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## A decay estimate for a class of hyperbolic pseudo-differential equations

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**Equazioni a derivate parziali.** — *A decay estimate for a class of hyperbolic pseudo-differential equations.* Nota di SANDRA LUCENTE e GUIDO ZILIOTTI, presentata (\*) dal Corrisp. S. Spagnolo.

ABSTRACT. — We consider the equation  $u_t - i\Lambda u = 0$ , where  $\Lambda = \lambda(D_x)$  is a first order pseudo-differential operator with real symbol  $\lambda(\xi)$ . Under a suitable convexity assumption on  $\lambda$  we find the decay properties for  $u(t, x)$ . These can be applied to the linear Maxwell system in anisotropic media and to the nonlinear Cauchy Problem  $u_t - i\Lambda u = f(u)$ ,  $u(0, x) = g(x)$ . If  $f(u)$  is a smooth function which satisfies  $f(u) \simeq |u|^p$  near  $u = 0$ , and  $g$  is small in suitably Sobolev norm, we prove global existence theorems provided  $p$  is greater than a critical exponent.

KEY WORDS: Decay estimate; Nonlinear hyperbolic equations; Small data.

RIASSUNTO. — *Su una stima di decadimento per una classe di equazioni iperboliche pseudo-differenziali.* In questo lavoro si considera l'equazione  $u_t - i\Lambda u = 0$  ove  $\Lambda = \lambda(D_x)$  è un operatore pseudo-differenziale del primo ordine con simbolo  $\lambda(\xi)$  reale. Opportune ipotesi di convessità sui sottolivelli del simbolo  $\lambda$  consentono di dimostrare il decadimento della soluzione  $u(t, x)$ . Questa stima si applica al sistema di Maxwell in mezzi anisotropi e al seguente problema nonlineare:  $u_t - i\Lambda u = f(u)$ ,  $u(0, x) = g(x)$ . Supponendo che  $f(u)$  sia una funzione regolare che in un intorno dell'origine è equivalente a  $|u|^p$ , si dimostra l'esistenza globale della soluzione quando il dato è piccolo in una opportuna norma di Sobolev e l'esponente  $p$  è sovracritico.

## INTRODUCTION

Many equations of the Mathematical Physics can be written in the form of a first order system

$$\partial_t v - \sum_{j=1}^n A_j \partial_j v = F$$

where  $A_j$  are self-adjoint matrices (see [1, 4]). Under a suitable assumption on the multiplicity of bicharacteristic surfaces such a system can be diagonalized and hence reduced to a finite numbers of simple scalar equations like

$$(0.1) \quad u_t - i\Lambda u = f(t, x)$$

where  $\Lambda = \lambda(D_x)$  is a first order pseudo-differential operator.

In this work we examine decay properties of the solution to the linear problem (0.1) with  $f = 0$  and initial data

$$u(0, x) = g(x)$$

where  $g \in C_0^\infty$ . For simplicity we consider the case when  $\Lambda$  is a pseudo-differential

(\*) Nella seduta del 23 aprile 1999.

operator of convolution type, namely

$$\Lambda f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \lambda(\xi) \widehat{f}(\xi) d\xi$$

where the symbol  $\lambda$  is a real function in  $C^\infty(\mathbb{R}^n \setminus \{0\})$  homogeneous of degree one. Our first goal is to obtain a generalization of the classical Von Wahl estimate (cf. [11]) for the wave equation

$$\begin{aligned} \square u &= 0 \\ u(0, x) &= g_0(x) \\ u_t(0, x) &= g_1(x). \end{aligned}$$

In that case one has

$$(0.2) \quad \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{2}} \left( \|g_0\|_{W[\frac{n}{2}]+1,1} + \|g_1\|_{W[\frac{n}{2}],1} \right).$$

In this direction our first result is

**THEOREM 0.1.** *Let  $k, n \in \mathbb{N}$  such that  $n$  is even or  $n = 4k + 3$ . Let  $\Lambda$  be a pseudo-differential operator with an elliptic real symbol  $\lambda(\xi)$  homogeneous of degree 1 which satisfies  $\lambda(\xi) = \lambda(-\xi)$  and either*

(i)  $\lambda(\xi) \geq 0$ , and  $\{\lambda(\xi) \leq 1\}$  is a strictly convex set of  $\mathbb{R}^n$   
or

(ii)  $\lambda(\xi) \leq 0$ , and  $\{\lambda(\xi) \geq -1\}$  is a strictly convex set of  $\mathbb{R}^n$ .

Let  $u(t, x)$  be the solution to the Cauchy Problem

$$(0.3) \quad \begin{aligned} u_t - i\Lambda u &= 0 \quad x \in \mathbb{R}^n \\ u(0, x) &= g(x) \end{aligned}$$

with  $g \in C_0^\infty(\mathbb{R}^n)$ . Then:

$$(0.4) \quad \|u(t)\|_\infty \leq Ct^{-\frac{n-1}{2}} \|g\|_{W[\frac{n}{2}]+1,1} \quad \forall t \geq 1.$$

We shall give a simple application of this theorem to the Maxwell system in anisotropic media, *i.e.*

$$\begin{aligned} \epsilon \partial_t E &= \text{rot} H \\ \partial_t H &= -\text{rot} E, \end{aligned}$$

where  $\epsilon$  is a diagonal matrix and  $E = (E_1, E_2, E_3)$ ,  $H = (H_1, H_2, H_3)$  are the electric and magnetic vector fields.

The main application of Theorem 0.1 is a global existence result for semilinear hyperbolic pseudo-differential equations. More precisely, we prove the following two theorems:

**THEOREM 0.2.** *Let  $\Lambda$  be a pseudo-differential operator with symbol  $\lambda(\xi)$  satisfying the hypotheses of Theorem 0.1. Let us consider the Cauchy Problem*

$$(0.5) \quad \begin{aligned} u_t - i\Lambda u &= \alpha u^3 \quad x \in \mathbb{R}^3, \alpha \in \mathbb{R} \\ u(0, x) &= g(x). \end{aligned}$$

Therefore, there exists some  $\varepsilon > 0$  such that for all  $g \in W^{2,1} \cap W^{2,2}$  satisfying

$$\|g\|_{W^{2,1}} + \|g\|_{W^{2,2}} < \varepsilon,$$

the Problem (0.5) has a unique global solution  $u : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $u \in L^\infty(\mathbb{R}_+, W^{2,2}(\mathbb{R}^3))$ .

**THEOREM 0.3.** *Let  $\Lambda$  be a pseudo-differential operator with symbol  $\lambda(\xi)$  which satisfies the assumptions of Theorem 0.1 Let us consider the Cauchy Problem*

$$(0.6) \quad \begin{aligned} u_t - i\Lambda u &= f(u) \quad x \in \mathbb{R}^n \\ u(0, x) &= g(x), \end{aligned}$$

where  $f$  is a smooth function, such that  $f(0) = 0$  and

$$(0.7) \quad |D^k f(u)| \leq C|u|^{p-k} \quad \text{near } u = 0, \quad \text{for } k = 0, 1, 2.$$

Suppose

$$n = 2, \quad p > 4, \quad \text{or } n = 3, \quad p > 3.$$

Therefore there exists  $\varepsilon > 0$  such that for all  $g \in W^{2,1} \cap W^{2,2}$  satisfying

$$\|g\|_{W^{2,1}} + \|g\|_{W^{2,2}} < \varepsilon,$$

the Problem (0.6) has a unique global solution  $u : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $u \in L^\infty(\mathbb{R}_+, W^{2,2}(\mathbb{R}^n))$ . In addition we have

$$(0.8) \quad \|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C(1+t)^{-\frac{n-1}{2}}.$$

We shall study only the case  $t \rightarrow +\infty$ , since the case  $t \rightarrow -\infty$  is similar.

Let us briefly discuss the hypotheses of these theorems.

First we require that the symbol  $\lambda(\xi)$  is real, because we need our equation to be hyperbolic. For instance we consider the symbols of the form

$$\lambda(\xi) = \pm \left( \sum a_j \xi_j^q \right)^{1/q}, \quad a_j > 0, \quad 1 < q < \infty.$$

Taking  $a_j = 1$ ,  $q = 2$  we have  $\Lambda = \sqrt{-\Delta}$ , so that our problem can be regarded as a wave type problem since

$$\partial_{tt} - \Delta = (\partial_t - i\sqrt{-\Delta})(\partial_t + i\sqrt{-\Delta}).$$

For this reason the decay rates in (0.2) and in (0.4) are the same. Moreover the critical exponent for our problem is

$$p_C = 1 + \frac{2}{n-1},$$

just like the critical exponent of the wave equation. Another similarity with wave type equations can be seen in the case  $n = 3$ : according to whether  $p > 3$  or  $p = 3$  two different approaches have to be considered, just like for the three-dimensional Klein Gordon equation (cf. [9]).

The plan of the work is the following. In Section 1 we study the decay properties for the solutions to (0.3) and we apply the decay result to the Maxwell system in

anisotropic media. In Section 2 we give some inequalities which will be essential for the proof of the nonlinear results. In Section 3 we prove Theorem 0.3 and we give a generalization for the cases  $n \geq 4$ . Finally, using a different technique, in Section 4 we establish Theorem 0.2.

Few remarks about the notations: we put  $\|f\|_p := \|f\|_{L^p}$  and we always use the Fourier transform with respect to the space variable only:

$$\widehat{f}(t, x) = \mathcal{F}(f)(t, x) = (2\pi)^{-n/2} \int e^{-ix \cdot \xi} f(t, \xi) d\xi.$$

1. VON WAHL TYPE ESTIMATE

We consider  $\lambda$  and  $g$  as in Theorem 0.1 and the corresponding Cauchy Problem (0.3). We suppose that  $\lambda$  is positive, since in the case  $\lambda \leq 0$  the same argument works. By Fourier transform we find

$$\begin{aligned} \widehat{u}' - i\lambda(\xi)\widehat{u} &= 0 \\ \widehat{u}(0, \xi) &= \widehat{g}(\xi); \end{aligned} \tag{1.1}$$

hence  $\widehat{u}(t, \xi) = e^{it\lambda(\xi)}\widehat{g}(\xi)$ , and finally

$$u(t, x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi + it\lambda(\xi)} \widehat{g}(\xi) d\xi. \tag{1.2}$$

In other words  $u = E * g$  where  $E$  is the distribution defined by the following oscillatory integral:

$$E(t, x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi + it\lambda(\xi)} d\xi. \tag{1.3}$$

The proof of a decay estimate for the solution  $u(t, x)$  is carried out with the aid of the theory of oscillatory integral operators.

DEFINITION 1.1. *Let  $\Phi \in C^\infty(\mathbb{R}^n)$  be a real function and  $a \in C_0^\infty(\mathbb{R}^n)$ . The integral*

$$I(\mu) = \int_{\mathbb{R}^n} e^{i\mu\Phi(x)} a(x) dx$$

*is called oscillatory integral with phase  $\Phi$  and amplitude  $a$ .*

The analysis of the asymptotic behavior for such integrals relies on the following two theorems (cf. [8]):

PROPOSITION 1.1. *Suppose that  $\Phi, a$  are as in Definition 1.1. If  $|\nabla_x \Phi| \geq c > 0$  for all  $x$  in  $\text{supp } a$ ,  $I(\mu)$  is a rapidly decreasing function; in particular*

$$\left| \int_{\mathbb{R}^n} e^{i\mu\Phi(x)} a(x) dx \right| \leq C_N \mu^{-N} \quad N = 1, 2, \dots, \quad \text{for } \mu > 1 \tag{1.4}$$

*where  $C_N$  depends only on  $c$ , provided  $\Phi$  and  $a$  are bounded in  $C^\infty(\mathbb{R}^n)$ .*

PROPOSITION 1.2 (Stationary phase method). *Suppose that  $\Phi, a$  are as in Definition 1.1. Let  $x_0$  be an isolated non-degenerate critical point of  $\Phi$  in  $\text{supp } a$ . Then we have*

$$(1.5) \quad I(\mu) = \frac{C e^{i\mu\Phi(x_0)} a(x_0)}{\mu^{\frac{n}{2}} \sqrt{\det \Phi''(x_0)}} + O\left(\frac{1}{\mu^{\frac{n}{2}+1}}\right) \quad \text{for } \mu > 0.$$

REMARK 1.1. In what follows we need a modification of Proposition 1.1. Let us consider  $\Phi, \lambda, a \in C^\infty(\mathbb{R}^n \setminus \{0\})$ . In particular  $\Phi, \lambda$  are positive homogeneous function of degree one and  $a$  is rapidly decreasing function in  $\{|x| \geq 1\}$ . We get

$$(1.6) \quad \left| \int_{\mathbb{R}^n} e^{i\mu\Phi(x)} a(x) \frac{dx}{\lambda^\theta(x)} \right| < \frac{C}{\mu^{n-\theta}} \quad \text{if } 0 \leq \theta < n.$$

By a rescaling argument, it is sufficient to prove the previous estimate in the case  $\mu = 1$ . We take  $\varphi$  a cutoff function such that  $0 \leq \varphi \leq 1$  and  $\varphi(x) = 1$  if  $|x| \leq 1$  and  $\varphi(x) = 0$  if  $|x| \geq 2$ . Then

$$\int_{\mathbb{R}^n} e^{i\Phi(x)} a(x) \frac{dx}{\lambda^\theta(x)} = \int_{\mathbb{R}^n} e^{i\Phi(x)} \varphi(x) a(x) \frac{dx}{\lambda^\theta(x)} + \int_{\mathbb{R}^n} e^{i\Phi(x)} (1 - \varphi(x)) a(x) \frac{dx}{\lambda^\theta(x)}.$$

Using the assumption  $0 \leq \theta < n$  we have

$$\left| \int_{\mathbb{R}^n} e^{i\Phi(x)} \varphi(x) a(x) \frac{dx}{\lambda^\theta(x)} \right| \leq C \|a\|_\infty.$$

As to the second term, let  $\Psi_1, \dots, \Psi_N$  be a partition of unity for  $S^{n-1}$  which satisfies

$$\exists \nu_i \quad \text{such that} \quad \forall x \in \text{supp } \Psi_i \quad \left| \frac{\partial \Phi}{\partial \nu_i} \right| \geq \frac{K}{2}$$

being  $K = \inf_{S^{n-1}} |\nabla \Phi|$ . To this partition we associate the corresponding conic partition of  $\mathbb{R}^n$ , denoted by  $\tilde{\Psi}_1, \dots, \tilde{\Psi}_N$ . Finally we get

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i\Phi(x)} (1 - \varphi(x)) a(x) \frac{dx}{\lambda^\theta(x)} &= \sum_{i=1}^N \int_{\mathbb{R}^n} e^{i\Phi(x)} \tilde{\Psi}_i(x) (1 - \varphi(x)) a(x) \frac{dx}{\lambda^\theta(x)} = \\ &= i \sum_{i=1}^N \int_{\mathbb{R}^n} e^{i\Phi(x)} \frac{\partial}{\partial \nu_i} \left( \frac{\tilde{\Psi}_i(x)}{\partial \nu_i \Phi(x)} (1 - \varphi(x)) \frac{a(x)}{\lambda^\theta(x)} \right) dx. \end{aligned}$$

We can iterate this computation and to each step we have a bounded quantity; infact we are far away from the origin, since  $1 - \varphi(x)$  is supported in  $\{|x| \geq 1\}$ .

In order to apply the previous results to derive an  $L^\infty - L^1$  estimate for the solution (1.2) the convexity and ellipticity assumptions on the symbol  $\lambda$  come into play.

Ellipticity in turn implies that  $\lambda(\xi) \neq 0$  for  $\xi \in \mathbb{R}^n \setminus \{0\}$ ; since  $\lambda$  is homogeneous we get

$$(1.7) \quad |\nabla \lambda(\xi)| \geq \frac{1}{c} > 0;$$

where  $1/c = \min_{|\xi|=1} |\nabla \lambda(\xi)|$ . From this and from the strict convexity of  $\{\lambda(\xi) \leq 1\}$ ,

we achieve that the level surface

$$(1.8) \quad \Sigma := \{\lambda(\xi) = 1\}$$

can be represented, in a neighborhood of each of its points, by means of a convex function with nondegenerate Hessian matrix.

Due to the symmetry of  $\lambda$  we get that  $\lambda$  is a norm, so we define the dual norm

$$(1.9) \quad \nu(x) = \sup_{\lambda(\xi)=1} x \cdot \xi.$$

In particular, using Lagrange multiplier Theorem, we find

$$\nu(\nabla\lambda(\xi)) = 1 \quad \forall \xi.$$

Now we are in a position to prove Theorem 0.1

Consider a partition of unity for  $\mathbb{S}^{n-1}$ , say  $\{\Phi_k\}_{k=1, \dots, n}$ , such that

$$(1.10) \quad x \in \text{supp } \Phi_k \text{ implies } |x_k| > C$$

for some constant  $0 < C < 1$  which depends only on the space dimension. Let  $b \in \mathbb{N}$  we write the solution (1.2) in the following form:

$$\begin{aligned} u(t, x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi + it\lambda(\xi)} d\xi g(y) dy = \\ &= (2\pi)^{-n} \sum_{k=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_k \left( \frac{\xi}{|\xi|} \right) e^{i(x-y)\cdot\xi + it\lambda(\xi)} d\xi g(y) dy = \\ &= (2\pi)^{-n} \sum_{k=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial y_k} \right)^b \left( \int_{\mathbb{R}^n} \Phi_k \left( \frac{\xi}{|\xi|} \right) \frac{e^{i(x-y)\cdot\xi + it\lambda(\xi)}}{(-i\xi_k)^b} d\xi \right) g(y) dy = \\ &= (2\pi)^{-n} (-i)^b \sum_{k=1}^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi_k \left( \frac{\xi}{|\xi|} \right) \frac{e^{i(x-y)\cdot\xi + it\lambda(\xi)}}{\xi_k^b} d\xi \left( \frac{\partial}{\partial y_k} \right)^b g(y) dy. \end{aligned}$$

From (1.10) we obtain that the function

$$(1.11) \quad \mu_k(\xi) := \Phi_k \left( \frac{\xi}{|\xi|} \right) \frac{\lambda(\xi)^b}{\xi_k^b}$$

is homogeneous of degree zero and bounded. Then we can write

$$(1.12) \quad u(t, x) = (2\pi)^{-n} (-i)^b \sum_{k=1}^n \int_{\mathbb{R}^n} K_k(y, t, x) \left( \frac{\partial}{\partial y_k} \right)^b g(y) dy$$

where

$$(1.13) \quad K_k(y, t, x) = \int_{\mathbb{R}^n} \mu_k(\xi) \frac{e^{i(x-y)\cdot\xi + it\lambda(\xi)}}{\lambda(\xi)^b} d\xi.$$

This integral has to be considered as the limit (in the distribution sense) for  $\varepsilon \rightarrow 0$  of



the quantities

$$K_{k,\varepsilon}(y, t, x) = \int_{\mathbb{R}^n} \mu_k(\xi) \frac{e^{-\varepsilon|\xi|^2 + i(x-y)\cdot\xi + it\lambda(\xi)}}{\lambda(\xi)^b} d\xi.$$

For simplicity of notation we omit to write the index  $k$  and the factor  $e^{-\varepsilon|\xi|^2}$ ; we remark only that all the constants in the estimates below are uniform with respect to  $\varepsilon$ . In order to estimate these kernels  $K(y, t, x)$ , it is necessary to consider two different cases:

$$(1.14) \quad \frac{\nu(x-y)}{t} < \frac{1}{2}$$

and

$$(1.15) \quad \frac{\nu(x-y)}{t} \geq \frac{1}{2}.$$

being  $\nu$  the function defined in (1.9).

Suppose (1.14) holds. We apply Proposition 1.1 and Remark 1.1 to estimate the kernel (1.13) with phase

$$\Phi_{x,y,t}(\xi) = \frac{x-y}{t} \cdot \xi + \lambda(\xi)$$

and parameter  $t$ . For the critical points we find  $\nu(x-y) = t$ . Hence no critical point verifies (1.14), and we have

$$(1.16) \quad \int_{\nu(x-y) \leq 1/2t} K(y, t, x) \sum_{|\alpha|=b} \left( \frac{\partial}{\partial y} \right)^\alpha g(y) dy \leq C_\theta t^{-n+b} \|g\|_{W^{b,1}}$$

where  $b = \lceil \frac{n}{2} \rceil + 1$ . In the case (1.15), we have to apply Proposition 1.2. Let be  $\Sigma$  defined in (1.8); we use the change of variable  $T : \mathbb{R}^n \rightarrow \mathbb{R} \times \Sigma$ , where  $T(\xi) = (\rho, \omega)$  with

$$(1.17) \quad \begin{cases} \rho = \lambda(\xi) \\ \omega = \frac{\xi}{\lambda(\xi)}. \end{cases}$$

By the coarea formula (cf. [3, 3.2.12])

$$\int_{\mathbb{R}^n} h(x) dx = \int_{-\infty}^{+\infty} \int_{\{f=\rho\}} h(y) |\nabla f(y)|^{-1} d\mathcal{H}^{n-1}(y) d\rho$$

which holds for any lipschitzian function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $|\nabla f| \neq 0$  and any integrable function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ , we then get

$$K(y, t, x) = \int_0^{+\infty} \rho^{n-1-b} \int_{\Sigma} e^{i\rho t \left( \frac{(x-y)\cdot\omega}{t} + 1 \right)} \mu(\omega) |\nabla \lambda(\omega)|^{-1} d\omega d\rho.$$

Now we deal with the kernel

$$(1.18) \quad K_{x,y}(\rho, t) = \int_{\Sigma} e^{i\rho t \left( \frac{\omega \cdot (x-y)}{t} + 1 \right)} \mu(\omega) |\nabla \lambda(\omega)|^{-1} d\omega$$

with phase

$$\Phi_{x,y,t}(\omega) = \frac{\omega \cdot (x-y)}{t} + 1$$

and parameter  $\rho t$ . We remark that  $\Phi_{x,y,t}(\omega)$  depends on  $x, y, t$ , and  $x \neq y$  since we assume (1.15). In order to apply the stationary phase method, we take a partition of unity for  $\Sigma$ : a finite sequence of function  $\varphi_j : \mathbb{R}^n \rightarrow \mathbb{R}$ . with sufficiently small  $\text{diam}(\text{supp } \varphi_j)$ . This allows us to describe the surface  $\Sigma \cap \text{supp } \varphi_j$  by means of

$$\omega_{k_j} = \psi_j(\omega_1, \dots, \omega_{k_j-1}, \omega_{k_j+1}, \dots, \omega_n)$$

for suitable  $\psi_j : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ . Since  $\{\lambda(\xi) \leq 1\}$  is strictly convex, we see that  $\nabla^2 \psi_j$ , the Hessian of  $\psi_j$ , is positive definite. Thus we get

$$K_{x,y}(\rho, t) = \sum_j \int_{\Sigma} e^{i\rho t \left(\frac{\omega \cdot (x-y)}{t} + 1\right)} \mu(\omega) |\nabla \lambda(\omega)|^{-1} \varphi_j(\omega) d\omega =: \sum_j K_{j,x,y}(\rho, t).$$

For any  $\xi \in \mathbb{R}^n$  we put  $\tilde{\xi} := (\xi_1, \xi_{k_j-1}, \xi_{k_j+1}, \dots, \xi_n)$ . We see that

$$\Phi_{x,y,t}(\omega) = \frac{\tilde{x} - \tilde{y}}{t} \cdot \tilde{\omega} + \frac{x_{k_j} - y_{k_j}}{t} \psi_j(\tilde{\omega}) + 1,$$

$$K_{j,x,y}(\rho, t) = \int_{\mathbb{R}^{n-1}} e^{i\rho t \Phi_{x,y,t}(\tilde{\omega}, \psi_j(\tilde{\omega}))} \varphi_j(\tilde{\omega}, \psi_j(\tilde{\omega})) \frac{\chi_j(\tilde{\omega}) \mu(\tilde{\omega}, \psi_j(\tilde{\omega}))}{|\nabla \lambda(\tilde{\omega}, \psi_j(\tilde{\omega}))|} d\tilde{\omega}$$

where  $\chi_j(\tilde{\omega}) = \sqrt{1 + |\nabla \psi_j(\tilde{\omega})|^2}$ . The critical points of  $\Phi_{x,y,t}(\omega)$  satisfy

$$(1.19) \quad \nabla \Phi_{x,y,t}(\omega) = \frac{\tilde{x} - \tilde{y}}{t} + \frac{x_{k_j} - y_{k_j}}{t} \nabla \psi_j(\tilde{\omega}) = 0.$$

From the assumption (1.15), we see that there are not critical points if  $x_{k_j} = y_{k_j}$ ; in this case applying Proposition 1.1 we conclude:

$$K_{j,x,y}(\rho, t) = O((\rho t)^{-N}) \quad \forall N \in \mathbb{N}.$$

The same holds if the support of  $\varphi_j$  does not contain critical points. In the critical points instead we have

$$(1.20) \quad \nabla \psi_j(\tilde{\omega}_j) = -\frac{\tilde{x} - \tilde{y}}{x_{k_j} - y_{k_j}}.$$

From the relation

$$\nabla^2 \Phi_{x,y,t}(\omega) = \frac{x_{k_j} - y_{k_j}}{t} \nabla^2 \psi_j(\tilde{\omega}),$$

it follows that the critical points are nondegenerate.

Since  $\Sigma$  is strictly convex and the phase is linear in  $\omega$ , there are only two critical points, denoted by  $\omega_+, \omega_-$ . These points satisfy  $\omega_+ = -\omega_-$  and

$$\begin{aligned} \Phi_{x,y,t}(\omega_+) &= \frac{\nu(x-y)}{t} + 1 \\ \Phi_{x,y,t}(\omega_-) &= -\frac{\nu(x-y)}{t} + 1. \end{aligned}$$

Denoted by  $\varphi_+, \varphi_-$  the functions such that  $\omega_\pm \in \text{supp } \varphi_\pm$ , we can take  $\varphi_\pm(\omega_\pm) = 1$  and the corresponding functions  $\psi_\pm$  satisfying  $\psi_+(x) = -\psi_-(-x)$ . The stationary phase theorem, in  $\mathbb{R}^{n-1}$ , yields

$$K_{\pm, x, y}(\rho, t) = C e^{i\frac{\pi}{4} \text{sgn} \nabla^2 \Phi(\omega_\pm)} \frac{e^{i\rho t \Phi(\omega_\pm)} \chi(\tilde{\omega}_\pm) \varphi_\pm(\omega_\pm) \mu(\omega_\pm)}{(\rho t)^{\frac{n-1}{2}} |\nabla \lambda(\omega_\pm)| \sqrt{\det \nabla^2 \Phi_{x, y, t}(\omega_\pm)}} + O\left((\rho t)^{-\frac{n+1}{2}}\right).$$

We observe that the constant in the error term  $O\left((\rho t)^{-(n+1)/2}\right)$  is independent of  $x, y$ ; indeed it is possible to write an asymptotic formula for  $K_{j, x, y}$  of the following type:

$$K_{\pm, x, y}(\rho, t) = (\rho t)^{-\frac{n-1}{2}} \sum_{k=0}^{\infty} a_k (\rho t)^{-k}$$

where each  $a_k$  depends on a finite number of derivatives of the amplitude and on the  $(\det \nabla^2 \Phi_{x, y, t}(\omega_\pm))^{-1/2}$  (cf. [8, chapter VIII]). The essential point is that

$$(1.21) \quad \frac{1}{\sqrt{\det \nabla^2 \Phi_{x, y, t}(\omega_\pm)}} \leq C$$

where  $C$  is independent of  $x, y, t$ . This follows from the relation

$$|x - y| = \sqrt{1 + |\nabla \psi_j(\tilde{\omega}_\pm)|^2} |x_{k_j} - y_{k_j}|$$

which can be combined with the assumption (1.15). To gain the conclusion according to whether  $n$  is even or  $n = 4k + 3$ , two different approaches are necessary.

In the case  $n$  even, we split the integral (1.13) into two parts, near the origin we take  $h = n/2$ , far away from the origin we take  $h = n/2 + 1$ . For the leading term this gives

$$|K(y, t, x)| \leq C \left| \int_0^1 \rho^{-1/2} e^{i\rho t \Phi(\omega_\pm)} d\rho \right| + \left| \int_1^\infty \rho^{-3/2} e^{i\rho t \Phi(\omega_\pm)} d\rho \right|$$

and these integrals are absolutely convergent. Combining this with (1.12), and having in mind (1.16), we get the estimate (0.4).

In the case  $n = 4k + 3$  we have  $e^{i\frac{\pi}{4} \text{sgn} \nabla^2 \Phi(\omega_+)} = -e^{i\frac{\pi}{4} \text{sgn} \nabla^2 \Phi(\omega_-)}$ ; this fact and the symmetry assumption on the symbol  $\lambda$  assure that the leading term in the asymptotic development is

$$K_+(\rho, t) + K_-(\rho, t) = C \frac{e^{i\rho(t+\nu(x-y))} - e^{i\rho(t-\nu(x-y))}}{(\rho t)^{\frac{n-1}{2}}} + O\left((\rho t)^{\frac{n-1}{2}}\right).$$

Since

$$\left| \int_0^\infty \frac{e^{i\rho(t+\mu(x-y))} - e^{i\rho(t-\mu(x-y))}}{\rho} d\rho \right| \leq C$$

the desired estimate holds.

If we take  $\lambda \leq 0$  a slight change in the proof shows that (0.4) still holds, the only difference being the change of the set  $\Sigma = \{\lambda = -1\}$  and the fact that the transformation in (1.17) is replaced by  $T(\xi) = (\rho, \omega)$  such that

$$\begin{cases} \rho = -\lambda(\xi) \\ \omega = -\frac{\xi}{\lambda(\xi)}. \end{cases}$$

We conclude this section applying Theorem 0.1 to a linear first order system: Maxwell system in anisotropic media (cf. [5]). Let  $a, b$  be strictly positive numbers. We denote by  $\epsilon$  the matrix

$$\epsilon := \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}$$

and consider the system

$$\begin{aligned} \epsilon \partial_t E &= \text{rot} H \\ \partial_t H &= -\text{rot} E. \end{aligned}$$

Here  $E = (E_1, E_2, E_3)$  and  $H = (H_1, H_2, H_3)$  are the electric and magnetic vector fields. Let  $u : \mathbb{R}^3 \rightarrow \mathbb{R}^6$  be defined as  $u = (E_1, E_2, E_3, H_1, H_2, H_3)$ . We consider the initial data  $g = (g_1, g_2, g_3, g_4, g_5, g_6)$  such that  $g_j \in C_0^\infty(\mathbb{R}^3)$  for all  $j = 1, \dots, 6$ . Hence we deal with the equation

$$A_0 \partial_t u - \sum_{j=1}^n A_j \partial_{x_j} u = 0$$

and then

$$(1.23) \quad A_0 \widehat{u}' - iA(\xi) \widehat{u} = 0$$

where  $A_0, A(\xi)$  are the  $6 \times 6$  matrices given by

$$A_0 := \begin{bmatrix} \epsilon & 0 \\ 0 & I \end{bmatrix}, \quad (\text{in block notation})$$

$$A(\xi) := \begin{bmatrix} 0 & 0 & 0 & 0 & -\xi_3 & \xi_2 \\ 0 & 0 & 0 & \xi_3 & 0 & -\xi_1 \\ 0 & 0 & 0 & -\xi_2 & \xi_1 & 0 \\ 0 & \xi_3 & -\xi_2 & 0 & 0 & 0 \\ -\xi_3 & 0 & \xi_1 & 0 & 0 & 0 \\ \xi_2 & -\xi_1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Denoting by  $A_0^{1/2}$  the diagonal matrix with non zero entries  $(\sqrt{a}, \sqrt{b}, \sqrt{b}, 1, 1, 1)$  and putting

$$\omega = A_0^{1/2} \widehat{u},$$

we can write (1.23) in the form

$$(1.24) \quad \widehat{\omega}' - iA_0^{-1/2}A(\xi)A_0^{1/2}\widehat{\omega} = 0.$$

The matrix  $A_0^{1/2}A(\xi)A_0^{-1/2}$  is symmetric, hence it is diagonalizable, with eigenvalues

$$\begin{aligned} \lambda_{1,2} &= 0 \\ \lambda_{3,4}(\xi) &= \pm \sqrt{b^2\xi_1^2 + a^2\xi_2^2 + a^2\xi_3^2} \\ \lambda_{5,6}(\xi) &= \pm b|\xi|. \end{aligned}$$

Except for  $\lambda_{1,2}(\xi)$ , these eigenvalues satisfy the assumptions of Theorem 0.1.

For any non zero eigenvalue  $\lambda_j$ , denote by  $\pi(\lambda_j)$  the projection on the corresponding eigenspace, so that

$$I = \pi_{\ker A} + \sum_{j=3}^6 \pi(\lambda_j).$$

Applying these projections to the equation (1.24) for  $j = 3, 4, 5, 6$ , we find

$$\left(\pi(\lambda_j)\widehat{\omega}\right)' - i\lambda_j\left(\pi(\lambda_j)\widehat{\omega}\right) = 0.$$

and hence

$$\|u_j(t)\|_\infty \leq C\|\mathcal{F}^{-1}\pi(\lambda_j)\widehat{\omega}\|_\infty \leq C(g)(1+t)^{-1}.$$

In conclusion, to establish a decay result for the vector  $u$ , we have to suppose that  $u_1 = u_2 = 0$ . This occurs if the initial data  $g$  verify  $\pi_{\ker A}(\widehat{g}) = 0$ . This condition of differential nature in the classical case  $\epsilon = I$  implies the other two physical Maxwell laws in the case of null charge density:

$$\operatorname{div}E = 0, \operatorname{div}H = 0.$$

Instead in our case we have to require a similar relation, *i.e.*

$$b\frac{\partial E_1}{\partial x_1} + a\frac{\partial E_2}{\partial x_2} + a\frac{\partial E_3}{\partial x_3} = 0, \operatorname{div}H = 0.$$

## 2. BASIC INEQUALITIES

In order to find the global solution to the nonlinear problem (0.6), we have to combine the linear estimate (0.4), obtained in the previous section, with suitable energy inequalities and nonlinear estimates in Sobolev spaces.

The *energy estimates* will be set in  $H^k(\mathbb{R}^n) := W^{k,2}(\mathbb{R}^n)$ ; it will be useful to recall that  $\|u(t)\|_{H^k} \simeq \|(1 + |\xi|^2)^{k/2}\widehat{u}\|_2$ .

For the homogeneous problem (0.3) we have the following conservation laws:

$$\|u(t)\|_{H^k} = \|g\|_{H^k} \quad \text{for any } k \in \mathbb{N}.$$

Let us consider the non homogeneous problem

$$(2.2) \quad \begin{aligned} u_t - i\Lambda u &= f \\ u(0, x) &= g(x), \end{aligned}$$

Multiplying by  $\bar{u}$  the equivalent problem

$$\begin{aligned} \hat{u}' - i\lambda(\xi)\hat{u} &= \hat{f} \\ \hat{u}(0, \xi) &= \hat{g}(\xi) \end{aligned}$$

we get:

$$\frac{d}{dt} \int |\hat{u}(t, \xi)|^2 d\xi = 2 \int \text{Re} \hat{f}(t, \xi) \bar{\hat{u}}(t, \xi) d\xi.$$

Gronwall Lemma then implies

$$\|u(t)\|_2 \leq \|g\|_2 + C \int_0^t \|f(\tau)\|_2 d\tau.$$

Similarly we find

$$(2.3) \quad \|u(t)\|_{H^k} \leq \|g\|_{H^k} + C \int_0^t \|f(\tau)\|_{H^k} d\tau.$$

Now we focus our attention on  $f(u)$  satisfying (0.7). A *Moser type inequality* allows us to estimate this nonlinear term in Sobolev spaces:

PROPOSITION 2.1 (cf. [7, 5.2.5, 5.3.2]). *Let  $1 < r < +\infty$ ,  $p > 1$  and  $k \in \mathbb{N}$ . Suppose*

$$(2.4) \quad k < p + 1/r,$$

then

$$(2.5) \quad \| |u|^p \|_{W^{k,r}(\mathbb{R}^n)} \leq C \|u\|_{W^{k,r}(\mathbb{R}^n)} \|u\|_{L^\infty(\mathbb{R}^n)}^{p-1}$$

for all  $u \in W_r^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Moreover, if  $p \in \mathbb{N}$  the previous inequality holds without restriction on  $k$ .

We underline that the previous estimate fails in the case  $r = 1$ . In order to estimate the quantity  $\| |u|^p \|_{W^{k,1}}$ , by means of  $L^\infty$  norm of  $u$ , we use *Gagliardo Nirenberg inequality*:

PROPOSITION 2.2 (cf. [6]). *Let us consider  $1 \leq r, l, q \leq \infty$ ,  $m \in \mathbb{N}$ . There exists a constant  $C > 0$  such that for all  $u \in W^{m,l}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$  the following inequality holds:*

$$(2.6) \quad \|D^\beta u\|_q \leq C \|u\|_{W^{m,l}}^{|\beta|/m} \|u\|_r^{1-|\beta|/m}.$$

Here

$$0 \leq |\beta| \leq m - 1, \quad \frac{1}{q} = \frac{|\beta|}{m} \frac{1}{l} + \left(1 - \frac{|\beta|}{m}\right) \frac{1}{r}.$$

This gives:

PROPOSITION 2.3. *Let us consider  $p > 2$ ,  $m \in \mathbb{N}$ . There exists a constant  $C > 0$  such that for all  $u \in H^m(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*

$$(2.7) \quad \| |u|^p \|_{W^{m,1}} \leq C \|u\|_\infty^{p-2} \|u\|_{H^m}^2.$$

PROOF. Let  $k \leq m$ . We begin observing that

$$\|D^k |u|^p\|_1 \leq \sum_{l=1}^k \sum_{\alpha_1 + \dots + \alpha_l = k} C_{\alpha_1, \dots, \alpha_l} \int |u|^{p-l} |D^{\alpha_1} u| \dots |D^{\alpha_l} u|.$$

In the case  $l = 1$  it suffices to use Schwartz inequality and obtain

$$\| |u|^{p-1} D^k u \|_1 \leq \|u\|_\infty^{p-2} \|u\|_{H^k}^2 \leq \|u\|_\infty^{p-2} \|u\|_{H^m}^2.$$

If  $l = 2$  and  $\alpha_1 + \alpha_2 = k$  we get

$$\| |u|^{p-2} D^{\alpha_1} u D^{\alpha_2} u \|_1 \leq \|u\|_\infty^{p-2} \|D^{\alpha_1} u\|_2 \|D^{\alpha_2} u\|_2 \leq \|u\|_\infty^{p-2} \|u\|_{H^m}^2$$

Suppose  $m, l \geq 3$ . Holder inequality yields

$$(2.8) \quad \| |u|^{p-l} |D^{\alpha_1} u| \dots |D^{\alpha_l} u| \|_1 \leq \|u\|_\infty^{p-l} \|D^{\alpha_1} u\|_2 \prod_{i=2}^l \|D^{\alpha_i} u\|_{p_i}$$

where the  $p_i$  satisfy  $\sum_i \frac{1}{p_i} = \frac{1}{2}$ . From Gagliardo Nirenberg inequality (2.6) we find

$$(2.9) \quad \|D^{\alpha_i} u\|_{p_i} \leq C \|u\|_{H^b}^{\alpha_i/b} \|u\|_{L^\infty}^{1-\alpha_i/b}$$

provided  $1/p_i = \alpha_i/(2b)$ . Hence  $l = k - \alpha_1$  and  $p_i = 2(l - \alpha_1)/\alpha_i$ . Combining (2.8) and (2.9) we get the conclusion.  $\square$

### 3. PROOF OF THEOREM 0.3

We recall that we deal with the Cauchy Problem

$$\begin{aligned} u_t - i\Lambda u + f(u) &= 0 \\ u(0, x) &= g(x) . \end{aligned}$$

where  $f$  is smooth,  $f(0) = 0$  and  $|f(u)| \simeq |u|^p$  near  $u = 0$ . Our aim is to prove that this problem admits a unique global solution. In order to do this we have to suppose  $p$  sufficiently large and  $g$  small in a suitable Sobolev norm.

Throughout the proof we use the space-time norm

$$M(t) := M(u(t)) = \sup_{0 \leq \tau \leq t} \{ \|u(\tau)\|_{H[\frac{p}{2}]+1} + (1 + \tau)^{\frac{n-1}{2}} \|u(\tau)\|_\infty \}.$$

Combining the energy estimate (2.3) and the Moser inequality (2.5) we find

$$(3.1) \quad \|u\|_{H[\frac{p}{2}]+1} \leq \|g\|_{H[\frac{p}{2}]+1} + 2M^p(t) \int_0^t (1 + \tau)^{-\frac{n-1}{2}(p-1)} d\tau.$$

The integral on the right converges for  $t \rightarrow \infty$ , provided

$$(3.2) \quad p > 1 + \frac{2}{n-1},$$

i.e. if  $p$  is supercritical. Now we use the representation formula

$$u(x, t) := E(t, x) * g + \int_0^t E(t-s) * f(s) ds$$

where  $E(t, x)$  is the fundamental solution defined in (1.3). Theorem 0.1 gives

$$\|u(t)\|_\infty \leq (1+t)^{-\frac{n-1}{2}} \|g\|_{W^{[\frac{n}{2}]+1,1}} + \int_0^t (1+t-s)^{-\frac{n-1}{2}} \| |u|^p \|_{W^{[\frac{n}{2}]+1,1}} ds,$$

hence, from Proposition 2.3, we find

$$\|u(t)\|_\infty \leq (1+t)^{-\frac{n-1}{2}} \|g\|_{W^{[\frac{n}{2}]+1,1}} + CM^p(t) \int_0^t (1+t-s)^{-\frac{n-1}{2}} (1+s)^{-\frac{n-1}{2}(p-2)} ds.$$

The last integral is not greater than  $(1+t)^{-(n-1)/2}$  in the following cases

$$(3.3) \quad \begin{aligned} n = 2, & \quad p > 4 \\ n = 3, & \quad p > 3 \\ n \geq 4, & \quad p \geq 3. \end{aligned}$$

In fact we can prove:

PROPOSITION 3.1. *Let  $\alpha, \beta \in \mathbb{R}_+$ .*

(i) *If  $\beta \geq \alpha > 1$  or  $\beta > 1 \geq \alpha$  then*

$$(3.4) \quad \int_0^t (1+t-s)^{-\alpha} (1+s)^{-\beta} ds \leq C_{\alpha,\beta} (1+t)^{-\alpha}.$$

(ii) *The previous estimate does not hold if  $0 < \alpha \leq 1$ , and  $0 < \beta \leq 1$ .*

PROOF. The case (i) is proved in the appendix of [2], in the more general form: if  $\max\{\alpha, \beta\} > 1$  then

$$\int_0^t (1+t-s)^{-\alpha} (1+s)^{-\beta} ds \leq C_{\alpha,\beta} (1+t)^{-\min\{\alpha,\beta\}}.$$

On the contrary, from the relation

$$\int_0^t (1+t-s)^{-1} (1+s)^{-1} ds = \frac{1}{(2+t)} \left[ \int_0^t \frac{1}{1+t-s} ds + \int_0^t \frac{1}{1+s} ds \right] = 2 \frac{\ln(1+t)}{2+t}$$

one can not obtain the desired decay. From this fact, we also deduce that (3.4) does not hold if  $0 < \alpha \leq 1$  and  $0 < \beta \leq 1$ .  $\square$

Coming back to the proof of our Theorem in the case  $n = 2, p > 4$  and  $n = 3, p > 3$  we get

$$(3.5) \quad M(t) \leq (\|g\|_{H^2} + \|g\|_{W^{2,1}}) + CM^p(t).$$



Suppose

$$(3.6) \quad \|g\|_{H^2} + \|g\|_{W^{2,1}} \leq \varepsilon;$$

applying a standard argument which involves contraction mapping principle, from (3.5) we obtain the unique global solution to our problem.

In particular for this solution we find

$$M(u(t)) \leq C\varepsilon.$$

This means  $u(t) \in W^{2,2}(\mathbb{R}^n)$  and

$$\|u(t)\|_\infty \leq C\varepsilon(1+t)^{-\frac{n-1}{2}}.$$

We can extend Theorem 0.3 to some cases of space dimension  $n \geq 4$  even or  $n = 4k + 3$ . In particular, we can use the same technique of this section assuming

$$p \in \mathbb{N}, \quad p \geq 3 \quad \text{or} \quad p \in \mathbb{R}, \quad p > [n/2] + 3/2.$$

These more restrictive condition on  $p$  follow from the assumption (3.3) and from the hypothesis (2.4) which allows to use Moser inequality. These assumptions are reasonable, in fact we need to estimate more derivatives as the space dimension grows.

It is obvious that in this case we require

$$|D^k f(u)| \simeq |u|^{p-k} \quad \text{near } u = 0 \quad \text{for } k \leq [n/2] + 1.$$

Finally if Cauchy data satisfy

$$\|g\|_{W^{[n/2]+1,1}} + \|g\|_{W^{[n/2]+1,2}} < \varepsilon$$

we have

$$M(t) \leq C\varepsilon + CM^p(t).$$

By the aid of contraction mapping principle we find the unique global solution to our problem; in particular  $u(t) \in W^{[n/2]+1,2}$  and  $\|u\|_\infty \leq C(1+t)^{-\frac{n-1}{2}}$ .

#### 4. PROOF OF THEOREM 0.2

Let  $g$  satisfies the assumption (3.6). The energy estimate and the Moser type inequality, proved in Section 2, give

$$\|u(t)\|_{H^2} \leq \varepsilon + C \int_0^t \|u\|_\infty^2 \|u\|_{H^2} ds,$$

while Von Wahl type estimate yields

$$(4.1) \quad \|u(t)\|_\infty \leq (1+t)^{-1}\varepsilon + \int_0^t (1+t-s)^{-1} \|u\|_\infty \|u\|_{H^2}^2 ds.$$

The argument used in the previous section does not work, since the integral

$$\int_0^t (1+t-s)^{-1}(1+s)^{-1} ds$$

has logarithmic factor. The idea is to change the norm  $M(u(t))$  which include the  $L^\infty$  space-time norm, with a quantity which involves the  $L^2$  norm in time and the  $L^\infty$  norm in space. In order to do this we introduce the functions:

$$X(t) := X(u(t)) = \left( \int_0^t \|u(s)\|_\infty^2 ds \right)^{1/2}$$

$$\Psi(t) := \Psi(u(t)) = \sup_{s \in [0, t]} \|u(s)\|_{H^2}.$$

The following inequalities then hold

$$(4.2) \quad \|u(\tau)\|_\infty^2 \leq 2\varepsilon^2(1 + t)^{-2} + 2 \left[ \int_0^\tau (1 + \tau - s)^{-1} \|u(s)\|_\infty \Psi^2(s) ds \right]^2,$$

$$(4.3) \quad \Psi(t) \leq \varepsilon + \int_0^t \|u(s)\|_\infty^2 \Psi(s) ds.$$

Since  $\Psi$  is an increasing function, from (4.3) it follows

$$(4.4) \quad \Psi(t) \leq \varepsilon + \Psi(t)X^2(t).$$

In order to derive a similar relation from (4.2) we use the following corollary of the boundedness of Hilbert transform in  $L^2$ .

LEMMA 4.1. *For all  $f \in L^2(\mathbb{R})$ , we put*

$$I(f)(\tau) = \int_0^\tau \frac{1}{1 + \tau - s} f(s) ds.$$

Therefore the operator  $I$  is bounded on  $L^2(0, t)$ , i.e.

$$\|I(f)\|_{L^2(0, t)} \leq C\|f\|_{L^2(0, t)}.$$

PROOF. It is well-known that

$$\mathcal{F} \left( \frac{1}{1 + x} \right) (\xi) = -i \sqrt{\frac{\pi}{2}} e^{i\xi} \operatorname{sgn} \xi.$$

Put  $K(x) = 1/(1 + x)$ . Using the Plancherel formula we find

$$\left\| \int_{-\infty}^{+\infty} \frac{1}{1 + t - s} f(s) ds \right\|_{L^2(\mathbb{R})} = \|K * f\|_{L^2(\mathbb{R})} = \|\widehat{K}\widehat{f}\|_{L^2(\mathbb{R})} = \sqrt{\frac{\pi}{2}} \|\widehat{f}\|_{L^2(\mathbb{R})} = \sqrt{\frac{\pi}{2}} \|f\|_{L^2(\mathbb{R})}.$$

Since

$$I(f)(t) = \int_{-\infty}^{+\infty} \frac{1}{1 + t - s} \chi_{(0, t)} f(s) ds$$

we conclude

$$\|I(f)\|_{L^2(0, t)} \leq C\|\chi_{(0, t)} f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(0, t)}. \quad \square$$

Now we come back to the proof of the global existence theorem. Integrating (4.2) we get

$$X^2(t) \leq C\varepsilon^2 + C\Psi^4(t) \int_0^t \left[ \int_0^\tau (1 + \tau - s)^{-1} \|u(s)\|_\infty ds \right]^2 d\tau,$$

whence, by the previous lemma

$$X^2(t) \leq C\varepsilon^2 + C\Psi^4(t) \int_0^t \|u(s)\|_\infty^2 ds.$$

Hence we get

$$(4.5) \quad X(t) \leq C\varepsilon + \Psi^2(t)X(t).$$

Finally we take the norm

$$M(u(t)) = \Psi(u(t)) + X(u(t)),$$

and from (4.4) and (4.5) we deduce

$$M(u(t)) \leq C\varepsilon + M^3(u(t)).$$

Then the contraction mapping principle yields a unique global solution to our problem, moreover this solution satisfies

$$M(u(t)) \leq C_1\varepsilon.$$

Combining this inequality with (4.1) we obtain

$$\|u(t)\|_\infty \leq C\varepsilon(1+t)^{-1} + C_1^2\varepsilon^2 \int_0^t (1+t-s)^{-1} \|u(s)\|_\infty ds.$$

Now we are in a position to apply Gronwall lemma; we conclude

$$\|u(t)\|_\infty \leq C\varepsilon(1+t)^{C_1^2\varepsilon^2-1}.$$

We observe that in this case we do not find the optimal decay rate  $\|u(t)\|_\infty \leq C(1+t)^{-1}$ .

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