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Gradient estimates and Harnack inequalities for solutions to the minimal surface equation

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Calcolo delle variazioni. — *Gradient estimates and Harnack inequalities for solutions to the minimal surface equation.* Nota (*) del Socio MARIO MIRANDA.

ABSTRACT. — A gradient estimate for solutions to the minimal surface equation can be proved by Partial Differential Equations methods, as in [2]. In such a case, the oscillation of the solution controls its gradient. In the article presented here, the estimate is derived from the Harnack type inequality established in [1]. In our case, the gradient is controlled by the area of the graph of the solution or by the integral of it. These new results are similar to the one announced by Ennio De Giorgi in [3].

KEY WORDS: Minimal surface; Gradient estimate; Harnack inequality; Generalized solution for the minimal surface equation.

RIASSUNTO. — *Stime del gradiente e diseguaglianze di Harnack per le soluzioni delle equazioni delle superfici minime.* Una stima del gradiente delle soluzioni dell'equazione delle superfici minime può essere ricavata, con i metodi funzionali applicati alle equazioni differenziali di tipo ellittico, come è stato fatto in [2]. In tale caso è l'oscillazione della funzione che controlla il gradiente. In questa breve *Nota* si dimostra che anche l'area del grafico della soluzione come l'integrale della stessa servono allo stesso scopo. La dimostrazione è ricavata da una diseguaglianza del tipo Harnack, stabilita in [1]. Ricordiamo che il ricorso alla diseguaglianza di Harnack per stimare il gradiente era stato preannunciato da Ennio De Giorgi in [3].

INTRODUCTION

Ennio De Giorgi [3] announced first the gradient estimate for solutions to the minimal surface equation in any dimension.

THEOREM 1 (De Giorgi). *For any integer* $n \geq 2$ *there exists a function*

$$g : [0\,, +\infty) \to [0\,, +\infty)\,,$$

such that, if $x_0 \in \mathbb{R}^n$, $\varrho > 0$, $f \in \mathcal{C}^2(\{x : |x - x_0| < \varrho\})$ *satisfies*

$$\operatorname{div}\left(\frac{Df(x)}{\sqrt{1 + |Df(x)|^2}}\right) = 0\,, \quad \forall |x - x_0| < \varrho\,, \tag{1}$$

then

$$|\operatorname{grad} f(x_0)| = |Df(x_0)| \leq g\left(\sup\{\varrho^{-1}|f(x) - f(x_0)| : |x - x_0| < \varrho\}\right). \tag{2}$$

A complete proof of this theorem was not written in [3]. A few months after De Giorgi's announcement, Enrico Bombieri, myself and De Giorgi himself proved the following

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THEOREM 2 (Bombieri-De Giorgi-Miranda). *If $n \geq 2$ and f satisfies* (1), *then*

$$|Df(x_0)| \leq c_1(n) \exp\left(c_2(n) \varrho^{-1} \sup_{|x-x_0|<\varrho} \big(f(x) - f(x_0)\big)\right), \tag{3}$$

where $c_1(n) \in \mathbb{R}$, $c_2(n) \in \mathbb{R}$.

The proof of this theorem was published in [2].

We present here two new estimates for the gradient, whose proves are independent of Bombieri-De Giorgi-Miranda's Theorem.

THEOREM 3. *For any integer $n \geq 2$, there exist two real functions g_1, g_2, defined on* $[0, +\infty)$, *such that: if f satisfies* (1), *then*

$$|Df(x_0)| \leq g_1\left(\varrho^{-n-1} \int_{|x-x_0|<\varrho} \big(f(x) - f(x_0)\big) \vee 0 \, dx \right), \tag{4}$$

$$|Df(x_0)| \leq g_2\left(\varrho^{-n} \int_{|x-x_0|<\varrho} \sqrt{1 + \left|D\left(\big(f(x) - f(x_0)\big) \vee 0\right)\right|^2} dx \right). \tag{5}$$

The proof of this theorem is a consequence of properties of generalized solutions to the minimal surface equation, introduced in [5]. These properties are a straightforward consequence of a Harnack type inequality proven by Bombieri and Enrico Giusti in [1].

1. THE HARNACK TYPE INEQUALITY

We present here this inequality and its consequence for generalized solutions, as it was done in [6].

THEOREM 4 (Bombieri-Giusti). *If E has minimal boundary in the open set A of $\mathbb{R}^{n+1}$, if B_ϱ is an open ball of $\mathbb{R}^{n+1}$, contained in A and u is a positive supersolution of a strongly elliptic linear equation on $\partial E \cap B_\varrho$,* i.e.

$$\int_{\partial E \cap B_\varrho} \sum_{i,j=1}^{n+1} a_{ij}(x) \delta_i u \delta_j \phi \, d\mathcal{H}^n \geq 0, \quad \forall \phi \geq 0, \quad \phi \in C_0^1(B_\varrho);$$

where δ_i are the tangential derivatives on ∂E. Then the following inequality holds

$$\int_{\partial E \cap B_{\varrho\varepsilon}} u \, d\mathcal{H}^n \leq c^{\sqrt{\Lambda}} \mathcal{H}^n(\partial E \cap B_\varrho) \inf_{\partial E \cap B_{\varrho\varepsilon}} u,$$

where $\Lambda = \sup_{i,j} \sup_x |a_{ij}(x)|$ and ε, c are positive real numbers depending on n only.

For the definitions of minimal boundaries and tangential derivatives see [4].

2. Generalized solutions to minimal surface equation

Definition 1. *A Lebesgue measurable function* $f : \Omega \to [-\infty, +\infty]$, *is a generalized solution to the minimal surface equation in the open set* $\Omega \subset \mathbb{R}^n$, *if the set*

$$E = \{(x, t) : x \in \Omega, t < f(x)\}$$

has minimal boundary in the open set $\Omega \times \mathbb{R}$ *of* $\mathbb{R}^{n+1}$.

If f is a generalized solution in Ω, then the sets

$$\mathcal{P} = \{x \in \Omega : f(x) = +\infty\}, \quad \mathcal{N} = \{x \in \Omega : f(x) = -\infty\},$$

do have minimal boundaries in Ω.

The following result, whose proof depends on Bombieri-Giusti's Theorem, gives a complete description of generalized solutions.

Theorem 5. *For any generalized solution* f *in* Ω, *there exists an open set* $G \subset \Omega$ *such that* $f_{|G}$ *is analytic and a solution to the minimal surface equation. Moreover*

$$\Omega = G \cup \mathcal{P} \cup \mathcal{N} \cup \left(\partial\mathcal{P} \cap \partial\mathcal{N} \cap \Omega\right),$$

$$\partial E \cap (\Omega \times \mathbb{R}) = \left\{ \operatorname{graph} f_{|G} \right\} \cup \left\{ \left(\partial\mathcal{P} \cap \partial\mathcal{N} \cap \Omega\right) \times \mathbb{R} \right\},$$

$f_{|G\cup\mathcal{P}\cup\mathcal{N}}$ *is a continuous function with values in* $[-\infty, +\infty]$.

3. Proof of the new gradient inequalities

Let $\{f_j\}$ be a sequence of solutions to the minimal surface equation in $B_\varrho(x_0) = \left\{x \in \mathbb{R}^n : |x - x_0| < \varrho\right\}$. Assume $f_j(x_0) = 0$ for every j and

$$\lim_{j\to\infty} |Df_j(x_0)| = +\infty. \tag{6}$$

Since $\{f_j\}$ is compact, with respect to the convergence almost everywhere in the set of all generalized solutions, we can assume that

$$f_j(x) \to f(x), \quad \text{almost all } x \in B_\varrho(x_0).$$

(6) implies that $(x_0, 0) \in \partial E \setminus \operatorname{graph} f_{|G}$. Therefore $\mathcal{P}$ and $\mathcal{N}$ cannot be empty. More precisely there exists $\varepsilon > 0$ such that

$$\operatorname{meas}\left(B_\varrho(x_0) \cap \mathcal{P}\right) > 0,$$

$$\operatorname{meas}\left(B_\varrho(x_0) \cap \mathcal{N}\right) > 0,$$

$$\operatorname{meas}\left(B_\varrho(x_0) \cap \mathcal{P}\right) + \operatorname{meas}\left(B_\varrho(x_0) \cap \mathcal{N}\right) = \operatorname{meas} B_\varrho(x_0).$$

So (6) is impossible, if either

$$\sup_j \left(\varrho^{-n-1} \int_{|x-x_0|<\varrho} f_j(x) \vee 0 \right) <+\infty \quad \text{or}$$

$$\sup_j \varrho^{-n} \int_{|x-x_0|<\varrho} \sqrt{1 + \left| D\left(f_j(x) \vee 0\right) \right|^2} dx <+\infty$$

are satisfied.

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