a(t, \xi) + b(t, \xi),$$

where

$$a(t, \xi) = \int_0^{\min\{\xi^{1/(\gamma_2-\gamma_1)}, t\}} T(s) f(t-s)\, ds,$$

$$b(t, \xi) = \int_{\min\{\xi^{1/(\gamma_2-\gamma_1)}, t\}}^t T(s) f(t-s)\, ds.$$

Our aim is to show that

$$(t,\xi)\mapsto \xi^{-\theta}\|a(t,\xi)\|_{Y_1}\in L^p\left((0\,,\,T)\times(0\,,1);dt\times\frac{d\xi}{\xi}\right),$$
$$(t,\xi)\mapsto \xi^{1-\theta}\|b(t,\xi)\|_{Y_2}\in L^p\left((0\,,\,T)\times(0\,,1);dt\times\frac{d\xi}{\xi}\right),$$

and their respective norms are estimated by $C\|f\|_{L^p((0\,,T);Y_0)}$.

In fact we have

$$\int_0^T dt\int_0^1 \xi^{-\theta p}\|a(t,\xi)\|_{Y_1}^p\frac{d\xi}{\xi}\le\int_0^T dt\int_0^1 \xi^{-\frac{(1-\gamma_1)p}{\gamma_2-\gamma_1}}\left(\int_0^{\min\{\xi^{1/(\gamma_2-\gamma_1)},t\}}\frac{c_1(s)}{s^{\gamma_1}}\|f(t-s)\|_{Y_0}ds\right)^p\frac{d\xi}{\xi}\le$$
$$\le\int_0^T dt\int_0^\infty \sigma^{-(1-\gamma_1)p}(\gamma_2-\gamma_1)\left(\int_0^\sigma\frac{c_1(s)}{s^{\gamma_1-1}}\chi_{(0\,,t)}(s)\|f(t-s)\|_{Y_0}\frac{ds}{s}\right)^p\frac{d\sigma}{\sigma}.$$

We use now (2.1($i$)) with $\alpha=1-\gamma_1$ and we get that the last integral is less or equal to

$$\frac{\gamma_2-\gamma_1}{(1-\gamma_1)^p}\int_0^T dt\int_0^\infty c_1(s)^p\chi_{(0\,,t)}(s)\|f(t-s)\|_{Y_0}^p\frac{ds}{s}=$$
$$=\frac{\gamma_2-\gamma_1}{(1-\gamma_1)^p}\int_0^T c_1(s)^p\frac{ds}{s}\int_s^T\|f(t-s)\|_{Y_0}^p dt\le\frac{\gamma_2-\gamma_1}{(1-\gamma_1)^p}\|c_1\|_{L_*^p(0\,,T)}^p\|f\|_{L^p(0\,,T;Y_0)}^p.$$

Therefore, $(t,\xi)\mapsto\xi^{-\theta}\|a(t,\xi)\|_{Y_1}\in L^p((0\,,\,T)\times(0\,,1);dt\times d\xi/\xi)$, with norm less or equal to $C\|f\|_{L^p(0\,,T;Y_0)}$.

Let us consider now the function $b$. We have

$$\int_0^T dt\int_0^1\xi^{(1-\theta)p}\|b(t,\xi)\|_{Y_2}^p\frac{d\xi}{\xi}\le\int_0^T dt\int_0^1\xi^{\frac{(\gamma_2-1)p}{\gamma_2-\gamma_1}}\left(\int_{\min\{\xi^{1/(\gamma_2-\gamma_1)},t\}}^t\frac{c_2(s)}{s^{\gamma_2}}\|f(t-s)\|_{Y_0}ds\right)^p\frac{d\xi}{\xi}\le$$
$$\le\int_0^T dt\int_0^\infty\sigma^{(\gamma_2-1)p}(\gamma_2-\gamma_1)\int_\sigma^\infty\frac{c_2(s)}{s^{\gamma_2-1}}\chi_{(0\,,t)}(s)\|f(t-s)\|_{Y_0}\frac{ds}{s}\Bigg)^p\frac{d\sigma}{\sigma}.$$

We use here (2.1($ii$)) with $\alpha=\gamma_2-1$ to estimate the last integral by

$$\frac{\gamma_2-\gamma_1}{(\gamma_2-1)^p}\int_0^T dt\int_0^t c_2(s)^p\|f(t-s)\|_{Y_0}^p\frac{ds}{s}=$$
$$=\frac{\gamma_2-\gamma_1}{(\gamma_2-1)^p}\int_0^T c_2(s)^p\frac{ds}{s}\int_s^T\|f(t-s)\|_{Y_0}^p dt\le\frac{\gamma_2-\gamma_1}{(\gamma_2-1)^p}\|c_2\|_{L_*^p(0\,,T)}^p\|f\|_{L^p(0\,,T;Y_0)}^p.$$

Therefore, also $(t,\xi)\mapsto\xi^{1-\theta}\|b(t,\xi)\|_{Y_2}$ belongs to $L^p((0\,,\,T)\times(0\,,1);dt\times d\xi/\xi)$, with norm less or equal to $C\|f\|_{L^p(0\,,T;Y_0)}$, and the statement follows.

REMARK 2.1. Theorem 1.1 may be extended without any important changement to evolution operators $U(t,s)$ satisfying, similarly to (1.2),

$$\left\{\begin{array}{l}(t-s)^{\gamma_i}\|U(t,s)\|_{L(Y_0\,,Y_i)}\le c_i(s)\,,\quad 0<s<t,\ \ i=1\,,2\,,\\[2mm] c_i\in L_*^p(0\,,\,T)\,,\quad i=1\,,2.\end{array}\right.\tag{2.2}$$

In this case the statement of Theorem 1.1 holds for the function $u(t)=\int_0^t U(t,s)f(s)ds$.

## 3. Examples and applications

The most obvious applications of Theorem 1.1 are optimal regularity results for (1.1) when $A$ generates an analytic semigroup in a Banach space $X$.

In such a case the assumptions of Theorem 1.1 are satisfied by $Y_0 = D_A(\theta, p)$, $Y_1 = D_A(\alpha, p)$, $Y_2 = D_A(\alpha + 1, p)$, with $0 \le \theta < \alpha < \theta + 1$, thanks to next Lemma 3.1.

The spaces $D_A(\theta, p)$ are defined as follows.

Assume first that $A$ is of negative type. For $\theta > 0$, $D_A(\theta, p)$ is the real interpolation space $(X, D(A^m))_{\theta/m,p}$, where $m$ is any integer $> \theta$. It is independent of $m$ and it coincides with the set of all $x \in X$ such that

$$\|x\|^{(m)}_{D_A(\theta,p)} = \left(\int_0^\infty (t^{m-\theta}\|A^m e^{tA}x\|)^p \frac{dt}{t}\right)^{1/p} < \infty.$$

See [23, §1.14.5]. As a canonical norm we shall consider $\|x\|_{D_A(\theta,p)} = \|x\|^{([\theta]+1)}_{D_A(\theta,p)}$, dropping the superscript $[\theta]+1$. Here $[\theta]$ is the integral part of $\theta$.

Let now $\theta = 0$. We denote by $D_A(0, p)$ the completion of $\{x \in X : \|x\|_{D_A(0,p)} < \infty\}$ with respect to the norm

$$\|x\|_{D_A(0,p)} = \left(\int_0^\infty (t\|Ae^{tA}x\|)^p \frac{dt}{t}\right)^{1/p}.$$

$D_A(0, p)$ is not in general an interpolation space, and it is not in general embedded in $X$. In any case the semigroup $T(t)$ has a natural extension to $D_A(0, p)$, which will be denoted again by $T(t)$. The domain of the generator of $T(t)$ in $D_A(0, p)$ is the space $D_A(1, p)$ characterized by

$$D_A(1, p) = \left\{x \in X : \|x\|_{D_A(1,p)} = \left(\int_0^\infty \|tA^2 e^{tA}x\|^p \frac{dt}{t}\right)^{1/p} < \infty\right\}.$$

See [8, 9].

Let now the type of $A$ be any real number, and fix $\theta \in [0, 1)$, $p \in [1, \infty)$. It is possible to show that for all $\omega \in \mathbb{R}$ such that $A_\omega = A - \omega I$ is of negative type the spaces $D_{A_\omega}(\theta, p)$ coincide, and their norms are equivalent. See [23, §1.14.5] for $\theta > 0$, [9] for $\theta = 0$. If the type of $A$ is nonnegative we choose $\omega = 1 + 2$ type of $A$ and we set $D_A(\theta, p) = D_{A_\omega}(\theta, p)$.

Next lemma was proved in [18] in the case $p = 2$, $\alpha < 1$; its generalization to any $p$ and $\alpha$ is easy but we write down the proof for the reader's convenience.

Lemma 3.1. *Let $A$ be the generator of an analytic semigroup in a Banach space $X$, and let $0 \le \theta < \alpha$, $1 \le p < \infty$, $T > 0$. Then there is $C = C(\theta, \alpha, T) > 0$ such that for every $f \in D_A(\theta, p)$*

$$\left(\int_0^T t^{(\alpha-\theta)p}\|T(t)f\|^p_{D_A(\alpha,p)} \frac{dt}{t}\right)^{1/p} \le C\|f\|_{D_A(\theta,p)}.$$

*In other words, the function $t \mapsto t^{(\alpha-\theta)}\|T(t)\|_{L(D_A(\theta,p),D_A(\alpha,p))}$ belongs to $L^p_*(0, T)$.*

PROOF. We may assume that $A$ is of negative type. Let $f \in D_A(\theta, p)$, let $m$ be any integer bigger than $\alpha$, and let $[\theta]$ denote the integral part of $\theta$. Then

$$\int_0^T t^{(\alpha-\theta)p} \| T(t)f\|^p_{D_A(\alpha, p)} \frac{dt}{t} = \int_0^T t^{(\alpha-\theta)p} \int_0^\infty (\xi^{m-\alpha} \|A^m T(t+\xi)f\|)^p \frac{d\xi}{\xi} \frac{dt}{t} =$$

$$= \int_0^T t^{(\alpha-\theta)p-1} \int_t^\infty ((s-t)^{m-\alpha-1/p} \|A^m T(s)f\|)^p \, ds\, dt \leq$$

$$\leq \int_0^\infty \|A^m T(s)f\|^p \int_0^s t^{(\alpha-\theta)p-1} (s-t)^{(m-\alpha)p-1} \, dt\, ds =$$

$$= \int_0^\infty s^{(m-\theta)p-1} \|A^{m-1-[\theta]} T(s/2)\|^p_{L(X)} \|A^{[\theta]+1} T(s/2)f\|^p \int_0^1 \sigma^{(\alpha-\theta)p-1} (1-\sigma)^{(m-\alpha)p-1} \, d\sigma\, ds \leq$$

$$\leq C \|f\|^p_{D_A(\theta, p)},$$

where we have used the estimates $\|A^k T(t)\|_{L(X)} \leq M_k t^{-k}$ for every $k \in \mathbb{N}$.

Some obvious remarks have to be made for $\theta = 0$, in the case where $D_A(0, p)$ is not embedded in $X$. The above estimates are correct if applied to any $f \in X$ such that $\|f\|_{D_A(0,p)}$ is finite. Every element of $D_A(0, p)$ is identified with a sequence $\{f_n\}$ of elements of $X$ which is a Cauchy sequence in the norm $\| \cdot \|_{D_A(0,p)}$; the final statement follows letting $n \to \infty$. □

We may now apply Theorem 1.1, with $Y_0 = D_A(\theta, p)$, $Y_1 = D_A(\alpha, p)$, $Y_2 = D_A(\alpha+1, p)$, $0 \leq \theta < \alpha < \theta + 1$ to get the following result.

THEOREM 3.2. *Let $A : D(A) \subset X \mapsto X$ be the generator of analytic semigroup $T(t)$ in $X$, let $1 \leq p < \infty$, $0 \leq \theta < \infty$, $T > 0$. Then for every $f \in L^p(0, T; D_A(\theta, p))$ the solution $u$ of* (1.1) *belongs to $L^p(0, T; D_A(\theta+1, p))$, and consequently to $W^{1,p}(0, T; D_A(\theta, p))$. There exists $C > 0$, independent of $f$, such that*

$$\|u\|_{L^p(0, T; D_A(\theta+1, p))} \leq C \|f\|_{L^p(0, T; D_A(\theta, p))}.$$

For $\theta > 0$ Theorem 3.2 has been proved in [5], with the aid of the theory of sum of commuting operators. Another independent proof may be found in [7]. For $\theta = 0$ Theorem 3.2 has been proved in [8], with the techniques of [7].

An obvious application of Theorem 3.2 is to initial boundary value problems for general $2m$ order elliptic operators with regular coefficients in a regular open set $\Omega \subset \mathbb{R}^n$, and boundary operators satisfying the Agmon-Douglis-Nirenberg conditions. Actually it is known [1, 12] that the realizations of such operators generate analytic semigroups in $X = L^q(\Omega)$ for every $q \geq 1$, and the interpolation spaces $D_A(\theta, p)$ for $\theta > 0$ have been completely characterized in [11, 13] as (subspaces of) Besov spaces, which coincide with fractional Sobolev spaces if $p = q$ for noncritical values of the parameters.

Since all this is well known, we skip it referring the reader to [5, 7, 9].

3.1. *An application to a non analytic semigroup.*

Let $T(t)$ be the Ornstein-Uhlenbeck semigroup defined in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, by

$$(3.1)\qquad \begin{cases} (T(t)\varphi)(x) = \dfrac{1}{(4\pi)^{n/2}(\mathrm{Det}\, Q_t)^{1/2}} \displaystyle\int_{\mathbb{R}^n} e^{-\frac{1}{4}\langle Q_t^{-1}y, y\rangle} \varphi(e^{tB}x - y)\,dy, \quad t > 0, \\ \\ T(0)\varphi = \varphi. \end{cases}$$

Here $Q = [q_{ij}]_{i,j=1,\dots,n}$ is any symmetric positive definite matrix, $B = [b_{ij}]_{i,j=1,\dots,n}$ is any nonzero matrix and

$$(3.2)\qquad Q_t = \int_0^t e^{sB} Q e^{sB^*}\, ds, \quad t \geq 0.$$

Its infinitesimal generator is the realization $A$ of the differential operator

$$\mathcal{A}\varphi = \sum_{i,j=1}^n q_{ij} D_{ij}\varphi + \sum_{i,j=1}^n b_{ij} x_j D_i \varphi$$

in $L^p(\mathbb{R}^n)$, and it is not an analytic semigroup (the counterexample in [22] for $n = 1$ may be easily extended to any $n$). However, it enjoys good smoothing properties, which we state in next Proposition 3.4. For the proof we need a lemma which may be of interest in itself.

LEMMA 3.3. *Let $T > 0$, and let $X$, $D$ be Banach spaces, with $D \subset X$. Assume that $A : [0, T] \mapsto L(D, X)$ is a $\gamma$-Hölder continuous function, with $\gamma > 0$, such that for every $t \in [0, T]$ $A(t) : D(A(t)) = D \mapsto X$ generates an analytic semigroup. Denote by $G(t, s)$ the relevant evolution operator. Then for $0 < \theta < \alpha < 1$ and $1 \leq p < \infty$ the function $t \mapsto t^{\alpha-\theta}\| G(t, 0)\|_{L((X,D)_{\theta,p},(X,D)_{\alpha,p})}$ belongs to $L^p_*(0, T)$.*

PROOF. We know from Lemma 3.1 that $t \mapsto t^{\alpha-\theta}\| e^{tA(0)}\|_{L((X,D)_{\theta,p},(X,D)_{\alpha,p})}$ belongs to $L^p_*(0, T)$, therefore it is sufficient to prove that $t \mapsto t^{\alpha-\theta}\| G(t, 0) - e^{tA(0)}\|_{L((X,D)_{\theta,p},(X,D)_{\alpha,p})}$ is in $L^p_*(0, T)$.

Let $x \in (X, D)_{\theta,p} = D_{A(0)}(\theta, p)$, and set

$$v(t) = G(t, 0)x - e^{tA(0)}x, \quad 0 \leq t \leq T.$$

$v$ satisfies

$$\begin{cases} v'(t) = A(0)v(t) + (A(t) - A(0))G(t, 0)x, \quad 0 < t \leq T, \\ v(0) = 0, \end{cases}$$

so that

$$v(t) = \int_0^t e^{(t-s)A(0)}(A(s) - A(0))G(s, 0)x\, ds, \quad 0 \leq t \leq T.$$

To estimate $v(t)$ we use the well known estimates

$$\| G(\sigma, 0)x\|_D \leq \frac{C}{\sigma^{1-\theta}}\|x\|_{(X,D)_{\theta,p}}, \quad \|e^{\sigma A(0)}y\|_{(X,D)_{\alpha,p}} \leq \frac{C}{\sigma^{\alpha}}\|y\|_X, \quad 0 < \sigma \leq T.$$

They imply

$$\|v(t)\|_{(X,D)_{\alpha,p}} \le C^2 \int_0^t \frac{1}{(t-s)^{\alpha} s^{1-\theta-\gamma}} ds \|x\|_{(X,D)_{\theta,p}} = C' t^{\gamma+\theta-\alpha} \|x\|_{(X,D)_{\theta,p}}, \quad 0 < t \le T.$$

It follows that $t \mapsto t^{\alpha-\theta}\|v(t)\|_{(X,D)_{\alpha,p}} \in L^p_*(0, T)$, with norm less or equal to $C\|x\|_{(X,D)_{\theta,p}}$, and the statement is proved. □

Proposition 3.4. *Let $T(t)$ be defined by (3.1). For $0 < \theta < \alpha < 2$ non integers and for every $T > 0$*

$$\text{(3.3)} \quad \begin{aligned} &(i) \qquad t \mapsto t^{(\alpha-\theta)/2} \| T(t)\|_{L(W^{\theta,p}(\mathbb{R}^n), W^{\alpha,p}(\mathbb{R}^n))} \in L^p_*(0, T), \\ &(ii) \qquad t \mapsto t^{1+(\alpha-\theta)/2} \| T(t)\|_{L(W^{\theta,p}(\mathbb{R}^n), W^{\alpha+2,p}(\mathbb{R}^n))} \in L^p_*(0, T). \end{aligned}$$

Proof. Let us introduce the family of operators $A(t) : D(A(t)) = W^{2,p}(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$,

$$A(t)u = \mathrm{Tr}(e^{tB} Q e^{tB^*} D^2 u), \quad t \ge 0.$$

For every $T > 0$ the function $[0, T] \mapsto L(W^{2,p}(\mathbb{R}^n), L^p(\mathbb{R}^n))$, $t \mapsto A(t)$ is Lipschitz continuous. Moreover each $A(t)$ is the realization of a constant coefficients elliptic operator in $L^p(\mathbb{R}^n)$, so it generates an analytic semigroup by [1], and $W^{\theta,p}(\mathbb{R}^n) = D_{A(t)}(\theta/2, p)$ if $\theta \ne 1$.

Let $G(t, s)$ be the evolution operator generated by the family $\{A(t) : t \ge 0\}$ in $L^p(\mathbb{R}^n)$. For every $\varphi \in L^p(\mathbb{R}^n)$ we have

$$\text{(3.4)} \qquad (T(t)\varphi)(x) = (G(t, 0)\varphi)(e^{tB}x), \quad t \ge 0, \ x \in \mathbb{R}^n.$$

Then statement $(i)$ is a consequence of Lemma 3.3.

Concerning $(ii)$, (3.4) implies that for every $\varphi \in W^{\theta,p}(\mathbb{R}^n) = D_{A(0)}(\theta/2, p)$, $T(t)\varphi \in D_{A(0)}(\theta/2 + 1, p) = W^{\theta+2,p}(\mathbb{R}^n)$ and there is $C > 0$ such that

$$\text{(3.5)} \qquad \| T(t)\varphi\|_{W^{\theta+2,p}} \le \frac{C}{t} \|\varphi\|_{W^{\theta,p}}, \quad 0 < t \le T.$$

Moreover we use the identity ($D^2$ stands for the Hessian matrix)

$$D^2 T(t)\psi = e^{tB^*} (T(t) D^2\psi) e^{tB}, \quad \text{for } \psi \in W^{2,p}(\mathbb{R}^n),$$

which can be readily deduced from (3.1), and which implies

$$\begin{aligned} &\int_0^T t^{(2+\alpha-\theta)p/2} \| T(t)\varphi\|^p_{W^{\alpha+2,p}} \frac{dt}{t} \le \\ &\le \int_0^T t^{(2+\alpha-\theta)p/2} \left( \| T(t)\varphi\|_{W^{2,p}} + \sum_{i,j=1}^n \| D_{ij} T(t/2) T(t/2)\varphi\|_{W^{\alpha,p}} \right)^p \frac{dt}{t} \le \\ &\le C\left( \|\varphi\|^p_{W^{\theta,p}} + \sum_{i,j=1}^n \int_0^T t^{(2+\alpha-\theta)p/2} \|(e^{tB^*/2}(T(t/2) D^2 T(t/2)\varphi) e^{tB/2})_{ij}\|^p_{W^{\alpha,p}} \frac{dt}{t} \right) \le \end{aligned}$$

$$\leq C\|\varphi\|^p_{W^{\theta,p}} + C\sum_{k,l=1}^{n}\int_0^T t^{(2+\alpha-\theta)p/2}\|T(t/2)D_{kl}T(t/2)\varphi\|^p_{W^{\alpha,p}}\frac{dt}{t} \leq$$

$$\leq C\|\varphi\|^p_{W^{s,p}} + C\int_0^T t^{(\alpha-\theta)p/2}(\|T(t/2)\|_{L(W^{\theta,p},W^{\alpha,p})}\|tT(t/2)\|_{L(W^{\theta,p},W^{\theta+2,p})}\|\varphi\|_{W^{\theta,p}})^p\frac{dt}{t} \leq$$

$$\leq C\|\varphi\|^p_{W^{\theta,p}}$$

(in the last inequality we have used both (3.3($i$)) and (3.5)). □

We may state the final result concerning problem

$$\text{(3.6)}\quad \begin{cases} u_t(t,x) = \sum_{i,j=1}^{n} q_{ij}D_{ij}u(t,x) + \sum_{i,j=1}^{n} b_{ij}x_jD_iu(t,x) + f(t,x), & 0 \leq t \leq T,\ x \in \mathbb{R}^n, \\ u(0,x) = 0, \quad x \in \mathbb{R}^n. \end{cases}$$

THEOREM 3.5. *Let* $f \in L^p(0,T;W^{\theta,p}(\mathbb{R}^n))$, *with* $0 < \theta < 1$, $1 < p < \infty$. *Then the solution* $u$ *of* (3.6) *belongs to* $L^p(0,T;W^{\theta+2,p}(\mathbb{R}^n))$, *and there is* $C > 0$, *independent of* $f$, *such that*

$$\|u\|_{L^p(0,T;W^{\theta+2,p}(\mathbb{R}^n))} \leq C\|f\|_{L^p(0,T;W^{\theta,p}(\mathbb{R}^n))}.$$

PROOF. It is sufficient to apply Theorem 1.1 with $Y_0 = W^{\theta,p}(\mathbb{R}^n)$, $Y_1 = W^{\alpha,p}(\mathbb{R}^n)$, $Y_2 = W^{\alpha+2,p}(\mathbb{R}^n)$ with $\theta < \alpha < 1$, taking into account Proposition 3.4. □

For a thorough study of the Ornstein-Uhlenbeck semigroup in spaces of continuous functions and in Hölder spaces rather than Lebesgue and Sobolev spaces, see [6].

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Dipartimento di Matematica
Università degli Studi di Parma
Via M. D'Azeglio, 85/A - 43100 Parma
lunardi@prmat.math.unipr.it