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On the nodal set of the second eigenfunction of the laplacian in symmetric domains in $\mathbb{R}^N$

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Equazioni a derivate parziali. — *On the nodal set of the second eigenfunction of the laplacian in symmetric domains in* $\mathbb{R}^N$. Nota di Lucio Damascelli, presentata (*) dal Socio A. Ambrosetti.

Abstract. — We present a simple proof of the fact that if Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is convex and symmetric with respect to k orthogonal directions, $1 \leq k \leq N$, then the nodal sets of the eigenfunctions of the laplacian corresponding to the eigenvalues $\lambda_2, \cdots, \lambda_{k+1}$ must intersect the boundary. This result was proved by Payne in the case $N = 2$ for the second eigenfunction, and by other authors in the case of convex domains in the plane, again for the second eigenfunction.

Key words: Second eigenfunction; Nodal set; Maximum principle.

Riassunto. — *Sull'insieme nodale della seconda autofunzione del laplaciano in un dominio simmetrico di* $\mathbb{R}^N$. Viene presentata una semplice dimostrazione del fatto che se Ω è un dominio limitato di $\mathbb{R}^N$, $N \geq 2$, convesso e simmetrico in k direzioni ortogonali, $1 \leq k \leq N$, allora gli insiemi nodali delle autofunzioni del laplaciano corrispondenti agli autovalori $\lambda_2, \cdots, \lambda_{k+1}$ hanno intersezione non vuota con la frontiera del dominio. Questo risultato era stato dimostrato da Payne nel caso $N = 2$ per la seconda autofunzione, e da altri autori nel caso di domini piani convessi, sempre per la seconda autofunzione.

1. Introduction and statement of the results

In this *Note* we consider the eigenvalue problem

$$(1.1) \qquad \begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain (*i.e.* connected open set) in $\mathbb{R}^N$. We will denote the sequence of the eigenvalues of (1.1) as $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ where each eigenvalue is counted according to its multiplicity, and we will denote by $\varphi_1, \varphi_2, \varphi_3, \ldots$ a corresponding basis of eigenfunctions.

It is well known that the first eigenvalue λ_1 is simple and that a corresponding eigenfunction does not change sign in Ω. All the other eigenfunctions must change sign and by the Courant's Nodal Line Theorem [3] the eigenfunction φ_k corresponding to the eigenvalue λ_k has at most k nodal domains, *i.e.* subdomains of Ω where φ_k does not change sign. In particular any second eigenfunction φ_2 has exactly two nodal domains.

Let $\mathcal{N} = \mathcal{N}_\varphi = \overline{\{x \in \Omega : \varphi(x) = 0\}}$ be the nodal set of an eigenfunction φ of problem (1.1). A conjecture about the topology of the second eigenfunction in the case $N = 2$ (when the nodal set is a curve) is the following (see [8, 10, 11]):

* The eigenfunction φ_2 cannot have a closed interior nodal curve.

(*) Nella seduta del 23 giugno 2000.

In [9] Payne proved the conjecture in the case of a domain $D \subset \mathbb{R}^2$ convex and symmetric with respect to a direction; then Lin [5] proved the conjecture when $\Omega \subset \mathbb{R}^2$ is a bounded smooth convex domain which is invariant under a rotation with angle $2\pi \frac{p}{q}$ where p and q are positive integers. Later Melas [7] proved the validity of the conjecture for a general bounded smooth convex domain in $\mathbb{R}^2$ and this result was then extended in [1] to general bounded convex domains in $\mathbb{R}^2$. More precisely these authors prove that the nodal line intersects the boundary at exactly two points.

In this paper we show that for any dimension $N \geq 2$ and a bounded (possibly nonsmooth) domain in $\mathbb{R}^N$ which is symmetric and convex in several directions the statement of the conjecture holds also for higher eigenfunctions. In particular in the case of a domain in $\mathbb{R}^N$ convex and symmetric in one direction we get again the result of Payne [9] with a new elementary proof which does not use any regularity result about the structure of the nodal set, as done in previous papers. More precisely we prove the following result.

Theorem 1.1. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *be a bounded domain,* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ *the sequence of the eigenvalues of the laplacian in* Ω *and* φ_j *the eigenfunction corresponding to the eigenvalue* λ_j.

If Ω *is symmetric and convex with respect to* k *orthogonal directions,* $1 \leq k \leq N$, *and* $\mathcal{N}_{\varphi_j} = \overline{\{x \in \Omega : \varphi_j(x) = 0\}}$ *is the nodal set of the eigenfunction* φ_j, *then* $\mathcal{N}_{\varphi_j} \cap \partial\Omega \neq \emptyset$, $2 \leq j \leq k+1$.

Of course if $\lambda_2 = \lambda_3 = \cdots = \lambda_{k+1}$ this is again a result about the second eigenfunctions, but for general domains it is possible that the multiplicity of the second eigenvalue is lower than k. As an example if we take the domain $\Omega = \Pi_{i=1}^N (-\frac{a_i}{2}, \frac{a_i}{2})$ then the eigenvalues are the numbers $\lambda_{k_1 \cdots k_N} = \sum_{i=1}^N \frac{k_i^2 \pi^2}{a_i^2}$, $k_i \in \mathbb{N} \setminus \{0\}$, with associated eigenfunctions

$$w_{k_1 \cdots k_N} = \left(\frac{2^N}{a_1 \cdots a_N} \right)^{\frac{1}{2}} \prod_{i=1}^N \sin \frac{k_i \pi (x_i + \frac{a_i}{2})}{a_i} .$$

In this case, depending on the choice of the dimensions a_i, the multiplicity of λ_2 can be any number between 1 and N, and one can get simple eigenvalues $\lambda_2 < \lambda_3 < < \cdots \lambda_{N+1}$.

Remark 1.1. When the Laplace operator is substituted by the operator $-\Delta - V(x)$ the conjecture (*) is not in general true, see [6].

In the case when Ω is a smooth convex domain in the plane Lin [5] also proved that the multiplicity of the second eigenvalue is at most two. Then Zhang [12] proved that this is true for any smooth and simply connected domain in $\mathbb{R}^2$.

Our method gives a simple proof of this multiplicity result in the case of a planar domain that need not to be convex, but which is convex and symmetric with respect to two orthogonal directions. Moreover the proof shows that it is likely to conjecture that the multiplicity of the second eigenvalue is at most N in domains in $\mathbb{R}^N$ which are convex and symmetric with respect to N orthogonal directions (see Remark 2.1). More precisely we prove the following theorem.

THEOREM 1.2. *Let Ω be a domain in $\mathbb{R}^2$ which is convex and symmetric with respect to 2 orthogonal directions,* e.g. *the x_i-directions, $i = 1, 2$. Then the multiplicity of the eigenvalue λ_2 is at most 2 and the corresponding eigenspace is spanned by eigenfunctions each of which is odd in one variable and even in the other.*

2. PROOFS

Throughout this section Ω will be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, convex in a direction, that for simplicity will be assumed to be the x_1-direction, and symmetric with respect to the hyperplane $T_0 = \{x \in \mathbb{R}^N : x_1 = 0\}$. If $t \in \mathbb{R}$ we define

$$\begin{aligned} \Sigma_t^- &= \{x \in \Omega : x_1 < t\}\,, \qquad t \leq 0 \\ \Sigma_t^+ &= \{x \in \Omega : x_1 > t\}\,, \qquad t \geq 0 \\ T_t &= \{x \in \mathbb{R}^N : x_1 = t\}\,, \quad t \in \mathbb{R} \end{aligned}$$

and we write $\Omega^- = \Sigma_0^-$, $\Omega^+ = \Sigma_0^+$.

Together with problem (1.1) we will consider the Dirichlet eigenvalue problem in $\Omega^- = \Sigma_0^-$

$$\text{(2.1)} \qquad \begin{cases} -\Delta\psi = \mu\psi & \text{in } \Omega^- \\ \psi = 0 & \text{on } \partial\Omega^- = (\partial\Omega^- \cap \partial\Omega) \cup (T_0 \cap \Omega) \end{cases}$$

and the mixed boundary eigenvalue problem in Ω^-

$$\text{(2.2)} \qquad \begin{cases} -\Delta\zeta = \nu\zeta & \text{in } \Omega^- \\ \zeta = 0 & \text{on } \partial\Omega^- \cap \partial\Omega \\ \dfrac{\partial\zeta}{\partial x_1} = 0 & \text{on } T_0 \cap \Omega\,. \end{cases}$$

We will denote by $\mu_1 \leq \mu_2 \leq \mu_3 \dots$ the sequence of eigenvalues of (2.1), each one counted according to its multiplicity, with the associate pairwise orthogonal eigenfunctions $\psi_1, \psi_2, \psi_3 \dots$, while for problem (2.2) we will use the notations $\nu_1 \leq \nu_2 \leq \nu_3 \dots$, $\zeta_1, \zeta_2, \zeta_3 \dots$, for the corresponding eigenvalues and eigenfunctions.

Our proofs will be based on the simple remarks contained in the next lemma.

LEMMA 2.1. (i) *If (ν_k, ζ_k) is a pair eigenvalue-eigenfunction for (2.2) and $\widetilde{\zeta}_k$ is the even extension of ζ_k to Ω,* i.e. *$\widetilde{\zeta}_k(x_1, x') = \zeta_k(-x_1, x')$ if $x = (x_1, x') \in \Omega^+$, then ν_k is an eigenvalue $\lambda_{\alpha(k)}$ of problem (1.1) with associate eigenfunction $\widetilde{\zeta}_k$. In particular $\nu_1 = \lambda_1$ and $\widetilde{\zeta}_1$ is a first eigenfunction.*

(ii) *If (μ_k, ψ_k) is a pair eigenvalue-eigenfunction for (2.1) and $\widetilde{\psi}_k$ is the odd extension of ψ_k to Ω,* i.e. *$\widetilde{\psi}_k(x_1, x') = -\psi_k(-x_1, x')$ if $x = (x_1, x') \in \Omega^+$, then μ_k is an eigenvalue $\lambda_{\beta(k)}$ of problem (1.1) with associate eigenfunction $\widetilde{\psi}_k$. In particular $\mu_1 = \lambda_{\beta(1)}$ with $\beta(1) \geq 2$ so that $\lambda_2 \leq \mu_1$ and, up to multiplications by a constant, $\widetilde{\psi}_1$ is the only eigenfunction for (1.1) corresponding to the eigenvalue $\lambda_{\beta(1)}$ which is odd in the x_1-variable.*

(iii) *All the eigenvalues of* (1.1) *are given by the collections* $\{\mu_j\}_j$, $\{\nu_k\}_k$, *and if* $\{\psi_j\}_j$, $\{\zeta_k\}_k$ *are orthonormal bases in* $L^2(\Omega^-)$ *respectively for problem* (2.1), (2.2), *then the collection* $\{\frac{1}{\sqrt{2}}\widetilde{\psi}_j\}_j \cup \{\frac{1}{\sqrt{2}}\widetilde{\zeta}_k\}_k$ *is an orthonormal basis in* $L^2(\Omega)$ *for problem* (1.1).

Proof. (*i*) It is clear that since $\frac{\partial \zeta_k}{\partial x_1} = 0$ on $T_0 \cap \Omega$ the even extension of ζ_k satisfies (1.1) in Ω with $\lambda = \nu_k$. Moreover the first eigenfunction ζ_1 does not change sign in Ω^-, so that $\widetilde{\zeta}_1$ does not change sign in Ω and it is therefore the first eigenfunction.

(*ii*) Since $\psi_k = 0$ on $T_0 \cap \Omega$, for $i = 2, \dots N$ we have $\frac{\partial \psi_k}{\partial x_i} = \frac{\partial^2 \psi_k}{\partial x_i^2} = 0$ on $T_0 \cap \Omega$, and from the equation we get $\frac{\partial^2 \psi_k}{\partial x_1^2} = 0$ on $T_0 \cap \Omega$. It follows easily that the odd extension of ψ_k satisfies (1.1) in Ω, with $\lambda = \mu_k$. Since $\widetilde{\psi}_1$ must change sign in Ω, its eigenvalue (with respect to (1.1)), which is μ_1, must be at least the second eigenvalue, so that $\lambda_2 \le \mu_1 = \lambda_{\beta(1)}$; moreover if ψ is another eigenfunction relative to $\lambda_{\beta(1)}$ which is odd in x_1, its restriction to Ω^- is an eigenfunction for (2.1) with eigenvalue μ_1, the first eigenvalue, so it has to be a multiple of ψ_1.

(*iii*) If φ is an eigenfunction and we define $\varphi_0(x_1, x') = \varphi(-x_1, x')$ for $x = (x_1, x') \in \Omega$, we can write φ as a sum $\varphi = \varphi^s + \varphi^a$ of a symmetric part $\varphi^s = \frac{1}{2}[\varphi + \varphi_0]$, even in x_1, and an antisymmetric part $\varphi^a = \frac{1}{2}[\varphi - \varphi_0]$, odd in x_1. In this way we can generate each eigenspace of each eigenvalue by functions which are either even or odd in the variable x_1, whose restrictions to Ω^- solve (2.2), respectively (2.1), and the conclusion follows easily. $\square$

We now prove a theorem which will give us immediately the conclusions of Theorem 1.1.

Theorem 2.1. *Let* $k \ge 2$ *and* $\varphi = \varphi_k$ *be an eigenfunction for problem* (1.1) *corresponding to an eigenvalue* $\lambda_k \le \mu_1$, *where* μ_1 *is the first eigenvalue of problem* (2.1). *Then* $\mathcal{N}_\varphi \cap \partial\Omega \neq \emptyset$.

Proof. As in the proof of Lemma 2.1 we can write φ as a sum $\varphi = \varphi^s + \varphi^a$ where φ^s is even in x_1 and φ^a is odd in x_1 and each of them is either zero or an eigenfunction with the same eigenvalue λ_k.

If $\varphi^s = 0$ then φ is odd in x_1, its restriction to Ω^- is a first eigenfunction for (2.1), its nodal set is $\overline{T_0 \cap \Omega}$ and there is nothing to prove.

Let us consider then the case when φ^s does not vanish, so that it is an eigenfunction relative to λ_k. Suppose that

$$\mathcal{N}_\varphi \cap \partial\Omega = \emptyset$$

so that there is a neighborhood I of $\partial\Omega$ where φ has the same sign. This implies that φ^s, which does not vanish and it is therefore an eigenfunction corresponding to the same eigenvalue, has the same property. So it is enough to show that this is not possible for symmetric eigenfunctions.

Let us suppose now by contradiction that φ is an eigenfunction corresponding to an eigenvalue $\lambda_k \le \mu_1$ $(k \ge 2)$, which is even in the x_1-variable and satisfies

$$\varphi > 0 \quad \text{in } I \cap \Omega \tag{2.3}$$

where I is a neighborhood of $\partial\Omega$.

Let Σ_t^- be defined for $t<0$ as in the beginning of this section, and for $x=(x_1\,,\,x')$ in Σ_t^- let $x_t=(2\,t-x_1\,,\,x')$ be the point obtained by reflecting x through the hyperplane T_t and let $u_t(x)=u(x_t)$, $x\in\Sigma_t^-$, be the reflected function.

By (2.3) if $\varepsilon>0$ is sufficiently small we have for $-\varepsilon<t<0$

$$w_t=\varphi-\varphi_t\le 0\,,\quad w_t\not\equiv 0\quad \text{on } \partial\Sigma_t^-\,. \tag{2.4}$$

By hypothesis $\lambda_k\le\mu_1$, so that the first eigenvalue of the operator $L=-\Delta-\lambda_k$ in Ω^- is greater than or equal to zero. Therefore if $t<0$ the first eigenvalue of L is strictly positive in Σ_t^- and this implies that the operator L satisfies the maximum principle in Σ_t^- (see [2]). Since $L\,w_t=0$ in Σ_t^-, we deduce from (2.4) that $w_t\le 0$ in Σ_t^-, and by the strong maximum principle we get that if $-\varepsilon<t<0$ then

$$\varphi<\varphi_t\quad\text{in}\quad\Sigma_t^-\,. \tag{2.5}$$

Since φ is even in x_1 and must change sign in Ω it has to change sign in Ω^-, and this contradicts (2.5). In fact consider a point $x_0=(s_0\,,\,x')\in\Omega^-$ with $\varphi(x_0)<0$, and let $(a\,,\,s_0)$ be the maximal interval of numbers $s<s_0$ such that $\varphi(s\,,\,x')<0$. If $\overline{x}=(a\,,\,x')$ then $\varphi(\overline{x})=\varphi_0(\overline{x})=0$, while for $t<0$ small we have that $\varphi_t(\overline{x})=\varphi(\overline{x}_t)<0$, contradicting (2.5). $\square$

Proof of Theorem 1.1. Suppose that Ω is convex and symmetric with respect to the directions $x_1\,,\ldots\,,x_k$, and let $\Omega_i^-=\{x\in\Omega : x_i<0\}$ and $\mu_1(\Omega_i^-)$ be the first eigenvalue of the Dirichlet eigenvalue problem in Ω_i . By Lemma 2.1 each $\mu_1(\Omega_i^-)$ is an eigenvalue $\lambda_{\gamma(i)}$, with $\gamma(i)\ge 2$, for (1.1), while an associated first eigenfunction ψ_i in Ω_i^- gives, by odd reflection, an eigenfunction $\widetilde{\psi}_i$ in Ω. Moreover each ψ_i, being a first eigenfunction in the domain Ω_i^- (which is symmetric in the x_j-directions, $j\ne i$), is even in the other variables x_j by Lemma 2.1 (*i*). So if j is different from i we have that $\widetilde{\psi}_i$ is L^2-orthogonal to $\widetilde{\psi}_j$. This means that the eigenfunctions $\widetilde{\psi}_i$, $i=1\,,\cdots k$, correspond to different eigenvalues, each one greater than the first. Therefore at least one of the eigenvalues $\mu_1(\Omega_i)$ must satisfy $\mu_1(\Omega_i)\ge\lambda_{k+1}$ ($\ge\lambda_j$ for each $j=2\,,\ldots\,,k+1$). Applying Theorem 2.1 with the direction x_1 substituted by the direction x_i, we get that for each $j=2\,,\ldots\,,k+1$, $\mathcal{N}_{\varphi_j}\cap\partial\Omega\ne\emptyset$. $\square$

Remark 2.1. Let Ω be a bounded domain in $\mathbb{R}^N$, convex and symmetric with respect to the directions x_i, $i=1\,,\ldots\,,N$. By Lemma 2.1, applied in all the coordinate directions, the eigenspace of the eigenvalue λ_2 has a basis consisting of functions which are either symmetric or antysymmetric with respect to each hyperplane $T_i=\{x\in\in\mathbb{R}^N : x_i=0\}$, $i=1\,,\ldots\,,N$. Moreover if such a function is odd with respect to a variable x_i, it is the only one with this property by Lemma 2.1 (*ii*) and its restriction to Ω_i^- is a first eigenfunction, so again by Lemma 2.1 (*i*), it is even in the other variables; therefore it is orthogonal in $L^2(\Omega)$ to other eigenfunctions which are odd with respect to different variables. So there exist at most N independent eigenfunctions which are odd in some variable x_i, they are authomatically pairwise orthogonal, and the remaining eigenfunctions in the previous basis are even with respect to all the variables x_i, $i=1\,,\cdots\,,N$.

Then if we could prove that a second eigenfunction cannot be even with respect to all the variables x_i, $i = 1, \dots, N$ we would obtain that the multiplicity of the second eigenvalue in such domains is at most N. We conjecture that this is true in any dimension, but we are able to prove it only when $N = 2$ (see the proof that follows).

Proof of Theorem 1.2. As already remarked we only have to show that a second eigenfunction cannot be 2-symmetric, *i.e.* even with respect to the variables x_1, x_2. Suppose now by contradiction that φ is a 2-symmetric second eigenfunfunction. We will show, using an argument similar to that of Theorem 3.1 in [4], that φ has a sign close to the boundary, which contradicts Theorem 1.1.

To begin with, we have that each connected component of the set $\widetilde{\Omega} = \{x \in \Omega : \varphi(x) \neq 0\}$ intersects every cap $\Omega_i^{\pm}$, $i = 1, 2$. In fact if a component C of $\widetilde{\Omega}$ were contained in some Ω_i^-, say Ω_1^-, then the restriction of φ to that component would be a first eigenfunction, *i.e.* the first eigenvalue, $\lambda_1(L, C)$, of the operator $L = -\Delta - \lambda_2$ in C would be 0. Then $C \neq \Omega_1^-$ because otherwise φ, being even in x_1, would be positive in Ω. So C would be a proper subset of Ω_1^- and since by Lemma 2.1 (*ii*) we have that $\lambda_2 \leq \mu_1$, which implies that $\lambda_1(L, \Omega_1^-) \geq 0$, we would deduce that $\lambda_1(L, C) > \lambda_1(L, \Omega_1^-) \geq 0 = \lambda_1(L, C)$, a contradiction.

We want to show now that in a neighborhood of the boundary φ has the same sign. Suppose, by contradiction, that this is not the case. By the Courant's nodal line theorem $\widetilde{\Omega}$ has two components. Then considering the component A_1 where $\varphi > 0$, for what we have just proved we can take four points in $A_1 \cap \Omega_i^{\pm}$, $i = 1, 2$, and connect them with a closed curve $\gamma_1 \subset A_1$ which is symmetric with respect to the coordinate axes. By the Jordan Curve Theorem $\Omega \setminus \gamma_1$ has two components, and in the «exterior» of γ_1, *i.e.* in the component touching the boundary, there are points where $\varphi < 0$, since we are supposing that φ has not the same sign close to the boundary. This means that the component A_2 of $\widetilde{\Omega}$ where $\varphi < 0$ is contained in the «exterior» of γ_1. But then, constructing as before a closed symmetric curve γ_2 in A_2, we can find points in the «exterior» of γ_2 where φ is positive, so there exists a component A_3 of $\widetilde{\Omega}$ different from A_1, because A_3 is contained in the exterior of γ_2. This contradicts the Courant's Nodal Line Theorem, which states that there are at most two components of $\widetilde{\Omega}$. So φ necessarily has the same sign in a neighborhood of the boundary, and this contradicts Theorem 1.1. $\square$

Remark 2.2. When $\Omega = B(0, R)$ is a ball in $\mathbb{R}^N$ it is well known that the multiciplity of the second eigenvalue is exactly N and the corresponding eigenspace is spanned by N eigenfunctions $\varphi_1, \cdots, \varphi_N$, where φ_i is odd in x_i and even in the other variables. This follows also from our previous remarks.

In fact suppose that a second eigenfunction φ is even in all variables x_i, $i = 1, \cdots, N$. Then $\nabla\varphi(0) = 0$. We now take any direction ν and define, for $x \in \Omega_\nu = \{x \in \Omega : x.\nu < 0\}$, $\varphi^\nu(x) = \varphi(x^\nu)$, where x^ν is the reflection of x through the hyperplane $T_\nu = \{x \in \Omega : x.\nu = 0\}$. Then $(-\Delta - \lambda_2)(\varphi - \varphi^\nu) = 0$ in Ω_ν, so either $\varphi - \varphi^\nu = 0$, which means that φ is symmetric with respect to T_ν, or the restriction ψ of $\varphi - \varphi^\nu$ to Ω_ν is an eigenfunction of the Dirichlet eigenvalue problem in Ω_ν corresponding to the eigenvalue λ_2. In this latter case necessarily ψ is the first eigenfunction in Ω_ν because by Lemma 2.1 the first eigenvalue in Ω_ν is not less than λ_2. But this possibility is excluded, since in this case ψ would be positive (or negative) in Ω_ν, and by the Hopf's lemma we would get that $\frac{\partial\psi}{\partial\nu} = 2\frac{\partial\varphi}{\partial\nu} < 0$ on T_ν, against the fact that $\nabla\varphi(0) = 0$. So φ is symmetric with respect to all the hyperplanes passing through the origin and it is therefore radial and this contradicts Theorem 1.1.

This means that in the basis for the eigenspace associated to λ_2 considered in Remark 2.1 there are only functions which are odd in some variable and there is at least one. So λ_2 is equal to $\mu_1(\Omega_i^-)$ for at least one $i \in \{1, \cdots, N\}$. Since each cap Ω_i^- can be transformed by a rotation into another cap Ω_j^-, $j \neq i$, we have that $\mu_1(\Omega_i^-)$ does not depend on $i \in \{1, \cdots, N\}$ and must therefore be equal to λ_2. The previous construction gives then N independent eigenfunctions $\varphi_2, \ldots \varphi_{N+1}$, where φ_j is odd in the variable x_{j-1} and even in the other variables, which span the eigenspace of λ_2.

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