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Analisi matematica. — *On analyticity of Ornstein-Uhlenbeck semigroups.* Nota di

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ABSTRACT. — Let (R_t) be a transition semigroup of the Hilbert space-valued nonsymmetric Ornstein-Uhlenbeck process and let μ denote its Gaussian invariant measure. We show that the semigroup (R_t) is analytic in $L^2(\mu)$ if and only if its generator is variational. In particular, we show that the transition semigroup of a finite dimensional Ornstein-Uhlenbeck process is analytic if and only if the Wiener process is nondegenerate.

KEY WORDS: Ornstein-Uhlenbeck semigroup; Bilinear form; Variational generator; Polynomial chaos; Second quantization.

RIASSUNTO. — *Sull'analiticità del semigruppo di Ornstein-Uhlenbeck.* Sia (R_t) un semigruppo di transizione di un processo di Ornstein-Uhlenbeck non simmetrico e a valori in uno spazio di Hilbert e sia μ la sua misura Gaussiana invariante. Proviamo che il semigruppo (R_t) è analitico in $L^2(\mu)$ se e solo se il suo generatore è variazionale. In particolare dimostriamo che il semigruppo di transizione di un processo di Ornstein-Uhlenbeck finito dimensionale è analitico se e soltanto se il processo di Wiener è non degenere.

0. INTRODUCTION

This work deals with properties of the solution to a linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = Lu(t, x)\,, \\ u(0\,, x) = \phi(x)\,, \qquad t \geq 0\,, \end{cases} \tag{0.1}$$

on a real separable Hilbert space H. The operator L in this equation stands for the so-called Ornstein-Uhlenbeck operator

$$L\phi(x) = \frac{1}{2}\mathrm{tr}\left(QD^2\phi(x)\right) + \langle Ax\,, D\phi(x)\rangle\,, \tag{0.2}$$

where $D\phi$ denotes the Fréchet derivative of a function $\phi : H \to \mathbf{R}$. We assume that A is a generator of the C_0-semigroup $S(t)$, $t \geq 0$, of bounded operators on H and Q is a bounded linear operator on H which is moreover symmetric and nonnegative. In this paper we require that

$$\int_0^\infty \mathrm{tr}\,(S(u)QS^*(u))\,du < \infty \qquad \text{and} \qquad \ker Q_\infty = \{0\}\,, \tag{A}$$

where

$$Q_\infty = \int_0^\infty S(u)QS^*(u)\,du\,,$$

and $\mathrm{tr}(T)$ stands for the trace of a nuclear operator on H. If (A) holds then we can define on H the family of Gaussian measures μ_t, $t \geq 0$, and μ with the mean zero and

(*) Nella seduta del 23 aprile 1999.

the covariance operators

$$Q_t = \int_0^t S(u)QS^*(u)du$$

and $Q_\infty = \int_0^\infty S(u)QS^*(u)du$ respectively.

If ϕ is a sufficiently smooth cylindrical function then $L\phi$ is well defined and (0.1) has a classical solution $u(t,\cdot) = R_t\phi(\cdot)$ given by the formula (see [6] for details)

$$R_t\phi(x) = \int_H \phi(S(t)x + y)\,\mu_t(dy).$$

Moreover, the measure μ is invariant for R_t for every $t \geq 0$, that is $\int_H R_t\phi(x)\mu(dx) = \int_H \phi(x)\mu(dx)$ and the family of operators $\{R_t,\ t \geq 0\}$, defines a strongly continuous semigroup of contractions on $L^2(H,\mu)$. It has been shown in [1] that L has a unique extension to a generator of the C_0-semigroup on $L^2(H,\mu)$ which coincides with (R_t). The semigroup (R_t) may be identified as the transition semigroup corresponding to the Ornstein-Uhlenbeck process on H. If, for $x \in H$, we define

$$Z^x(t) = S(t)x + \int_0^t S(t-s)dW(s),$$

where W is a Wiener process on H with the covariance operator Q, then $R_t\phi(x) = E\phi\left(Z^x(t)\right)$ (see [6] for details).

The aim of this paper is to give necessary and sufficient conditions for analyticity of the semigroup (R_t). Note that this property does not hold in the space of continuous and vanishing at infinity functions even if H is finite dimensional (see [5]). The first results on the analyticity of nonsymmetric Ornstein-Uhlenbeck semigroup can be found in [11]. Recently, sufficient conditions were given in [9] for the finite dimensional case and in [8] for an arbitrary Hilbert space. The approach in [11] and [8] was to impose conditions on the semigroup (R_t) which assure that the generator L is variational. In this paper we justify this approach. Namely, we show that the semigroup (R_t) is analytic if and only if L is variational. In fact, we show that one of the sufficient conditions given in [8] when properly reformulated turns out to be a sector condition and is necessary for the analyticity of the Ornstein-Uhlenbeck semigroup (R_t). In other words the Ornstein-Uhlenbeck semigroup is analytic if and only if its generator defines a nonsymmetric Dirichlet form. The general theory of such processes can be found in [10] (see also references therein).

The proof is based on the fact shown in [2] that the Ornstein-Uhlenbeck semigroup can be obtained as a result of the second quantization procedure $\Gamma\left(S_0^*(t)\right)$ applied to a properly defined C_0-semigroup $S_0^*(t)$ acting on H (see Section 1 for definition). We show that this property holds also for a holomorphic extension of (R_t) (if exists) to a sector in a complex plane. It is well known that the second quantization operator $\Gamma(T)$ of the operator T is bounded on $L^2(H,\mu)$ if and only if T is a contraction on H. Therefore, to prove the aforementioned result it remains to apply to the semigroup $S_0^*(t)$ the characterization of holomorphic contraction semigroup given in [10].

Section 1 below contains some auxiliary results on the Wiener-Ito decomposition of the space $L^2(H,\mu)$ and the representation of the semigroup (R_t) as a second quantization operator. In Section 2 we show that the semigroup (R_t) is analytic if and only if the generator A_0^* of the semigroup $S_0^*(t)$ satisfies the sector condition. We provide also some necessary conditions for analyticity and in some cases we derive more explicit sufficient conditions for analyticity extending earlier results of [9, 7]. In particular we show that if $\dim H < \infty$ then (R_t) is analytic if and only if Q is nondegenerate. Sufficiency of this condition in finite dimension has been shown by a different method in [9].

We finish this section with a remark on notation. In what follows we use the same notation $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ for the norm and inner product in the spaces H and $L^2(H,\mu)$. The relevant meaning will be obvious from the context.

1. Wiener-Ito decomposition and the Ornstein-Uhlenbeck semigroup

We shall enunciate and prove theorems without repeating the assumptions on the operator L (given by (0.2)) made earlier in the Introduction.

Let $H_0 = Q_\infty^{1/2}(H)$ be the Reproducing Kernel Hilbert Space of the measure μ. The space H_0 endowed with the norm $\|x\|_0 = \left\|Q_\infty^{-1/2}x\right\|$ is continuously and densely imbedded into H. The operator

$$S_0^*(t) = Q_\infty^{1/2} S^*(t) Q_\infty^{-1/2}$$

is clearly well defined and bounded on H_0. Moreover, it has been shown in [2] that the space H_0 is invariant for the semigroup $S(t)$:

$$S(t)(H_0) \subset H_0$$

for every $t \geq 0$ and therefore $S_0^*(t)$ can be extended to a bounded operator $\overline{S_0^*(t)}$ on H (see [2] for details). For the reader's convenience we repeat the relevant properties of the operators $S_0^*(t)$ in the lemma below.

Lemma 1.1. *The family of operators* $\{S_0^*(t),\ t \geq 0\}$ *defines a strongly continuous semigroup on* H_0. *Its generator* A_0^* *has the domain*

$$\operatorname{dom}\left(A_0^*\right) = Q_\infty^{1/2}\left(\operatorname{dom}\left(A^*\right)\right)$$

and $A_0^* h = Q_\infty^{1/2} A^* Q_\infty^{-1/2} h$ *for* $h \in \operatorname{dom}\left(A_0^*\right)$. *The family of operators* $(\overline{S_0^*(t)})$ *defines a strongly continuous semigroup of contractions on* H *and the generator* A_0^* *is the part of the generator* $\overline{A_0^*}$ *of* $(\overline{S_0^*(t)})$ *in* H_0. *Finally,* $S(t)H_0 \subset H_0$ *for every* $t \geq 0$, *the family of operators* $S_0(t) = Q_\infty^{-1/2} S(t) Q_\infty^{1/2}$, $t \geq 0$, *defines a strongly continuous semigroup of contractions on* H *and the semigroup* $(\overline{S_0^*(t)})$ *is adjoint to the semigroup* $(S_0(t))$.

Proof. See [2].

In the sequel we use the same notation S_0^* for the semigroup S_0^* on H_0 and $\overline{S_0^*}$ on H.

Let H be a separable real Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathbf{R}}$ and let $H_{\mathbf{C}}$ denote its complexification $H_{\mathbf{C}} = H + iH$ endowed with the inner product

$$\langle h + ik, x + iy\rangle_{\mathbf{C}} = \langle h, x\rangle_{\mathbf{R}} + \langle k, y\rangle_{\mathbf{R}} + i\left(\langle h, y\rangle_{\mathbf{R}} - \langle k, x\rangle_{\mathbf{R}}\right).$$

From now on we omit the subscript $\mathbf{C}$ and denote by $\langle\cdot,\cdot\rangle$ the inner product in $H_{\mathbf{C}}$. By $L^2_{\mathbf{C}}(H,\mu)$ we denote the space of $\mathbf{C}$-valued square integrable functions defined on H. For every $h \in H_0$ we define a linear function on H

$$\phi_h(x) = \left\langle x, Q_\infty^{-1/2} h\right\rangle.$$

If $h = h_1 + ih_2$ with $h_1, h_2 \in H_0$ then $\phi_h = \phi_{h_1} + i\phi_{h_2}$.

Let $\mathcal{H}_{\leq n}$ denote the closed subspace of $L^2_{\mathbf{C}}(H,\mu)$ spanned by all products $\phi_{h_1}\cdot\ldots\cdot\phi_{h_m}$ of order $m \leq n$ of the functions $\phi_{h_1},\ldots,\phi_{h_m}$,where $h_1,\ldots,h_m \in H_0 + iH_0$ and let $\mathcal{H}_n$ be the orthogonal complement of $\mathcal{H}_{\leq n-1}$ in $\mathcal{H}_{\leq n}$. Then the Ito-Wiener decomposition says that

$$L^2_{\mathbf{C}}(H,\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_0$ is the space generated by constants. For $h \in H_0$ we define the function

$$E_h = \exp\left(\phi_h - \frac{1}{2}\|h\|^2\right).$$

The family $\left\{E_h : h \in H_0\right\}$ is linearly dense in $L^2_{\mathbf{C}}(H,\mu)$.

We will recall now some basic properties of the operator of second quantization as defined for example in [12], see also [2]. Let I_n be the orthogonal projection of $L^2_{\mathbf{C}}(H,\mu)$ onto $\mathcal{H}_n$. If T is a bounded operator on $H_{\mathbf{C}}$ then we define the operator $\Gamma_n(T) : \mathcal{H}_n \to \mathcal{H}_n$ for $n \geq 1$ by the formula

$$\Gamma_n(T) I_n(\phi_{h_1}\ldots\phi_{h_n}) = I_n(\phi_{Th_1}\ldots\phi_{Th_n}). \tag{1.1}$$

For $n = 0$ we put $\Gamma_0(T)1 = 1$. The operator Γ_n can be extended to the whole of $\mathcal{H}_n$ and $\|\Gamma_n(T)\| = \|T\|^n$. If $\|T\| \leq 1$ then the formula

$$\Gamma(T)\phi = \sum_{n\geq 0}\Gamma_n(T) I_n(\phi) \tag{1.2}$$

defines a bounded operator on $L^2_{\mathbf{C}}(H,\mu)$ and $\|\Gamma(T)\| = 1$. If $\|T\| > 1$ then the operator $\Gamma(T)$ defined by (1.2) is necessarily unbounded. Let T_1 and T_2 be two contractions on $H_{\mathbf{C}}$. Then by (1.1)

$$\Gamma\left(T_1 T_2\right) = \Gamma\left(T_1\right)\Gamma\left(T_2\right). \tag{1.3}$$

A bounded operator $T : H \to H$ is extended to H_C by the formula $T^C(h + ik) = = Th + iTk$, $h, k \in H$. If $T : H \to H$ is a contraction on H then $T^{\mathbf{C}}$ is a contraction on $H_{\mathbf{C}}$ and the operator $\Gamma\left(T^{\mathbf{C}}\right)$ is well defined. Let $\Gamma^{\mathbf{R}}(T)$ be the second quantization operator defined by the formula (1.2) in the space $L^2_{\mathbf{R}}(H,\mu)$ and let $\Gamma^{\mathbf{C}}(T)(\phi+i\psi)= =\Gamma^{\mathbf{R}}(T)\phi+i\Gamma^{\mathbf{R}}(T)\psi$. Then

$$\Gamma^{\mathbf{C}}(T)=\Gamma\left(T^{\mathbf{C}}\right). \tag{1.4}$$

Indeed, for $h\in H_0$

$$\Gamma^{\mathbf{C}}(T)E_h=\Gamma^{\mathbf{R}}(T)E_h=\Gamma\left(T^{\mathbf{C}}\right)E_h$$

and (1.4) follows from the density of $\mathrm{lin}\{E_h:\ h\in H_0\}$.

Proposition 1.2. *For all* $t\geq 0$ $R^{\mathbf{C}}_t=\Gamma\left(S^*_0(t)^{\mathbf{C}}\right)$.

Proof. By Theorem 1 in [2] $R_tE_h=\Gamma(S^*_0(t))E_h$, hence the proposition follows from (1.4).

The existence of invariant measure for the semigroup (R_t) implies (see [6, Theorem 11.7]) that

$$\langle A^*x,y\rangle+\langle A^*y,x\rangle=-\langle Qx,y\rangle \tag{1.5}$$

for all $x,y\in\mathrm{dom}(A^*)$. We will use the notation $K=Q^{1/2}_\infty(\mathrm{dom}(A^*))$. Note that by (A) K is dense in H and it can be easily seen by Lemma 1.1 and the Core Theorem that K is a core for the operator A^*_0 acting in H. Putting $x=Q^{-1/2}_\infty h$ and $y=Q^{-1/2}_\infty k$ for $h,k\in K$, we can rewrite (1.1) in the form

$$\left\langle A^*_0h,k\right\rangle+\left\langle A^*_0k,h\right\rangle=-\langle Vh,Vk\rangle, \tag{1.6}$$

where $V=Q^{1/2}Q^{-1/2}_\infty$ is an operator in H with the domain $\mathrm{dom}(V)=K$.

By $\mathcal{P}(K)$ we denote the subspace of $L^2(H,\mu)$ spanned by all functions of the form ϕ^n_h, where $n\geq 0$ and $h\in K$. Then we define

$$D_Q\phi^n_h=Q^{1/2}D\phi^n_h=n\phi^{n-1}_hVh$$

and extend this definition to the whole of $\mathcal{P}(K)$ by linearity. The operator L defined by (0.2) may be rewritten in the form

$$L\phi(x)=\frac{1}{2}\mathrm{tr}\left(QD^2\phi(x)\right)+\left\langle Q^{-1/2}_\infty x,A^*_0Q^{1/2}_\infty D\phi(x)\right\rangle$$

and thereby

$$L\phi^n_h=\frac{n(n-1)}{2}\|Vh\|^2\phi^{n-2}_h+n\phi^{n-1}_h\phi_{A^*_0h}$$

is well defined for $h\in K$ and $n\geq 2$. Clearly, $L\phi_h=\phi_{A^*_0h}$ and $L1=0$. Hence, L extends to $\mathcal{P}(K)$ by linearity.

The next lemma is a minor variation of the result proved in [1] hence we omit the proof.

Lemma 1.3. *The operator* $(L,\mathcal{P}(K))$ *has a unique extension to a generator of a* C_0*-semigroup on* $L^2(H,\mu)$ *which may be identified with* (R_t).

We introduce now the bilinear form

$$\mathcal{E}(\phi, \psi) = \langle -L\phi, \psi \rangle$$

with the domain $\operatorname{dom}(\mathcal{E}) = \mathcal{P}(K)$. The symmetric part of $\mathcal{E}$

$$\mathcal{E}_s(\phi, \psi) = \frac{1}{2}(\mathcal{E}(\phi, \psi) + \mathcal{E}(\psi, \phi))$$

will be considered on the same domain $\mathcal{P}(K)$. Note that

$$\mathcal{E}\left(\phi_h, \phi_k\right) = \left\langle -A_0^* h, k \right\rangle, \quad h, k \in K.$$

If $\phi, \psi \in \mathcal{P}(K)$ then by the standard calculation

$$L(\phi\psi)(x) = \phi(x) L\psi(x) + \psi(x) L\phi(x) + \left\langle D_Q\phi(x), D_Q\psi(x) \right\rangle.$$

Therefore, integrating the above with respect to μ and taking into account that $\langle L\phi, 1\rangle = = 0$ we find that

$$\mathcal{E}_s(\phi, \psi) = \frac{1}{2}\left\langle D_Q\phi, D_Q\psi \right\rangle, \quad \phi, \psi \in \mathcal{P}(K).$$

In particular it follows from (1.6) that

$$\mathcal{E}_s(\phi_h, \phi_k) = \frac{1}{2}\langle Vh, Vk \rangle.$$

2. Analyticity

For $a > 0$ we define a sector

$$s(a) = \{z \in \mathbf{C} : |\operatorname{Im} z| \leq \operatorname{aRe} z\}.$$

We will be using the following definition of the holomorphic semigroup.

Definition 2.1. *The family* $\{T(z) : z \in s(a)\}$ *of bounded operators on a Hilbert space* $H_{\mathbf{C}}$ *is called a holomorphic semigroup with the sector* $s(a)$ *if*

(*i*) $T(0) = I$,

(*ii*) $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in s(a)$,

(*iii*) *for every* $h \in H_{\mathbf{C}}$

$$\lim_{z \to 0} \|T(z)h - h\| = 0$$

provided $z \in s(\tilde{a})$ *with any* $\tilde{a} < a$.

(*iv*) *the function* $z \to \langle T(z)h, k \rangle$ *is analytic in the interior of* $s(a)$ *for all* $h, k \in H_{\mathbf{C}}$.

(*v*) *if* $\|T(z)\| \leq 1$ *for all* $z \in s(a)$ *then we say that* $\{T(z) : z \in s(a)\}$ *is a holomorphic semigroup of contractions.*

Theorem 2.2. *The semigroup* $(R_t^{\mathbf{C}})$ *is a restriction of a holomorphic semigroup if and only if there exists* $a > 0$ *such that*

$$\left|\left\langle A_0^* h, k \right\rangle_{\mathbf{R}}\right| \leq a\|Vh\|\|Vk\| \tag{2.1}$$

for all $h, k \in K$. *Moreover, if* (2.1) *holds then* $s\left(\frac{1}{2a}\right)$ *is the analyticity sector for the semigroups* $\left(R_t^{\mathbf{C}}\right)$ *and* $\left(S_0^{*\mathbf{C}}(t)\right)$, $\left\|R_z^{\mathbf{C}}\right\| = 1$ *for* $z \in s\left(\frac{1}{2a}\right)$, *and*

$$|\langle L\phi, \psi\rangle| \leq a\left\|D_Q\phi\right\|\left\|D_Q\psi\right\| \tag{2.2}$$

for $\phi, \psi \in \operatorname{dom}(L)$.

Proof. *Sufficiency.* Assume that (2.1) holds. By (1.6)

$$\|Vh\|^2 = -2\left\langle A_0^* h, h\right\rangle, \quad h \in K. \tag{2.3}$$

Let $h \in \operatorname{dom}\left(A_0^*\right)$ and let $\left(h_n\right) \subset K$ be such a sequence that $h_n \to h$ and $A_0^* h_n \to A_0^* h$ in H. Such a sequence exists because K is a core for A_0^*. Then

$$\left\|V\left(h_n - h_m\right)\right\|^2 = -2\left\langle A_0^*\left(h_n - h_m\right), h_n - h_m\right\rangle$$

and it follows that V can be extended to $\operatorname{dom}\left(A_0^*\right)$. Denoting still this extension by V we find that (2.1) and (2.3) hold for $h, k \in \operatorname{dom}\left(A_0^*\right)$. Note that (2.1) is a condition for continuity of the bilinear form associated to the generator A_0^*. Hence by Corollary I.2.21 in [10] $(S_0^*(t)^{\mathbf{C}})$ is a restriction of the holomorphic semigroup of contractions $S_0^*(z)$, $z \in s\left(\frac{1}{2a}\right)$. Therefore, the operator $\Gamma\left(S_0^*(z)\right)$ is well defined in $L^2_{\mathbf{C}}(H, \mu)$ and $\left\|\Gamma\left(S_0^*(z)\right)\right\| = 1$. The function $t \to R_t^{\mathbf{C}}$ is a restriction of the function

$$R_z = \Gamma\left(S_0^*(z)\right)$$

defined for $z \in s\left(\frac{1}{2a}\right)$ to the halfaxis $t > 0$. It remains to show that R_z is a holomorphic semigroup. The semigroup property follows immediately from (1.3). We shall show now that for $\phi \in L^2_{\mathbf{C}}(H, \mu)$ and $z \in s(\widetilde{a})$, $\widetilde{a} < \frac{1}{2a}$

$$\lim_{z \to 0}\left\|R_z\phi - \phi\right\| = 0. \tag{2.4}$$

By uniform boundedness it is enough to check this property for the monomials $I_n\left(\phi_h^n\right)$ for $n \geq 1$ and $h \in H_0$. Since $\|\phi_h^n = C_n\|h\|^n$ for all $n \geq 0$ and $h \in H_0$ we obtain for a certain $C > 0$

$$\begin{aligned}\left\|R_z I_n\left(\phi_h^n\right) - I_n\left(\phi_h^n\right)\right\| &= \left\|I_n\left(\phi^n_{S_0^*(z)h}\right) - I_n\left(\phi_h^n\right)\right\| \leq \\ &\leq \left\|\phi^n_{S_0^*(z)h} - \phi_h^n\right\| \leq \sum_{k=1}^{n-1}\binom{n}{k}\left\|\phi^k_{S_0^*(z)h-h}\right\|\left\|\phi_h^{n-k}\right\| \leq C\left\|S_0^*(z)h - h\right\|\|h\|^{n-1}\end{aligned}$$

and (2.4) follows. Finally, since $\|R_z\| \leq 1$ it is enough to show that for all $h, k \in H_0$ the function

$$t \to \left\langle R_t^{\mathbf{C}} E_h, E_k\right\rangle$$

extends to a holomorphic function on the interior of the sector $s\left(\frac{1}{2a}\right)$. Indeed, we have

$$\left\langle R_t^{\mathbf{C}} E_h, E_k\right\rangle = \left\langle R_t E_h, E_k\right\rangle = \left\langle E_{S_0^*(t)h}, E_k\right\rangle = \exp\left(\left\langle S_0^*(t)h, k\right\rangle\right)$$

and the proof of sufficiency is finished. Moreover if $R_z = \Gamma\left(S_0^*(z)\right)$ then $R_z\mathbf{1} = \mathbf{1}$ and therefore $\left\|R_z\right\| = 1$. It follows easily from the proof of sufficiency that $R_t^{\mathbf{C}}$ extends to a holomorphic semigroup in the sector $s\left(\frac{1}{2a}\right)$ if and only if $(S_0^*(t)^{\mathbf{C}})$ extends to holomorphic semigroup of contractions in the same sector $s\left(\frac{1}{2a}\right)$. Finally, because the semigroup $(R_t^{\mathbf{C}})$ is a restriction of the holomorphic semigroup of contractions, Corollary I.2.21 in [10] yields (2.2).

Necessity. Let $T(z)$, $z \in s\left(\frac{1}{2a}\right)$ be a holomorphic semigroup extending $\{R_t^{\mathbf{C}},\ t \geq 0\}$ for a certain $a > 0$. Then for any $n \geq 1$, $\phi \in \mathcal{H}_n$ and $\psi \in \mathcal{H}_n^{\perp}$ Proposition 1.2 yields $\langle T(z)\phi, \psi\rangle = 0$. In particular, the semigroup $T(z)$ when restricted to $\mathcal{H}_1$ is a holomorphic extension of the C_0-semigroup $R_t^{\mathbf{C}} I_1$ and because

$$R_t^{\mathbf{C}} I_1 \phi_h = \phi_{S_0^*(t)h}$$

we find that the semigroup $(S_0^*(t)^{\mathbf{C}})$ can be extended to a holomorphic semigroup in the sector $s\left(\frac{1}{2a}\right)$. Similar argument shows that for all $n > 1$ the semigroup $R_t^{\mathbf{C}} I_n = \Gamma_n\left(S_0^*(t)^{\mathbf{C}}\right)$ is a restriction of the holomorphic semigroup $\Gamma_n\left(S_0^*(z)\right)$ and therefore

$$T(z) I_n = \Gamma_n\left(S_0^*(z)\right).$$

Assume that $\left\|S_0^*(z)\right\| > 1$ for a certain $z \in s\left(\frac{1}{2a}\right)$. Then taking into account that $\left\|T(z)I_n\right\| = \left\|S_0^*(z)\right\|^n$ we obtain

$$\lim_{n\to\infty} \left\|T(z)I_n\right\| = \infty.$$

Hence $T(z)$ is unbounded on $L^2_{\mathbf{C}}(H,\mu)$ which gives the desired contradiction. It follows that $\left\|S_0^*(z)\right\| \leq 1$ for all $z \in s\left(\frac{1}{2a}\right)$ and therefore invoking again Corollary I.2.21 in [10] and (1.2) we obtain

$$\left|\langle A_0^* h, k\rangle_{\mathbf{R}}\right| \leq a\sqrt{\langle -A_0^* h, h\rangle_{\mathbf{R}}}\sqrt{\langle -A_0^* k, k\rangle_{\mathbf{R}}} = a\,\|Vh\|\,\|Vk\|\ ,$$

for $h, k \in K$.

Remark 2.3. Let $W_Q^{1,2}(H,\mu)$ denote the completion of $\mathcal{P}(K)$ with respect to the norm

$$\|\phi\|^2_{W_Q^{1,2}} = \|\phi\|^2 + \left\|Q^{1/2} D\phi\right\|^2.$$

If $\left(R_t\right)$ is analytic then by Theorem 2.2 above and Proposition 3.3 in [10] the bilinear form $\mathcal{E}_s$ is closable in $L^2(H,\mu)$ and its domain may be identified with $W_Q^{1,2}(H,\mu)$ which is a subspace of $L^2(H,\mu)$ in this case. Equivalently, the operator V with the domain K is closable in H and $H_0 \subset \mathrm{dom}\left(\overline{V}\right)$.

Corollary 2.4. *Assume that* (2.1) *is satisfied. Then the following conditions hold.*

(*i*) $\ker Q = \{0\}$.

(*ii*) $Q_\infty(H) \subset \mathrm{dom}\,(A)$.

Proof. (*i*) Note first that the operator $V = Q^{1/2} Q_\infty^{-1/2}$ is well defined on H_0. Assume that $Qx = 0$ for a certain $x \in H$. Then $x = Q_\infty^{-1/2} h$ for a certain $h \in H$ and

$Vh = 0$. Let $(h_n) \subset K$ be chosen in such a way that $x_n = Q_\infty^{-1/2} h_n$ converges to x. Then

$$h_n \to h \quad \text{and} \quad Vh_n \to 0.$$

It follows from (2.1) that

$$\left|\langle A_0^* h_n, k\rangle_{\mathbf{R}}\right| \le a \left\| Vh_n \right\| \| Vk\| \tag{2.5}$$

for all $k \in K$. Moreover,

$$S_0^*(t)h_n - h_n = \int_0^t S_0^*(s) A_0^* h_n \, ds,$$

and therefore

$$S_0^*(t)h - h = \lim_{n\to\infty} \int_0^t S_0^*(s) A_0^* h_n \, ds.$$

On the other hand, for any $k \in K$

$$\lim_{n\to\infty} \int_0^t \langle S_0^*(s) A_0^* h_n, k\rangle_{\mathbf{R}} \, ds = 0$$

by (2.5) and this yields $S_0^*(t)h = h$ for all $t \ge 0$. The identity $Q_t = Q_\infty - S(t)Q_\infty S^*(t)$ implies

$$\langle Q_t x, x\rangle_{\mathbf{R}} = \|h\|^2 - \left\|S_0^*(t)h\right\|^2 = 0$$

and therefore

$$\langle Q_\infty x, x\rangle_{\mathbf{R}} = \lim_{t\to\infty} \langle Q_t x, x\rangle_{\mathbf{R}} = 0$$

which gives $x = 0$.

(*ii*) Let $h = Q_\infty^{1/2} x$ and $k = Q_\infty^{1/2} y$ for some $x, y \in \operatorname{dom}(A^*)$, a dense subset of H. Then (2.1) yields

$$\left|\langle Q_\infty A^* x, y\rangle_{\mathbf{R}}\right| \le a \|Q\| \|x\| \|y\|$$

and this implies boundedness of the operator $Q_\infty A^*$.

The next corollary is an extension of the result proved in [7] by a completely different method.

COROLLARY 2.5. *Assume that the operator Q has bounded inverse. Then the semigroup (R_t) is analytic if and only if*

$$Q_\infty(H) \subset \operatorname{dom}(A) . \tag{2.6}$$

PROOF. In view of Corollary 2.4 it is enough to prove sufficiency. Taking into account invertibility of Q and invoking Lemma 1.1 we obtain for $h, k \in K$

$$\begin{aligned} \left|\langle A_0^* h, k\rangle_{\mathbf{R}}\right| &= \left|\left\langle Q_\infty A^* Q_\infty^{-1/2} h, Q_\infty^{-1/2} k\right\rangle_{\mathbf{R}}\right| \le \\ &\le \left\|Q_\infty A^*\right\| \left\|Q_\infty^{-1/2} h\right\| \left\|Q_\infty^{-1/2} k\right\| \le c \left\|Q^{1/2} Q_\infty^{-1/2} h\right\| \left\|Q^{1/2} Q_\infty^{-1/2} k\right\| \end{aligned}$$

and the proof corollary follows.

REMARK. It has been proved in [4] that if A generates an analytic semigroup of contractions on H and $Q = I$ then (2.6) holds.

COROLLARY 2.6. *Assume that* $\dim H < \infty$. *Then* (R_t) *is analytic if and only if* Q *is invertible.*

PROOF. If $\dim H < \infty$ then (2.6) is trivially satisfied and sufficiency follows from Corollary 2.5. Necessity follows immediately from (*ii*) of Corollary 2.4.

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REFERENCES

[1] A. CHOJNOWSKA-MICHALIK - B. GOLDYS, *Existence, uniqueness and invariant measures for stochastic semi-linear equations on Hilbert spaces*. Probability Theory and Related Fields, 102, 1995, 331-356.
[2] A. CHOJNOWSKA-MICHALIK - B. GOLDYS, *Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator*. J. Math. Kyoto Univ., 36, 1996, 481-498.
[3] A. CHOJNOWSKA-MICHALIK - B. GOLDYS, *On regularity properties of nonsymmetric Ornstein-Uhlenbeck semigroup in L^p spaces*. Stochastics and Stochastics Rep., 59, 1996, 183-209.
[4] G. DA PRATO, *Null controllability and strong Feller property of Markov transition semigroups*. Nonlinear Analysis, Theory, Methods & Applications, 25, 1995, 941-949.
[5] G. DA PRATO - A. LUNARDI, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*. J. Funct. Anal., 131, 1995, 94-114.
[6] G. DA PRATO - J. ZABCZYK, *Stochastic equations in infinite dimensions*. Cambridge University Press, 1992.
[7] G. DA PRATO - J. ZABCZYK, *Ergodicity for infinite dimensional systems*. Cambridge University Press, 1996.
[8] M. FUHRMAN, *Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces*. Studia Mathematica, 115, 1995, 53-71.
[9] A. LUNARDI, *On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures*. Trans. Amer. Math. Soc. 349 (1997), 155-169.
[10] Z. M. MA - M. RÖCKNER, *Dirichlet forms*. Springer, 1992.
[11] B. SCHMULAND, *Non-symmetric Ornstein-Uhlenbeck processes in Banach space via Dirichlet forms*. Can. J. Math., 45, 1993, 1324-1338.
[12] B. SIMON, *The $P(\phi)_2$ Euclidean (quantum) field theory*. Princeton University Press 1974.

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