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A Lecture on Noncommutative Geometry

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A LECTURE ON NONCOMMUTATIVE GEOMETRY

ABSTRACT. — The origin of Noncommutative Geometry is twofold. On the one hand there is a wealth of examples of spaces whose coordinate algebra is no longer commutative but which have obvious geometric meaning. The first examples came from phase space in quantum mechanics but there are many others, such as the leaf spaces of foliations, duals of nonabelian discrete groups, the space of Penrose tilings, the Noncommutative torus which plays a role in M-theory compactification and finally the Adele class space which is a natural geometric space carrying an action of the analogue of the Frobenius for global fields of zero characteristic. On the other hand the stretching of geometric thinking imposed by passing to Noncommutative spaces forces one to rethink about most of our familiar notions. The difficulty is not to add arbitrarily the adjective quantum behind our familiar geometric language but to develop far reaching extensions of classical concepts. Several of these new developments are described below with emphasis on the surprises from the noncommutative world.

KEY WORDS: Noncommutative Geometry; Operator algebras; Index theory.

Let me start by comparing two simple theorems. The first one, due to Frank Morley, deals with planar geometry and is one of the few results about the geometry of triangles that was apparently unknown to the Greek mathematicians. You take an arbitrary triangle ABC and you trisect each angle, then you consider the intersection of consecutive trisectors, and obtain another triangle $\alpha\beta\gamma$ (fig. 1). Now Morley's theorem, which he found around 1899, says the following,

THEOREM. *Whichever triangle ABC you start with, the triangle $\alpha\beta\gamma$ is always equilateral.*

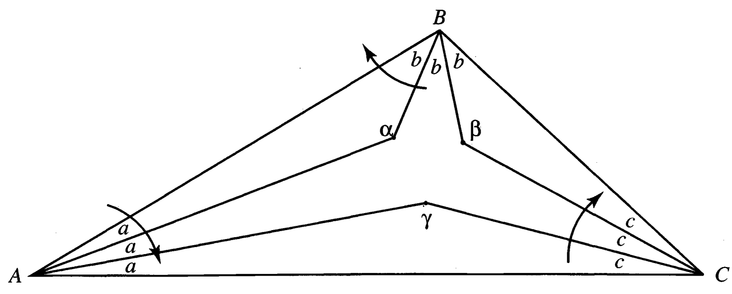

Fig. 1.

The second theorem is purely algebraic. You start with an arbitrary field k and consider three elements f, g, h of the affine group G of k: the affine group is just the group of transformations of the line of the form $x \to ax + b$; we can view it as the group of 2×2 matrices $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ where $a \in k$, $a \neq 0$, $b \in k$. This group looks very trivial because it is solvable. For $g \in G$ we let,

$$\delta(g) = a \in k^* . \tag{1}$$

By construction δ is a morphism from G to the multiplicative group k^* of non zero elements of k, and the subgroup $T = \operatorname{Ker} \delta$ is the group of translations, *i.e.* the additive group of k.

Each $g \in G$ defines a transformation,

$$g(x) = ax + b \qquad \forall\, x \in k , \tag{2}$$

and if $a \neq 1$ it admits one and only one fixed point,

$$\operatorname{fix}(g) = \frac{b}{1-a} . \tag{3}$$

Let us state the following simple fact,

THEOREM. *Let $f, g, h \in G$ be such that fg, gh, hf and fgh are* not *translations and let $j = \delta(fgh)$. The following two conditions are equivalent,*

a) $f^3 g^3 h^3 = 1$.

b) $j^3 = 1$ *and* $\alpha + j\beta + j^2\gamma = 0$ *where* $\alpha = \operatorname{fix}(fg)$, $\beta = \operatorname{fix}(gh)$, $\gamma = \operatorname{fix}(hf)$.

Now let us compare these two theorems. The first is very appealing geometrically and involves the obvious perception of geometry from the visual areas of the brain. But somehow if you try to prove it and take it the wrong way you might have a hard time. The second is algebraic and as such it involves the language. It can be checked by a high school student, it is a completely straightforward computation: you just compute the a part and the b part of the 2×2 matrix and you get the result. Now, what is true is that this second statement immediately implies the first as follows: you let the field k be the field of complex numbers and from the triangle you get three elements of the affine group which satisfy the equation $f^3 g^3 h^3 = 1$: what are they? just take for f the rotation centered at A and whose angle is twice the trisected angle a and similarly for g and h; now f^3 is a product of two symmetries along the sides of the triangle, and $f^3 g^3 h^3$ is obtained by applying twice the symmetry around each side of the triangle so of course you get $f^3 g^3 h^3 = 1$. Moreover for the same reason $\alpha = \operatorname{fix}(fg)$, $\beta = \operatorname{fix}(gh)$, $\gamma = \operatorname{fix}(hf)$ are the intersections of trisectors. Thus you get $\alpha + j\beta + j^2\gamma = 0$, but this is a well known condition which used to be taught in high school and characterizes equilateral triangles in euclidean geometry, when you consider α, β and γ as complex numbers.

This is an example of the type of duality which I want to use all the time between on the one hand the visual perception (where the geometrical facts can be sort of obvious) and on the other hand the algebraic understanding. What I mean is that, provided you can write things in algebraic terms, then somehow you enhance their power and you make them applicable in totally different circumstances. For instance the above theorem holds for a finite field, it holds for instance for the field F_4 which has cubic roots of unity and so on. So somehow, passing from the geometrical intuition to the algebraic formulation allows to increase the power of the original obvious fact and it's a little bit like the role of language with respect to perception, it allows to go back and forth and to make progress.

I chose my first example in order to say something about euclidean geometry: let us then consider non euclidean geometry. You know probably that a model of non euclidean (planar) geometry (perhaps the simplest model) in which it is completely obvious that through a point outside a line pass several parallel lines is the Klein model.

The points in this model of geometry form the inside of an ordinary ellipse (fig. 2), while the lines of the geometry are simply the intersections of the ordinary straight lines with the inside of the ellipse. Of course then you have to give, as in Euclid's axioms, the notions of congruence of segments and of angles. So in other words you have to provide a measurement of distances and a measurement of angles like the angle (LOL') (fig. 2). In the model the distance between two points A,B, is given by the logarithm of the cross ratio of the four points A, B, a, b where a, b are the points at infinity, *i.e.* on the ellipse, on the line A,B. Similarly the angle (LOL') is given by the imaginary part of $\log(LL'; l\, l')$ where l and l' are the tangents to the ellipse from O (they are imaginary).

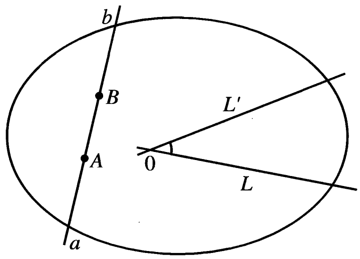

Fig. 2.

Now the discovery of non-euclidean geometry was first motivated by the perplexing role of Euclid's axiom of unique parallel, and it was at first considered as a rather esoteric type of construction, yielding to a counterexample. But it quickly became an object of great mathematical power and served as a basic example in the further development of geometry. With a slight oversimplification, the two directions that geometry took from this discovery can be summarized like this.

The first idea is that what is behind the power and the beauty of this example is its homogeneity, namely its large symmetry group, which is here the group of projective transformations preserving the ellipse. So of course this led to the development of the theory of Lie groups, to the Klein program. That theory is tractable because one is dealing with finite dimensional Lie algebras, which can be understood and classified in simple terms.

But another very powerful view of geometry came from the work of Gauss and Riemann. They formulated the idea of the intrinsic geometry of a curved space independently of its embedding in Euclidean space. This allowed to understand that spaces with nonconstant curvature, in which rigid motion is no longer possible, are just as geometric as the more symmetric ones. The general idea of manifold was introduced by Riemann in his well known memoir (cf. [28]) on the hypothesis of geometry. He explains there how to label (locally) the points of a manifold M by finitely many real numbers x^μ. One proceeds by induction on the dimension n of the manifold and reduces from n to $n-1$ using a real valued function f on M and the level hyper-surfaces $f(x)=a$ of f. The next key notion is that of the line element *i.e.* the unit of length which, provided one can carry it around, allows to measure distances in the small. Thanks to the existence of the infinitesimal calculus, this led him first of all to the simplest formula for the line element ds in local terms

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu . \tag{4}$$

But the most important point is that this new framework was not just an arbitrary generalization of geometry, inasmuch as most of the concepts which were present either in euclidean or non euclidean geometry continued to make sense, while it considerably increased the number of available interesting examples. In particular the idea of straight lines does make sense and is governed, thanks to the infinitesimal calculus, by the equation of geodesics,

$$\frac{d^2 x^\mu}{dt^2} = -\frac{1}{2} g^{\mu\alpha}(g_{\alpha\nu,\rho} + g_{\alpha\rho,\nu} - g_{\nu\rho,\alpha}) \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \tag{5}$$

At first if you step back and consider these two generalizations of geometry, the homogeneous one and the Riemannian one you might be more inclined to prefer the beauty of the homogeneous one to the generality of the other.

In fact, and as long as we consider geometry as intimately related to our model of space, Einstein's general relativity gave a very clear victory to Riemann's point of view. The simplest way to see the superiority of that point of view is to understand the following simple fact: you first need to stretch a little bit your imagination by accepting that the $g_{\mu\nu}$'s do not necessarily correspond to a positive quadratic form, you should allow for instance the metric $dx^2 + dy^2 + dz^2 - dt^2$ defining Minkowski's space.

The point then is that if one just alters a little bit the metric of Minkowski's space by replacing the coefficient $g_{00} = -1$ of dt^2 by

$$g_{00} = -(1 + 2V(x, y, z))$$

and leaving the other components unchanged (so $g_{11} = g_{22} = g_{33} = 1$ and the other components are zero) then when you write down the equation of geodesics you find Newton's law for the motion of a body in the gravitational field which is defined by the potential function V (cf. [29] for the more precise formulation).

This makes it very clear that one would loose a lot by being very conservative and only caring about homogeneous geometries. In these geometries where the $g_{\mu\nu}$ are variable, motion of a rigid body is no longer possible, but the variability which is around is in fact exactly what is needed to be able to model geometrically, by the straight lines of a geometrical substratum, crucial physical laws such as motion in a Newtonian potential.

That this geometrical substratum is independent of which type of particle, which type of body is moving, is the content of the equivalence principle *i.e.* the equality between inertial mass and gravitational mass.

Of course, all these things are fairly standard. Now let me turn to the origin of noncommutative geometry. It can be traced back essentially not to the beginning of quantum mechanics with Planck, but to the understanding of the conceptual meaning of the basic laws of spectroscopy, in particular by Heisenberg.

The very bare fact which came directly from experimental findings in spectroscopy and was unveiled by Heisenberg (and then understood at a more mathematical level by Born, Jordan, Dirac and the physicists of the 1920's) is that whereas when you are dealing with a manifold you can parametrize (locally) its points x by real numbers $x_1, x_2, \ldots$ so that you have n real numbers that specify completely the situation of the system, when you turn to the phase space of a quantum mechanical system, even of the simplest kind, the coordinates, namely these real numbers $x_1, x_2, \ldots$ and so on, that you would like to use to parametrize points, actually do not commute.

So in fact what happens is that from the simplest examples of quantum mechanics one finds that the familiar duality of algebraic geometry between a space and its algebra of coordinates (*i.e.* the algebra of functions on that space) is really too restrictive even to model the phase space of very simple physical systems. What we are enticed to do, then, is to stretch this duality so that the algebra of coordinates on a space, this algebra of variable numbers $x_1, x_2, \ldots$ is no longer required to be commutative.

What I shall now do is to give you examples and a general principle which show that this phenomenon is by no means limited to quantum mechanics. It is in fact a fairly general phenomenon and the general principle which is behind it is roughly speaking the following:

«Noncommutative spaces are naturally generated by the operation of quotient».

Many of the spaces, many of the sets that we are used to consider are not really defined by listing their points but they are defined by identifications, their elements are given as equivalence classes; I mean you start with a set that is much larger than the one you want to consider, and in fact you cut this set into equivalence classes, you identify the points which belong to the same class.

Now there are two ways of proceeding at the algebraic level in order to identify two points a and b in a given space M. There is a first way to proceed which is to take only those functions f on M which take the same value at the points a and b, namely that satisfy $f(a) = f(b)$. Thus the usual algebra of functions associated to the quotient is

$$\mathcal{A} = \{f; f(a) = f(b)\}. \tag{6}$$

There is however another way of describing, in an algebraic manner, the above quotient operation. It consists, instead of taking the subalgebra given by (6), to adjoin to the algebra of functions on $\{a, b\}$ the identification of a with b. The obtained algebra is the algebra of two by two matrices

$$\mathcal{B} = \left\{ f = \begin{bmatrix} f_{aa} & f_{ab} \\ f_{ba} & f_{bb} \end{bmatrix} \right\}. \tag{7}$$

This second way of taking the quotient can be described as follows: do not impose on functions to have the same value at a and b, but allow the two points a and b to speak to each other. In order to do that, just add the «off diagonal» matrix elements f_{ab} and f_{ba}. Now, the effect of these off-diagonal matrix elements will be to coalesce these two points into one point in the spectrum of the algebra, namely in the space of its irreducible representations. Indeed the algebra is that of two by two matrices and the irreducible representation corresponding to a (associated to the pure state a) has now become equivalent to the representation coming from b.

Now in this very simple example of two points, the two different algebras that one obtains are still equivalent. The equivalence between $\mathcal{A}$ and $\mathcal{B}$ is called Morita equivalence [54] and means that the corresponding categories of right modules are equivalent. There is of course an obvious difference between $\mathcal{A}$ and $\mathcal{B}$ namely $\mathcal{B}$ is no longer commutative.

So let me now give you a next simple example, extremely simple too, in which the two ways of doing things do not give you the same answer, even up to Morita equivalence. This example is the following: geometrically you just take two intervals, $[0, 1]$, and you just want to identify the insides of these two intervals but you do not want to identify the end points (fig. 3).

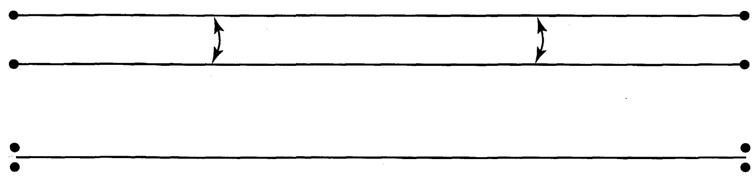

Fig. 3.

Now, let us do this in the first way as above. Thus we take C^∞ functions on $M = [0, 1] \times \{0, 1\}$ and we only consider functions which have equal values on the points $(x, 0)$ and $(x, 1)$ for $x \in]0, 1[$. Since the functions are continuous, and are equal inside the intervals they will also be equal at the end points. Thus what we get is simply the algebra $\mathcal{A} = C^\infty([0, 1])$, whose spectrum is the interval $[0, 1]$.

Now, however, let us take the quotient using the second way, what do we get? Well, we get C^∞ maps from the interval $[0, 1]$ to 2×2 matrices , but we don't get all C^∞ such maps, we only get C^∞ maps which have diagonal matrices as boundary values. So the value of the matrix at 0 is a diagonal matrix and the value of the matrix at 1 is a diagonal matrix. Now, if you know a little bit of Fourier analysis you will find that this second way gives you an algebra $\mathcal{B}$ which is quite interesting, it is the group ring of the dihedral group, namely of the free product of two groups with 2 elements,

Z_2, which is also the semi-direct product of $\mathbb{Z}$ by Z_2, and this algebra is quite different from the first one, and certainly not Morita equivalent to $C^\infty([0\,,\,1])$.

In general, when we consider more complicated examples of quotient spaces it is no longer true that the two algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. The first operation (6) is of a cohomological flavor while the second (7) keeps a closer contact with the quotient space. The general theory is thus the extension of the familiar duality of algebraic geometry to

$$\text{Quotient spaces} \leftrightarrow \text{Noncommutative algebra}. \tag{8}$$

It rests mainly on the richness of the examples and on the extension of geometrical concepts to the noncommutative case.

Let us thus proceed to a much more difficult example where obviously the first manner of taking the quotient doesn't work but the second does. This example is the following: consider the 2-torus

$$M = \mathbb{R}^2/\mathbb{Z}^2\,. \tag{9}$$

The space X which we contemplate is the space of solutions of the differential equation,

$$dx = \theta\, dy \qquad x\,,y \in \mathbb{R}/\mathbb{Z} \tag{10}$$

where $\theta \in]0\,,\,1[$ is a fixed irrational number (fig. 4).

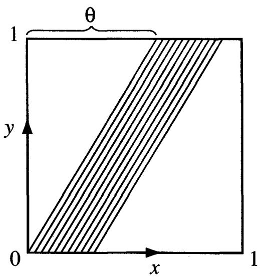

Fig. 4.

Thus the space we are interested in here is just the space of leaves of the foliation defined by the differential equation. We can label such a leaf by a point of the transversal given by $y = 0$ which is a circle $S^1 = \mathbb{R}/\mathbb{Z}$, but clearly two points of the transversal which differ by an integer multiple of θ give rise to the same leaf. Thus

$$X = S^1/\theta\mathbb{Z} \tag{11}$$

i.e. X is the quotient of S^1 by the equivalence relation which identifies any two points on the orbits of the irrational rotation

$$R_\theta x = x + \theta \quad \bmod 1\,. \tag{12}$$

When we deal with S^1 as a space in the various categories (smooth, topological, measurable) it is perfectly described by the corresponding algebra of functions,

$$C^\infty(S^1) \subset C(S^1) \subset L^\infty(S^1)\,. \tag{13}$$

When one applies the operation (6) to pass to the quotient, one finds, irrespective of which category one works with, the trivial answer

$$\mathcal{A} = \mathbb{C}. \tag{14}$$

The operation (7) however gives very interesting algebras, by no means Morita equivalent to $\mathbb{C}$. In the above situation of a quotient by a group action the operation (7) is the construction of the crossed product familiar to algebraist from the theory of central simple algebras.

An element of $\mathcal{B}$ is given by a power series

$$b = \sum_{n \in \mathbb{Z}} b_n U^n \tag{15}$$

where each b_n is an element of the algebra (13), while the multiplication rule is given by

$$U h U^{-1} = h \circ R_\theta^{-1}. \tag{16}$$

Now the algebra (13) is generated by the function V on S^1,

$$V(\alpha) = \exp(2\pi i \alpha) \qquad \alpha \in S^1 \tag{17}$$

and it follows that $\mathcal{B}$ admits the generating system (U, V) with presentation given by the relation

$$UVU^{-1} = \lambda^{-1} V \qquad \lambda = \exp 2\pi i \theta. \tag{18}$$

Thus, if for instance we work in the smooth category a generic element b of $\mathcal{B}$ is given by a power series

$$b = \sum_{\mathbb{Z}^2} b_{nm} U^n V^m \ , \ b \in \mathcal{S}(\mathbb{Z}^2) \tag{19}$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences of rapid decay on $\mathbb{Z}^2$.

This algebra is by no means Morita equivalent to $\mathbb{C}$ and has a very rich and interesting algebraic structure. It is (canonically up to Morita equivalence) associated to the foliation (10) and the interplay between the geometry of the foliation and the algebraic structure of $\mathcal{B}$ begins by noticing that to a *closed transversal* of the foliation corresponds canonically a *finite projective module* over $\mathcal{B}$.

From the transversal $x = 0$, one obtains the following right module over $\mathcal{B}$. The underlying linear space is the usual Schwartz space,

$$\mathcal{S}(\mathbb{R}) = \{\xi \, , \, \xi(s) \in \mathbb{C} \quad \forall \, s \in \mathbb{R}\} \tag{20}$$

of smooth functions on the real line all of whose derivatives are of rapid decay.

The right module structure is given by the action of the generators U, V

$$(\xi U)(s) = \xi(s + \theta) \ , \ (\xi V)(s) = e^{2\pi i s} \xi(s) \quad \forall \, s \in \mathbb{R}. \tag{21}$$

One of course checks the relation (18), and it is a beautiful fact that as a right module over $\mathcal{B}$ the space $\mathcal{S}(\mathbb{R})$ is *finitely generated* and *projective* (*i.e.* complements to a free module). It follows that it has the correct algebraic atributes to deserve the name of «noncommutative vector bundle» according to the dictionary,

Space	Algebra
Vector bundle	Finite projective module.

The concrete description of the general finite projective modules over $\mathcal{A}_\theta$ is obtained by combining the results of [48, 73, 74]. They are classified up to isomorphism by a pair of integers (p, q) such that $p + q\theta \geq 0$ and the corresponding modules $\mathcal{H}^\theta_{p,q}$ are obtained by the above construction from the transversals given by closed geodesics of the torus M.

The algebraic counterpart of a vector bundle is its space of smooth sections $C^\infty(X, E)$ and one can in particular compute its dimension by computing the trace of the identity endomorphism of E. If one applies this method in the above noncommutative example, one finds

$$\dim_B(\mathcal{S}) = \theta \,. \tag{22}$$

The appearance of non integral dimension is very exciting and displays a basic feature of von Neumann algebras of type II. The dimension of a vector bundle is the only invariant that remains when one looks from the measure theoretic point of view (*i.e.* when one takes the third algebra in (13)). The von Neumann algebra which describes the quotient space X from the measure theoretic point of view is the crossed product,

$$R = L^\infty(S^1) \rtimes_{R_\theta} \mathbb{Z} \tag{23}$$

and is the well known hyperfinite factor of type II_1. In particular the classification of finite projective modules $\mathcal{E}$ over R is given by a positive real number, the Murray and von Neumann *dimension*,

$$\dim_R(\mathcal{E}) \in \mathbb{R}_+ \,. \tag{24}$$

The next surprise is that even though the *dimension* of the above module is irrational, when we compute the analogue of the first Chern class, *i.e.* of the integral of the curvature of the vector bundle, we obtain an integer. Indeed the two commuting vector fields which span the tangent space for an ordinary (commutative) 2-torus correspond algebraically to two commuting derivations of the algebra of smooth functions. These derivations continue to make sense when the generators U and V of $C^\infty(\mathbb{T}^2)$ no longer commute but satisfy (18) so that they generate $C^\infty(\mathbb{T}^2_\theta)$. They are given by the same formulas as in the commutative case,

$$\delta_1 = 2\pi i U \frac{\partial}{\partial U} \;, \quad \delta_2 = 2\pi i V \frac{\partial}{\partial V} \tag{25}$$

so that $\delta_1\left(\sum b_{nm} U^n V^m\right) = 2\pi i \sum n b_{nm} U^n V^m$ and similarly for δ_2. One still has of course

$$\delta_1 \delta_2 = \delta_2 \delta_1 \tag{26}$$

and the δ_j are still derivations of the algebra $\mathcal{B} = C^\infty(\mathbb{T}^2_\theta)$,

$$\delta_j(bb') = \delta_j(b)b' + b\delta_j(b') \quad \forall\, b, b' \in \mathcal{B}. \tag{27}$$

The analogues of the notions of connection and curvature of vector bundles are straightforward to obtain since a connection is just given by the associated covariant differentiation ∇ on the space of smooth sections. Thus here it is given by a pair of linear operators,

$$\nabla_j : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \tag{28}$$

such that

$$\nabla_j(\xi b) = (\nabla_j \xi)b + \xi\delta_j(b) \quad \forall\, \xi \in \mathcal{S}\ ,\, b \in \mathcal{B}. \tag{29}$$

One checks that, as in the usual case, the trace of the curvature $\Omega = \nabla_1\nabla_2 - \nabla_2\nabla_1$, is independent of the choice of the connection. Now the remarkable fact here is that (up to the correct powers of $2\pi i$) the integral curvature of $\mathcal{S}$ is an integer. In fact for the following choice of connection the curvature Ω is constant, equal to $\frac{1}{\theta}$ so that the irrational number θ disappears in the integral curvature, $\theta \times \frac{1}{\theta}$

$$(\nabla_1\xi)(s) = -\frac{2\pi i s}{\theta}\,\xi(s) \quad (\nabla_2\xi)(s) = \xi'(s). \tag{30}$$

With this integrality, one could get the wrong impression that the algebra $\mathcal{B} = C^\infty(\mathbb{T}^2_\theta)$ looks very similar to the algebra $C^\infty(\mathbb{T}^2)$ of smooth functions on the 2-torus. A striking difference is obtained by looking at the range of Morse functions. The range of a Morse function on $\mathbb{T}^2$ is of course a connected interval. For the above noncommutative torus $\mathbb{T}^2_\theta$ the range of a Morse function is the spectrum of a real valued function such as

$$h = U + U^* + \mu(V + V^*) \tag{31}$$

and it can be a Cantor set, *i.e.* have infinitely many disconnected pieces. This shows that the one dimensional shadows of our space $\mathbb{T}^2_\theta$ are truly different from what they are in the commutative case. The above noncommutative torus $\mathbb{T}^2_\theta$ is the simplest example of noncommutative manifold, it arises naturally not only from foliations but also from the Brillouin zone in the Quantum Hall effect as understood by J. Bellissard, and in M-theory as we shall see next. Indeed both the noncommutative tori and the components ∇_j of the Yang-Mills connections occur naturally in the classification of the BPS states in M-theory [61]. In the matrix formulation of M-theory the basic equations to obtain periodicity of two of the basic coordinates X_i turn out to be the following

$$U_i X_j U_i^{-1} = X_j + a\delta^j_i, \quad i = 1, 2 \tag{32}$$

where the U_i are unitary gauge transformations.

The multiplicative commutator $U_1 U_2 U_1^{-1} U_2^{-1}$ is then central and in the irreducible case its scalar value $\lambda = \exp 2\pi i\theta$ brings in the algebra of coordinates on the noncommutative torus. The X_j are then the components of the Yang-Mills connections. It is quite remarkable that the same picture emerged from the other information one has about M-theory concerning its relation with 11 dimensional supergravity and that string theory dualities could be interpreted using Morita equivalence. The latter [54] relates the values of θ on an orbit of $SL(2,\mathbb{Z})$ and this type of relation, which is obvious from the foliation point of view, would be invisible in a purely deformation theoretic perturbative expansion like the one given by the Moyal product.

We shall come back later to the natural moduli space for the noncommutative tori and to the metric aspect when we have the correct general tools. But we first need to dispell the impression that the noncommutative torus, because of its ubiquity, is the only example of noncommutative space.

A very large class of examples is provided first of all by the duals of discrete groups: I showed you already the dual of the dihedral group which was truly very simple, but as soon as you go to more complicated groups you find that duals of discrete groups are exactly like leaf spaces of foliations. It is important to understand at least on one example why the dual of a discrete non abelian group is in essence a quotient space, like the one we analysed above for foliations. Thus, let Γ be the solvable group $\Gamma = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$ obtained as the semi-direct product of the additive group $\mathbb{Z}^2$ by the action of $\mathbb{Z}$ given by the matrix

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \in SL(2,\mathbb{Z})\,. \tag{33}$$

By Mackey's theory of induced representations one gets that each element x of the 2-torus $\widehat{\mathbb{Z}}^2$ which is the Pontrjagin dual of the group $\mathbb{Z}^2$, determines by induction an irreducible representation of Γ,

$$\pi_x \in \mathrm{Irrep}(\Gamma)\,. \tag{34}$$

Moreover the two representations π_x and π_y associated to $x, y \in \widehat{\mathbb{Z}}^2$, are equivalent iff x and y are on the same orbit of the (transposed) transformation T of $\widehat{\mathbb{Z}}^2$. One does not get all irreducible representations of Γ by this procedure but the quotient space

$$X = \widehat{\mathbb{Z}}^2/\sim \ , \ x \sim y \text{ iff } y = T^n x \text{ for some } n \tag{35}$$

does correspond in the above sense to the group ring $\mathcal{A} = \mathbb{C}\Gamma$ of Γ.

Here is a more complete list of basic examples,

Space of leaves of foliations
Space of irreducible representations of discrete groups
Space of Penrose tilings of the plane
Brillouin zone in the quantum Hall effect
Phase space in quantum mechanics
Space time

Then there is of course deformation theory and in particular the deformation of phase space or of Poisson manifolds which is another rich source of examples.

But let me still mention another space which I might have little time to talk about. It is the space of Adele classes, which is a noncommutative space whose understanding is intimately related to the location of the zeros of Hecke L-functions in the number field case.

Thus you can see that there are plenty of examples of noncommutative spaces that beg our understanding but which are very difficult to comprehend.

The reason why I started working in noncommutative geometry is that I knew that even at the level of measure theory, already there, at this very coarse level, things were becoming highly non trivial in the general noncommutative case. When you look at an ordinary space and you do measure theory, you use the Lebesgue theory which is a beautiful theory, but all spaces are the same, there is nothing really happening in ordinary measure theory. This is not at all the case in noncommutative measure theory. What happens there is very surprising. It is an absolutely fascinating fact that when you take a non commutative algebra M from the measure theory point of view, such an algebra evolves with time!

What I mean is that it admits a god-given time evolution given by a canonical group homomorphism [1],

$$\delta : \mathbb{R} \to \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M) \tag{36}$$

from the additive group $\mathbb{R}$ to the center of the group of automorphism classes of M modulo inner automorphisms.

This homomorphism is provided by the uniqueness of the, a priori state dependent, modular automorphism group of a state. Together with the earlier work of Powers, Araki-Woods and Krieger it was the beginning of a long story which eventually led to the complete classification [1-9] of approximately finite dimensional factors (also called hyperfinite).

They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* \ , \tag{37}$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$ in the sense of G. Mackey, *i.e.* an ergodic action of $\mathbb{R}_+^*$.

I realized many years afterwards what was the meaning of this theory.

In fact one can interpret this classification as being the correct local class field theory for Archimedian local fields. Local class field theory is interesting and non trivial for local fields which are non archimedean, like p-adic numbers and so on, because such fields K are very far from being algebraically closed. One lets K_{ab} be the maximal abelian extension of K and W_K be the subgroup of the Galois group $\mathrm{Gal}(K_{\mathrm{ab}} : K)$ whose elements induce on the maximal unramified extension $K_{\mathrm{un}} \subset K_{\mathrm{ab}}$ an integral power of the Frobenius automorphism. One endows W_K with the locally compact topology dictated by the exact sequence of groups

$$1 \to \mathrm{Gal}(K_{\mathrm{ab}} : K_{\mathrm{un}}) \to W_K \to \mathrm{Mod}(K) \to 1 \ , \tag{38}$$

and the main result of local class field theory asserts the existence of a canonical isomorphism

$$W_K \xrightarrow{\sim} K^* \tag{39}$$

compatible with the module. Thus in that case one has an interesting theory, given by the above correspondence between a Galois group and the multiplicative group of the field.

But there is nothing like that when you look at class field over complex numbers, because the field of complex numbers is algebraically closed, so there is apparently nothing going on. Now it turns out that the theory of factors gives the correct replacement for the missing Brauer theory at archimedian places. This can be seen as follows. To say that a field is algebraically closed can be formulated in terms of representation theory. It is equivalent to say that when we look at representations of semisimple objects (groups, algebras...) over this field any such representation can be decomposed in direct sums of multiples of representations whose commutant is the scalars. Of course this does hold when we take *finite dimensional* representations over $\mathbb{C}$. But as soon as we take infinite dimensional representations a new phenomenon occurs and it is in perfect analogy with the unramified extensions of p-adic fields. Non trivial factors occur, they are commutants of representations in Hilbert space and though they have trivial center, they are not Morita equivalent to $\mathbb{C}$. As we saw these factors have an invariant, which is called their module, very analogous to the module for local fields; this module is not really a subgroup of $\mathbb{R}_+^*$, it's a virtual subgroup of $\mathbb{R}_+^*$ in the sense that it is a flow. And for factors which are approximately finite dimensional this flow turns out to give exactly a complete invariant of factors and all flows occur. This analogy goes quite far and yields a spectral interpretation of the zeros of L-functions in number theory. I'll just refer to [10].

So, we see that noncommutative measure theory is already highly non trivial, and thus we have all reasons to believe that if one goes further in the natural hierarchy of features of a space, one will discover really interesting new phenomenas.

Measure theory is indeed a very coarse way of looking at a space, and a finer and finer picture is obtained by going up in the following hierarchy of points of view:

Riemannian geometry
Differential geometry
Topology
Measure theory

The development of the topological ideas was prompted by the work of Israel Gel'fand, whose C^*-algebras give the required framework for noncommutative topology. The two main driving forces were the Novikov conjecture on homotopy invariance of higher signatures of ordinary manifolds as well as the Atiyah-Singer Index theorem. It has led, through the work of Atiyah, Singer, Brown, Douglas, Fillmore, Miscenko and Kasparov [11-15] to the recognition that not only the Atiyah-Hirzebruch K-theory but more importantly the dual K-homology admit Hilbert space techniques and functional analysis as their natural framework. The cycles in the K-homology group $K_*(X)$ of a compact space X are indeed given by Fredholm representations of the C^*-algebra A of continuous functions on X. The central tool is the Kasparov bivariant K-theory. A basic example of C^*-algebra to which the theory applies is the group ring of a discrete group and restricting oneself to commutative algebras is an obviously undesirable assumption.

For a C^*-algebra A, let $K_0(A)$, $K_1(A)$ be its K theory groups. Thus $K_0(A)$ is the algebraic K_0 theory of the ring A and $K_1(A)$ is the algebraic K_0 theory of the ring $A \otimes C_0(\mathbb{R}) = C_0(\mathbb{R}, A)$. If $A \to B$ is a morphism of C^*-algebras, then there are induced homomorphisms of abelian groups $K_i(A) \to K_i(B)$. Bott periodicity provides a six term K theory exact sequence for each exact sequence $0 \to J \to A \to B \to 0$ of C^*-algebras and excision shows that the K groups involved in the exact sequence only depend on the respective C^*-algebras. As an exercise to appreciate the power of this abstract tool one should for instance use the six term K theory exact sequence to give a short proof of the Jordan curve theorem.

Discrete groups, Lie groups, group actions and foliations give rise through their convolution algebra to a canonical C^*-algebra, and hence to K theory groups. The analytical meaning of these K theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0\, C_r^*\, G$ where $C_r^*\, G$ is the reduced C^*-algebra of G. Thus an explicit determination of the K theory in this case in particular involves an enumeration of the discrete series.

We introduced with P. Baum [16] a geometrically defined K theory which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In the case of semi-simple Lie groups it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [17]. Using elliptic operators we constructed a natural map μ from our geometrically defined K theory groups to the above analytic (*i.e.* C^*-algebra) K theory groups. Much progress has been made in the past years to determine the range of validity of the isomorphism between the geometrically defined K theory groups and the above analytic (*i.e.* C^*-algebra) K theory groups. We refer to the three Bourbaki seminars [18-20] for an update on this topic and for a precise account of the various contributions. Among the most important contributions are those of Kasparov and Higson who showed that the conjectured isomorphism holds for amenable groups. It also holds for real semi-simple Lie groups thanks in particular to the work of A. Wassermann. Moreover the recent work of V. Lafforgue crossed the barrier of property T, showing that it holds for cocompact subgroups of rank one Lie groups and also of $SL(3, \mathbb{R})$ or of p-adic Lie groups. He also gave the first general conceptual proof of the isomorphism for real or p-adic semi-simple Lie groups. The proof of the isomorphism is certainly accessible for all connected locally compact groups. The proof by G. Yu of the analogue (due to J. Roe) of the conjecture in the context of coarse geometry for metric spaces which are uniformly embeddable in Hilbert space, and the work of G. Skandalis J. L. Tu,

J. Roe and N. Higson on the groupoid case got very striking consequences such as the injectivity of the map μ for exact $C_r^*(\Gamma)$ due to Kaminker, Guentner and Ozawa, but recent progress due to Gromov, Higson, Lafforgue and Skandalis gives counterexamples to the general conjecture for locally compact groupoids for the simple reason that the functor $G \to K_0(C_r^*(G))$ is not half exact, unlike the functor given by the geometric group. This makes the general problem of computing $K(C_r^*(G))$ really interesting. It shows that besides determining the large class of locally compact groups for which the original conjecture is valid, one should understand how to take homological algebra into account to deal with the correct general formulation. It also raises many integrality questions in cyclic cohomology of both discrete groups and foliations since a number of natural cyclic cocycles take integral values on the range of the map μ from the geometric group to the analytic group [27].

The development of differential geometric ideas, including de Rham homology, connections and curvature of vector bundles, etc. ... took place during the eighties thanks to cyclic cohomology which came from two different horizons [21-25]. This led for instance to the proof of the Novikov conjecture for hyperbolic groups [26], but got many other applications. Basically, by extending the Chern-Weil characteristic classes to the general framework it allows for many concrete computations of differential geometric nature on noncommutative spaces. It also showed the depth of the relation between the above classification of factors and the geometry of foliations. For instance, using cyclic cohomology together with the following simple fact,

(40) «A connected group can only act trivially on a homotopy invariant cohomology theory»,

one proves (cf. [27]) that for any codimension one foliation F of a compact manifold V with non vanishing Godbillon-Vey class one has,

$$\mathrm{Mod}(M) \text{ has finite covolume in } \mathbb{R}_+^* \,, \tag{41}$$

where $M = L^\infty(V, F)$ and a virtual subgroup of finite covolume is a flow with a finite invariant measure.

In order to understand what cyclic cohomology is about, it is worthwile to prove, as an exercise, the following simple fact:

«Let $\mathcal{A}$ be an algebra and φ a trilinear form on $\mathcal{A}$ such that 1) $\varphi(a_0, a_1, a_2) = \varphi(a_1, a_2, a_0)$ $\forall a_j \in \mathcal{A}$. 2) $\varphi(a_0 a_1, a_2, a_3) - \varphi(a_0, a_1 a_2, a_3) + \varphi(a_0, a_1, a_2 a_3) - \varphi(a_2 a_0, a_1, a_2) = 0$ $\forall a_j \in \mathcal{A}$. Then the scalar $\varphi_n(E, E, E)$ is invariant under homotopy for projectors (idempotents) $E \in M_n(\mathcal{A})$»

(here φ has been uniquely extended to $M_n(\mathcal{A})$ using the trace on $M_n(\mathbb{C})$, *i.e.* $\varphi_n = \varphi \otimes \mathrm{Trace}$).

This fact is not difficult to prove, the point is that a deformation of idempotents is always isospectral,

$$\dot{E} = [X, E] \text{ for some } X \in M_n(\mathcal{A}) \,. \tag{42}$$

When we take $\mathcal{A} = C^\infty(M)$ for a manifold M and let

$$\varphi(f^0, f^1, f^2) = \langle C, f^0 df^1 \wedge df^2 \rangle \quad \forall f^j \in \mathcal{A} \tag{43}$$

where C is a 2-dimensional closed de Rham current, the invariant given by the lemma is equal to (up to normalisation)

$$\langle C, c_1(E) \rangle \tag{44}$$

where c_1 is the first chern class of the vector bundle E on M whose fiber at $x \in M$ is the range of $E(x) \in M_n(\mathbb{C})$. In this example we see that for any permutation of $\{0, 1, 2\}$ one has:

$$\varphi(f^{\sigma(0)}, f^{\sigma(1)}, f^{\sigma(2)}) = \varepsilon(\sigma)\varphi(f^0, f^1, f^2) \tag{45}$$

where $\varepsilon(\sigma)$ is the signature of the permutation. However when we extend φ to $M_n(\mathcal{A})$ as $\varphi_n = \varphi \otimes \mathrm{Tr}$,

$$\varphi_n(f^0 \otimes \mu^0, f^1 \otimes \mu^1, f^2 \otimes \mu^2) = \varphi(f^0, f^1, f^2)\mathrm{Tr}(\mu^0 \mu^1 \mu^2) \tag{46}$$

the property (45) only survives for *cyclic* permutations. This is at the origin of the name, *cyclic cohomology*, given to the corresponding cohomology theory.

In the example of the noncommutative torus, the cyclic cocycle that was giving an integral invariant is

$$\varphi(b^0, b^1, b^2) = \tau(b^0(\delta_1(b^1)\delta_2(b^2) - \delta_2(b^1)\delta_1(b^2))) \tag{47}$$

where τ is the unique trace,

$$\tau(b) = b_{00} \text{ for } b = \sum b_{nm} U^n W^m . \tag{48}$$

The pairing given by the lemma then gives the Hall conductivity when applied to a spectral projection of the Hamiltonian (see [38] for an account of the work of J. Bellissard).

We then obtain in general the beginning of a dictionary relating usual geometrical notions to their algebraic counterpart in such a way that the latter is meaningful in the general noncommutative situation.

Space	Algebra
Vector bundle	Finite projective module
Differential form	(Class of) Hochschild cycle
DeRham current	(Class of) Hochschild cocycle
DeRham homology	Cyclic cohomology
Chern Weil theory	Pairing $\langle K(\mathcal{A}), HC(\mathcal{A}) \rangle$

Of course writing down such a dictionary that translates standard notions of geometry in terms which do not involve commutativity, only gives a very naive idea of noncommutative geometry, because we already saw with measure theory that what is interesting is not only to translate, but to see completely new phenomenas.

For instance, unlike the de Rham cohomology, its noncommutative replacement which is cyclic cohomology is not graded but filtered. Also it inherits from the Chern character map a natural integral lattice. We shall see now that these two features play a basic role in the description of the natural moduli space (or more precisely, its covering Teichmüller space, together with a natural action of $SL(2, \mathbb{Z})$ on this space) for the noncommutative tori $\mathbb{T}^2_\theta$. The discussion parallels the description of the moduli space of elliptic curves but we shall find that our moduli space is the boundary of the latter space.

We first observe that as the parameter $\theta \in \mathbb{R}/\mathbb{Z}$ varies from 0 to 1 and if we follow up the finite projective modules $\mathcal{H}^\theta_{p,q}$ we get a monodromy, using the isomorphism $\mathbb{T}^2_\theta \sim \mathbb{T}^2_{\theta+1}$. The computation shows that this monodromy is given by the transformation $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ *i.e.*, $x \to x + y$, $y \to y$ in terms of the (x, y) coordinates in the K group K_0. This shows that in order to follow the θ-dependence of the K group, we should consider the algebra $\mathcal{A}$ together with a choice of isomorphism,

$$K_0(\mathcal{A}) \overset{\rho}{\simeq} \mathbb{Z}^2 , \quad \rho \text{ (trivial module)} = (1, 0) . \tag{49}$$

Exactly as the Jacobian of an elliptic curve appears as a quotient of the $(1, 0)$ part of the cohomology by the lattice of integral classes, we can associate canonically to $\mathcal{A}$ the following data:

1) The ordinary two dimensional torus $\mathbb{T} = HC_{\text{even}}(\mathcal{A})/K_0(\mathcal{A})$ quotient of the cyclic homology of $\mathcal{A}$ by the image of K theory under the Chern character map.

2) The foliation F (of the above torus) given by the natural filtration of cyclic homology (dual to the filtration of HC^{even}).

3) The transversal T to the foliation given by the geodesic joining 0 to the class $[1] \in K_0$ of the trivial bundle.

It turns out that the algebra associated (as in (7)) to the foliation F, and the transversal T is isomorphic to $\mathcal{A}$, and that a purely geometric construction associates to every element $\alpha \in K_0$ its canonical representative from the transversal given by the geodesic joining 0 to α. (Elements of the algebra associated to the transversal T are just matrices $a(i, j)$ where the indices (i, j) are arbitrary pairs of elements i, j of T which belong to the same leaf. The algebraic rules are the same as for ordinary matrices. Elements of the module associated to another transversal T' are rectangular matrices, and the dimension of the module is the transverse measure of T'.)

This gives the correct description of the modules $\mathcal{H}_{p,q}$. The above is in perfect analogy with the isomorphism of an elliptic curve with its Jacobian. The striking difference is that we use the *even* cohomology and K group instead of the odd ones.

It shows that, using the isomorphism ρ, the whole situation is described by a foliation $dx = \theta dy$ of $\mathbb{R}^2$ where the exact value of θ (not only modulo 1) does matter now. Now the space of translation invariant foliations of $\mathbb{R}^2$ is the boundary N of the space M of translation invariant conformal structures on $\mathbb{R}^2$, and with $\mathbb{Z}^2 \subset \mathbb{R}^2$ a fixed lattice, they both inherit an action of $SL(2,\mathbb{Z})$. We now describe this action precisely in terms of the pair $(\mathcal{A},\rho)$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$. Let $\mathcal{E} = \mathcal{H}_{p,q}$ where $(p,q) = \pm(d,-c)$, we define a new algebra $\mathcal{A}'$ as the commutant of $\mathcal{A}$ in $\mathcal{E}$, *i.e.* as

$$\mathcal{A}' = \mathrm{End}_{\mathcal{A}}(\mathcal{E}). \tag{50}$$

It turns out (this follows from Morita equivalence) that there is a canonical map μ from $K_0(\mathcal{A}')$ to $K_0(\mathcal{A})$ (obtained as a tensor product over $\mathcal{A}'$) and the isomorphism $\rho' : K_0(\mathcal{A}') \simeq \mathbb{Z}^2$ is obtained by

$$\rho' = g \circ \rho \circ \mu. \tag{51}$$

This gives an action of $SL(2,\mathbb{Z})$ on pairs $(\mathcal{A},\rho)$ with irrational θ (the new value of θ is $(a\theta+b)/(c\theta+d)$ and for rational values one has to add a point at ∞).

Finally note that the above action of $SL(2,\mathbb{Z})$ on the parameter θ lies beyond the purely formal realm of deformation theory in which θ is treated as a formal deformation parameter. This is a key point in which noncommutative geometry should be distinguished from formal attempts to deform standard geometry.

This is all I want to say about the soft part of differential geometry, namely the part which deals with external vector bundles and with Chern-Weil theory an so on and so forth. Of course this doesn't touch in any way the real problem which is to find the framework for geometry itself.

The central notion of noncommutative geometry comes from the identification of the noncommutative analogue of the two basic concepts in Riemann's formulation of Geometry, namely those of manifold and of infinitesimal line element. Both of these noncommutative analogues are of spectral nature and combine to give rise to the notion of spectral triple and spectral manifold, which will be described below. We shall first describe an operator theoretic framework for the calculus of infinitesimals which will provide a natural home for the line element ds.

I first have to make a little excursion, and I want it as naive as possible. I want to turn back to an extremely naive question about what is an infinitesimal. Let me first explain one answer that was proposed for this intuitive idea of infinitesimal and let me explain why this answer is not satisfactory and then give another answer which hopefully is satisfactory. So, I remember quite a long time ago to have seen an answer which was proposed by non standard analysis. The book I was reading was starting from the following problem:

You play a game of throwing darts at some target called Ω (fig. 5) and the question which is asked is the following: what is the probability $dp(x)$ that actually when you send the dart you land exactly at a given point $x \in \Omega$? Then the following argument was given: certainly this probability $dp(x)$ is smaller than $1/2$ because you can cut the target into two equal halves, only one of which contains x. For the same reason $dp(x)$ is smaller than $1/4$, and so on and so forth. So what you find out is that $dp(x)$ is smaller than any positive real number ϵ. On the other hand, if you give the answer that $dp(x)$ is 0, this is not really satisfactory, because whenever you send the dart you will land somewhere. So now, if you ask a mathematician about this naive question,

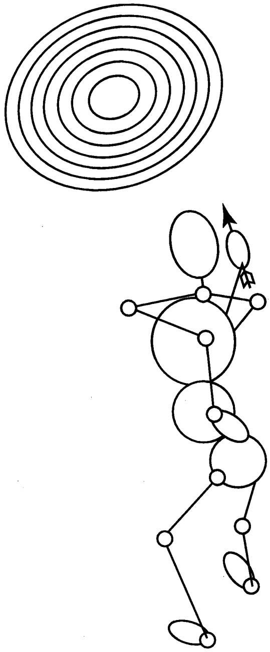

Fig. 5. – This drawing is borrowed from the lecture by David Mumford in the same volume.

he might very well answer: well, $dp(x)$ is a 2-form, or it's a measure, or something like that. But then you can try to ask him more precise questions, for instance «what is the exponential of $-\frac{1}{dp(x)}$». And then it will be hard for him to give a satisfactory answer, because you know that the Taylor expansion of this function is just flat. Now this book claimed to give an answer, and it was what is called a non standard number. So I worked on this theory for some time, learning some logics, until eventually I realized there was a very bad obstruction preventing one to get concrete answers. It is the following: it's a little lemma that one can easily prove, that if you are given a non standard number you can canonically produce a subset of the interval which is not Lebesgue measurable. Now we know from logic (from results of Paul Cohen and Solovay) that it will forever be impossible to produce explicitely a subset of the real numbers, of the interval $[0, 1]$, say, that is not Lebesgue measurable. So, what this says is that for instance in this example, nobody will actually be able to name a non standard number. A nonstandard number is some sort of chimera which is impossible to grasp and certainly not a concrete object. In fact when you look at nonstandard analysis you find out that except for the use of the ultraproducts, which is very nice, it just shifts the order in logic by one step; it's not doing much more. Now, what I want to explain is that to this very naive question there is a very beautiful and simple answer which is provided by quantum mechanics. This answer will be obtained just by going through the usual dictionary of quantum mechanics, but looking at it more closely. So, let us thus look at the first two lines of the following dictionary which translates classical notions into the language of operators in the Hilbert space $\mathcal{H}$:

Complex variable	Operator in $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order α	Compact operator with characteristic values μ_n satisfying $\mu_n = O(n^{-\alpha})$, $n \to \infty$
Integral of an infinitesimal of order 1	$⨍ T =$ Coefficient of logarithmic divergence in the trace of T.

We find that a variable (variable in the intuitive sense) should be thought of as an operator in Hilbert space. Thus the set of values of the variable is the spectrum of the operator, and the number of times a value is reached is the spectral multiplicity and so on and so forth. A real variable should be thought of as a self-adjoint operator. Thus for instance a self-adjoint operator has only real spectrum and we can act on it by any measurable function. Now you can act on complex variables only by holomorphic functions, of course, and this is similar, this is exactly what happens for operators. Well, we have known that for years and these two lines are completely standard. Now there is a third line which I want to look at, which is that there is in this dictionary a perfect place for what is an infinitesimal, namely for something which is smaller than ϵ for any ϵ, without being zero. Of course if you require that the operator norm is smaller than ϵ for any ϵ, you'll get nowhere. But one can be more subtle and ask that for any ϵ positive you can condition the operator by a finite number of linear conditions, so that you drop its norm to less than ϵ. This is a well known characterization of the compact operators in Hilbert space and they are the obvious candidates for infinitesimals. The basic rules of infinitesimals are easy to check, for instance if you add two compact operators, it is still compact, if you multiply by something bounded it is still compact, and so on, it's an ideal. Now, there is also an obvious notion of order for our infinitesimals. The size of the infinitesimal $T \in \mathcal{K}$ is governed by the order of decay of the sequence of characteristic values $\mu_n = \mu_n(T)$ as $n \to \infty$. In particular, for all real positive α the following condition defines infinitesimals of order α:

$$\mu_n(T) = O(n^{-\alpha}) \qquad \text{when } n \to \infty \tag{52}$$

(*i.e.* there exists $C > 0$ such that $\mu_n(T) \leq Cn^{-\alpha} \quad \forall\, n \geq 1$). Infinitesimals of order α also form a two-sided ideal and moreover,

$$T_j \text{ of order } \alpha_j \Rightarrow T_1 T_2 \text{ of order } \alpha_1 + \alpha_2 . \tag{53}$$

Now let me stop at this point and say one thing. Since the size of an infinitesimal is measured by the sequence $\mu_n \to 0$ it might seem that one does not need the operator formalism at all, and that it would be enough to replace the ideal $\mathcal{K}$ in $\mathcal{L}(\mathcal{H})$ by the ideal

$c_0(\mathbb{N})$ of sequences converging to zero in the algebra $\ell^\infty(\mathbb{N})$ of bounded sequences. A variable would just be a bounded sequence, and an infinitesimal a sequence μ_n, $\mu_n \to 0$. However, this commutative version does not allow for the existence of variables with range a continuum since all elements of $\ell^\infty(\mathbb{N})$ have a point spectrum and a discrete spectral measure. Thus, this is obviously wrong, because then you exclude variables which have continuous spectrum, you exclude variable which have Lebesgue spectrum. And it turns out that it is only because of non commutativity that you can have coexistence of infinitesimals with variables which have continuous spectrum. So this is exactly the place where this non commutativity comes as a great help. As we shall see shortly, it is precisely this lack of commutativity between the line element and the coordinates on a space that will provide the measurement of distances.

Now, the differential I won't talk about, but another key new ingredient in this dictionary is the integral $⨍$ which is slightly tricky to define. It is obtained by the following analysis, mainly due to Dixmier [30], of the logarithmic divergence of the partial traces

(54) $$\mathrm{Trace}_N(T) = \sum_0^{N-1} \mu_n(T) \ , \quad T \geq 0\,.$$

(In fact, it is useful to define $\mathrm{Trace}_\Lambda(T)$ for any positive real $\Lambda > 0$ by piecewise affine interpolation for noninteger Λ).

Define for all order 1 operators $T \geq 0$

(55) $$\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_e^\Lambda \frac{\mathrm{Trace}_\mu(T)}{\log \mu} \, \frac{d\mu}{\mu}$$

which is the Cesaro mean of the function $\frac{\mathrm{Trace}_\mu(T)}{\log \mu}$ over the scaling group $\mathbb{R}_+^*$.
For $T \geq 0$, an infinitesimal of order 1, one has

(56) $$\mathrm{Trace}_\Lambda(T) \leq C \log \Lambda$$

so that $\tau_\Lambda(T)$ is bounded. The essential property is the following *asymptotic additivity* of the coefficient $\tau_\Lambda(T)$ of the logarithmic divergence (56):

(57) $$|\tau_\Lambda(T_1 + T_2) - \tau_\Lambda(T_1) - \tau_\Lambda(T_2)| \leq 3C \, \frac{\log(\log \Lambda)}{\log \Lambda}$$

for $T_j \geq 0$.

An easy consequence of (57) is that any limit point τ of the nonlinear functionals τ_Λ for $\Lambda \to \infty$ defines a positive and linear trace on the two-sided ideal of infinitesimals of order 1.

In practice the choice of the limit point τ is irrelevant because in all important examples T is a *measurable* operator, *i.e.*:

(58) $$\tau_\Lambda(T) \text{ converges when } \Lambda \to \infty\,.$$

Thus the value $\tau(T)$ is independent of the choice of the limit point τ and is denoted

$$\int\!\!\!\!\!\!- T. \tag{59}$$

The first interesting example is provided by pseudodifferential operators T on a differentiable manifold M. When T is of order 1 in the above sense, it is measurable and $\int\!\!\!\!- T$ is the non-commutative residue of T [31]. It has a local expression in terms of the distribution kernel $k(x,y)$, $x, y \in M$. For T of order 1 the kernel $k(x,y)$ diverges logarithmically near the diagonal,

$$k(x,y) = -a(x) \log|x-y| + 0(1) \text{ (for } \; y \to x) \tag{60}$$

where $a(x)$ is a 1-density independent of the choice of Riemannian distance $|x-y|$. Then one has (up to normalization),

$$\int\!\!\!\!\!\!- T = \int_M a(x). \tag{61}$$

The right hand side of this formula makes sense for all pseudodifferential operators (cf. [31]) since one can easily see that the kernel of such an operator is asymptotically of the form

$$k(x,y) = \sum a_k(x, x-y) - a(x) \log|x-y| + 0(1) \tag{62}$$

where $a_k(x,\xi)$ is homogeneous of degree $-k$ in ξ, and the 1-density $a(x)$ is defined intrinsically since the logarithm does not mix with rational terms under a change of local coordinates.

The same principle of extension of $\int\!\!\!\!-$ to infinitesimals of order < 1 works for hypoelliptic operators and more generally for spectral triples whose dimension spectrum is simple, as we shall see below.

This framework gives a natural home for the analogue of the infinitesimal line element ds of Riemannian geometry, but we need first to exhibit its compatibility with the notion of manifold. So far we just described the framework to think about these problems. For instance when we go back to our initial naive question about the target and the darts, we find that quantum mechanics gives us an obvious infinitesimal which answers the question: it is the inverse of the Dirichlet Laplacian for the domain Ω. Thus there is now a clear meaning for the exponential of $\frac{-1}{dp}$, that's the well known heat kernel which is an infinitesimal of arbitrarily large order as we expected from the Taylor expansion.

From the H. Weyl theorem on the asymptotic behavior of eigenvalues of Δ it follows that dp is of order 1, and that given a function f on Ω the product $f\,dp$ is measurable, while

$$\int\!\!\!\!\!\!- f\,dp = \int_\Omega f(x_1, x_2)\, dx_1 \wedge dx_2 \tag{63}$$

gives the ordinary integral of f with respect to the measure given by the area of the target.

With this tool, with this understanding of the analogue of the infinitesimal calculus, we are now ready to talk about the line element ds but we still need first to answer the question: what is a manifold?

In ordinary geometry of course you can give a manifold by a cooking recipe, by charts and local diffeomorphisms, and one could be tempted to propose an analoguous cooking recipe in the noncommutative case. This is pretty much what is achieved by the general construction of the algebras of foliations and it is a good test of any general idea that it should at least cover that large class of examples.

But at a more conceptual level, it was recognized long ago by geometers that the main quality of the homotopy type of an oriented manifold, is to satisfy Poincaré duality not only in ordinary homology but also in K-homology. Poincaré duality in ordinary homology is not sufficient to describe homotopy type of manifolds [32] but D. Sullivan [33] showed (in the simply connected PL case of dimension ≥ 5 ignoring 2-torsion) that it is sufficient to replace ordinary homology by KO-homology.

The characteristic property of *differentiable manifolds* which is carried over to the noncommutative case is *Poincaré duality* in K-homology [53].

Moreover, as we saw above, K-homology admits a fairly simple definition in terms of Hilbert space and Fredholm representations of algebras.

In the general framework of Noncommutative Geometry the confluence of the Hilbert space incarnation of the two notions of metric and fundamental class for a manifold led very naturally to define a geometric space as given by a *spectral triple:*

$$(\mathcal{A}\,,\mathcal{H}\,,D) \tag{64}$$

where $\mathcal{A}$ is an involutive algebra of operators in a Hilbert space $\mathcal{H}$ and D is a selfadjoint operator on $\mathcal{H}$. The involutive algebra $\mathcal{A}$ corresponds to a given space M like in the classical duality «Space $\leftrightarrow$ Algebra» in algebraic geometry. The infinitesimal line element in Riemannian geometry is given by the equality

$$ds = 1/D\,, \tag{65}$$

which expresses the infinitesimal line element ds as the inverse of the Dirac operator D, hence under suitable boundary conditions as a propagator, so that the above equation can be suggestively symbolised as follows, using the notations of physicists for propagators in Feynman graphs,

$$ds = \times\!\!-\!\!\!-\!\!\!-\!\!\times \tag{66}$$

The significance of D is twofold. On the one hand it defines the metric by the above equation, on the other hand its homotopy class represents the K-homology fundamental class of the space under consideration. The exact measurement of distances is performed as follows, instead of measuring distances between points using the standard formula

$$d(x\,,y) = \mathrm{Inf}\{\text{Length } \gamma \,|\, \gamma \text{ is a path between } x \text{ and } y\} \tag{67}$$

where

$$\text{Length}\ \gamma = \int_\gamma ds\,, \tag{68}$$

we measure distances between states φ, ψ on $\bar{\mathcal{A}}$ by a dual formula. This dual formula involves *sup* instead of *inf* and does not use paths in the space

$$d(\varphi, \psi) = \text{Sup}\,\{|\varphi(a) - \psi(a)|\ ;\ a \in \mathcal{A}\ ,\ \|[D, a]\| \leq 1\}. \tag{69}$$

A state, is a normalized positive linear form on $\mathcal{A}$ such that $\varphi(1) = 1$,

$$\varphi : \bar{\mathcal{A}} \to \mathbb{C}\ ,\ \varphi(a^* a) \geq 0\ ,\quad \forall\, a \in \bar{\mathcal{A}}\ ,\ \varphi(1) = 1\,. \tag{70}$$

In the commutative case the points of the space coincide with the characters of the algebra or equivalently with its pure states (*i.e.* the extreme points of the convex compact set of states). As it should, this formula gives the geodesic distance in the Riemannian case. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ associated to a compact Riemannian manifold M, K-oriented by a spin structure, is given by the representation

$$(f\,\xi)(x) = f(x)\,\xi(x) \qquad \forall\, x \in M\ ,\ f \in \mathcal{A}\ ,\ \xi \in \mathcal{H} \tag{71}$$

of the algebra $\mathcal{A}$ of functions on M in the Hilbert space

$$\mathcal{H} = L^2(M, S) \tag{72}$$

of square integrable sections of the spinor bundle. The operator D is the Dirac operator (cf. [34]). The commutator $[D, f]$, for $f \in \mathcal{A} = C^\infty(M)$ is the Clifford multiplication by the gradient ∇f and its operator norm is:

$$\|[D, f]\| = \ \text{Sup}_{x\in M} \|\nabla f(x)\| = \text{Lipschitz norm}\ f\,. \tag{73}$$

Let $x, y \in M$ and φ, ψ be the corresponding characters: $\varphi(f) = f(x)$, $\psi(f) = f(y)$ for all $f \in \mathcal{A}$. Then formula (69) gives the same result as formula (67), *i.e.* it gives the geodesic distance between x and y.

Unlike the formula (67) the dual formula (69) makes sense in general, namely, for example for discrete spaces and even for totally disconnected spaces.

The second role of the operator D is to define the fundamental class of the space X in K-homology, according to the following table,

Space X	Algebra $\mathcal{A}$
$K_1(X)$	Stable homotopy class of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$
$K_0(X)$	Stable homotopy class of $\mathbb{Z}/2$ graded spectral triple

(*i.e.* for K_0 we suppose that $\mathcal{H}$ is $\mathbb{Z}/2$-graded by γ, where $\gamma = \gamma^*$, $\gamma^2 = 1$ and $\gamma a = a\gamma \quad \forall\ a \in \mathcal{A}$, $\gamma D = -D\gamma$).

We can make a few test of this general framework for noncommutative geometry. Thus we can check for instance that we easily recover the volume form of the

Riemannian metric by the equality

$$\text{(74)} \qquad ⨍ f\, |ds|^n = \int_{M_n} f \sqrt{g}\; d^n x$$

but the first interesting point is that besides this coherence with the usual computations there are new simple questions we can ask now such as «what is the two-dimensional measure of a four manifold» in other words «what is its area?». Thus one should compute

$$\text{(75)} \qquad ⨍ ds^2 .$$

It is obvious from invariant theory that this should be proportional to the Hilbert-Einstein action but doing the direct computation is a worthwile exercise (cf. [51, 52]), the exact result being

$$\text{(76)} \qquad ⨍ ds^2 = \frac{-1}{96\pi^2} \int_{M_4} r \sqrt{g}\; d^4 x$$

where as above $dv = \sqrt{g}\; d^4x$ is the volume form, $ds = D^{-1}$ the length element, *i.e.* the inverse of the Dirac operator and r is the scalar curvature.

There is an equally simple formula for the Yang-Mills action in general. The analogue of the Yang-Mills action functional and the classification of Yang-Mills connections on the noncommutative tori were developped in [60], with the primary goal of finding a «manifold shadow» for these noncommutative spaces. These moduli spaces turned out indeed to fit this purpose perfectly, allowing for instance to find the usual Riemannian space of gauge equivalence classes of Yang-Mills connections as an invariant of the noncommutative metric. We refer to [38] for the construction of the metrics on noncommutative tori from the conceptual point of view and to [50] for the check that all natural axioms of NCG are fulfilled in that case.

Now this is rather simple still. The power of the general theory comes from deeper general theorems such as the local computation of the analogue of Pontrjagin classes: *i.e.* of the components of the cyclic cocycle which is the Chern character of the K-homology class of D and which make sense in general. This result allows, using the infinitesimal calculus, to go from local to global in the general framework of spectral triples $(\mathcal{A}, \mathcal{H}, D)$.

The Fredholm index of the operator D determines (we only look at the odd case for simplicity but there are similar formulas in the even case) an additive map $K_1(\mathcal{A}) \xrightarrow{\varphi} \mathbb{Z}$ given by the equality

$$\text{(77)} \qquad \varphi([u]) = \mathrm{Index}\,(PuP) \quad , \quad u \in GL_1(\mathcal{A})$$

where P is the projector $P = \frac{1+F}{2}$, $F = \mathrm{Sign}\,(D)$.

It is an easy fact that this map is computed by the pairing of $K_1(\mathcal{A})$ with the following cyclic cocycle

$$\text{(78)} \qquad \tau(a^0, \dots , a^n) = \mathrm{Trace}\,(a^0[F, a^1] \dots [F, a^n]) \qquad \forall\, a^j \in \mathcal{A}$$

where $F = \text{Sign } D$ and we assume that the dimension p of our space is finite, which means that $(D+i)^{-1}$ is of order $1/p$, also $n \geq p$ is an odd integer. There are similar formulas involving the grading γ in the even case, and it is quite satisfactory [35, 36] that both cyclic cohomology and the Chern character formula adapt to the infinite dimensional case in which the only hypothesis is that $\exp(-D^2)$ is a trace class operator.

The cocycle τ is however nonlocal in general because the formula (78) involves the ordinary trace instead of the local trace $⨍$ and it is crucial to obtain a local form of the above cocycle.

This problem is solved by a general theorem [37] which we now describe. We make the following regularity hypothesis on $(\mathcal{A}, \mathcal{H}, D)$

$$a \text{ and } [D, a] \in \cap \operatorname{Dom} \delta^k, \quad \forall a \in \mathcal{A} \tag{79}$$

where δ is the derivation $\delta(T) = [|D|, T]$ for any operator T.

We let $\mathcal{B}$ denote the algebra generated by $\delta^k(a)$, $\delta^k([D, a])$. The usual notion of *dimension* of a space is replaced by the *dimension spectrum* which is a subset of $\mathbb{C}$. The precise definition of the dimension spectrum is the subset $\Sigma \subset \mathbb{C}$ of singularities of the analytic functions

$$\zeta_b(z) = \operatorname{Trace}(b|D|^{-z}) \qquad \operatorname{Re} z > p \ , \ \ b \in \mathcal{B}. \tag{80}$$

The dimension spectrum of an ordinary manifold M is the set $\{0, 1, \ldots, n\}$, $n = \dim M$; it is simple. Multiplicities appear for singular manifolds. Cantor sets provide examples of complex points $z \notin \mathbb{R}$ in the dimension spectrum.

We assume that Σ is discrete and simple, *i.e.* that ζ_b can be extended to $\mathbb{C}/\Sigma$ with simple poles in Σ.

We refer to [37] for the case of a spectrum with multiplicities. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple satisfying the hypothesis (79) and (80). The local index theorem is the following [37]:

1. The equality
$$⨍ P = \operatorname{Res}_{z=0} \operatorname{Trace}(P|D|^{-z})$$
defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, where $z \in \mathbb{C}$.
2. There is only a finite number of non-zero terms in the following formula which defines the odd components $(\varphi_n)_{n=1,3,\ldots}$ of a cocycle in the bicomplex (b, B) of $\mathcal{A}$,
$$\varphi_n(a^0, \ldots, a^n) = \sum_k c_{n,k} ⨍ a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)} |D|^{-n-2|k|} \qquad \forall\, a^j \in \mathcal{A}$$
where the following notations are used: $T^{(k)} = \nabla^k(T)$ and $\nabla(T) = D^2 T - TD^2$, k is a multi-index, $|k| = k_1 + \ldots + k_n$,
$$c_{n,k} = (-1)^{|k|} \sqrt{2i}(k_1! \ldots k_n!)^{-1}\, ((k_1+1) \ldots (k_1 + k_2 + \ldots + k_n + n))^{-1}\, \Gamma\Big(|k| + \frac{n}{2}\Big).$$

3. The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_1(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_1(\mathcal{A})$.

In general of course the explicit computation of this cocycle is extremely difficult to perform: but what is however very important is that all the terms of this formula are local, because the integral $⨍$ is coming from the logarithmic divergency, and whenever you compute, it always gives local contributions. The first test is the computation of these Pontrjagin classes in the case of foliations which as explained in [37] do fit with our general framework. It turned out that the computation was quite complicated even in the case of codimension 1 foliations: there were innumerable terms to be computed; this could be done by hand, like 3 weeks of hard work, but of course it was hopeless to try to proceed by brute force for the general case. Now the answer was found for the general case [39], but quite surprisingly it generated a Hopf algebra which only depends on the codimension of the foliation and which organizes the computation. It also dictated the correct generalization of cyclic cohomology for Hopf algebras [41]. Now one of the most interesting recent progress is that this Hopf algebra turns out to be tightly related with another Hopf algebra, discovered by Dirk Kreimer, which organizes the computation of renormalization in quantum field theory.

Dirk Kreimer showed [42-45] that for any quantum field theory, the combinatorics of Feynman graphs is governed by a Hopf algebra $\mathcal{H}$ whose antipode involves the same algebraic operations as in the Bogoliubov-Parasiuk-Hepp recursion and the Zimmermann forest formula.

His Hopf algebra is commutative as an algebra and we showed in [46] that it is the dual Hopf algebra of the envelopping algebra of a Lie algebra $\underline{G}$ whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of $\mathcal{H}$.

We also showed that, using dimensional regularization, the bare (unrenormalized) theory gives rise to a loop

$$\gamma(z) \in G \ , \qquad z \in C \tag{81}$$

where C is a small circle of complex dimensions around the integer dimension D of space-time. Our main result [47, 48] which relies on all the previous work of Dirk is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of γ.

The Birkhoff decomposition is the factorization

$$\gamma(z) = \gamma_-(z)^{-1}\, \gamma_+(z) \qquad z \in C \tag{82}$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and $\gamma_\pm$ are loops with values in G,

$$\gamma(z) \in G \qquad \forall\, z \in \mathbb{C}$$

and $\gamma_\pm$ are boundary values of holomorphic maps (still denoted by the same symbol)

$$\gamma_\pm : C_\pm \to G\,. \tag{83}$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (82) is unique (under suitable regularity conditions).

When G is a simply connected nilpotent complex Lie group the existence (and uniqueness) of the Birkhoff decomposition (82) is valid for any γ. When the loop $\gamma : C \to G$ extends to a holomorphic loop: $C_+ \to G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z \in C_+$ the evaluation,

$$\gamma \to \gamma_+(z) \in G \tag{84}$$

is a natural principle to extract a finite value from the singular expression $\gamma(z)$. This extraction of finite values coincides with the removal of the pole part when G is the additive group $\mathbb{C}$ of complex numbers and the loop γ is meromorphic inside C_+ with z as its only singularity.

As I mentioned earlier our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of the loop given by the unrenormalized theory γ.

We showed that the group G is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms. In fact the relation that we uncovered in [49] between the Hopf algebra of Feynman graphs and the Hopf algebra of coordinates on the group of formal diffeomorphisms of the dimensionless coupling constants of the theory allows to formulate the following corollary which for simplicity deals with the case of a single dimensionless coupling constant.

Let the unrenormalized effective coupling constant $g_{\rm eff}(\varepsilon)$ be viewed as a formal power series in g and let $g_{\rm eff}(\varepsilon) = g_{{\rm eff}_+}(\varepsilon)\,(g_{{\rm eff}_-}(\varepsilon))^{-1}$ be its (opposite) Birkhoff decomposition in the group of formal diffeomorphisms. Then the loop $g_{{\rm eff}_-}(\varepsilon)$ is the bare coupling constant and $g_{{\rm eff}_+}(0)$ is the renormalized effective coupling.

Finally there is yet another very important test of our general framework of geometry which is its compatibility with what we know of space-time. What we have done so far is to stretch the usual framework of ordinary geometry beyond its commutative restrictions (set theoretic restrictions) and of course now it's not perhaps a bad idea to test it with what we know about physics and to try to find a better model of space-time within this new framework. The best way is to start with the hard core information one has from physics and that can be summarized by a Lagrangian. This Lagrangian is the Einstein Lagrangian plus the standard model Lagrangian. I am not going to write it down, it's a very complicated expression since just the standard model Lagrangian comprises five types of terms. But one can start understanding something by looking at the symmetry group of this Lagrangian. Now, if it were just the Einstein theory, the symmetry group of the Lagrangian would just be, by the equivalence principle, the diffeomorphism group of the space-time manifold. But because of the standard model piece the symmetry group of this Lagrangian is not just the diffeomorphism group, because the gauge theory has another huge symmetry group which is the group of maps from the manifold to the small gauge group, namely $U_1 \times SU_2 \times SU_3$ as far as we know. Thus, the symmetry group G of the full Lagrangian is neither the diffeomorphism group nor the group of gauge transformations of second kind nor their product, but it is their semi-direct product. It is exactly like what happens with the Poincaré group where you have translations and Lorentz transformations, so it is the semi-direct product of these two subgroups. Now we can ask a very simple question: would there be some space X so that this group G would be equal to Diff(X)? If such a space would exist, then we would have some chance to actually geometrize completely the theory, namely to be able to say that it's pure gravity on the space X. Now, if you look for the space X among ordinary manifolds, you have no chance since by a result of John Mather the diffeomorphism group of a (connected) manifold is a simple group. A simple group cannot have a nontrivial normal subgroup, so you cannot have this structure of semi-direct product.

However, we can use our dictionary, and in this dictionary if we browse through it, we find that what corresponds to diffeomorphisms for a non commutative space is just the group $\mathrm{Aut}^+(\mathcal{A})$ of automorphisms of the algebra of coordinates $\mathcal{A}$, which preserve the fundamental class in K-homology, *i.e.* the automorphisms α of the involutive algebra $\mathcal{A}$, which are implemented by a unitary operator U in $\mathcal{H}$ commuting with the real structure J [53]

$$\alpha(x) = U\,x\,U^{-1} \qquad \forall\, x \in \mathcal{A}\ .$$

Now there is a beautiful fact which is that when an algebra is not commutative, then among its automorphisms there are very trivial ones, there are automorphisms which are there for free, I mean the inner ones, which associate to an element x of the algebra the element uxu^{-1}. Of course uxu^{-1} is not, in general equal to x because the algebra is not commutative, and these automorphisms form a normal subgroup of the group of automorphisms. Thus you see that the group $\mathrm{Aut}^+(\mathcal{A})$ has the same type of structure, namely it has a normal subgroup of internal automorphisms and it has a quotient. Now it turns out that there is one very natural non commutative algebra $\mathcal{A}$ whose group of internal automorphisms corresponds to the group of gauge transformations and the quotient $\mathrm{Aut}^+(\mathcal{A})/\,\mathrm{Int}(\mathcal{A})$ corresponds exactly to diffeomorphisms. It is amusing that the physics vocabulary is actually the same as the mathematical vocabulary. Namely in physics you talk about internal symmetries and in mathematics you talk about inner automorphisms, you could call them internal automorphisms. Now the corresponding space is a product $M \times F$ of an ordinary manifold M by a finite noncommutative space F. The corresponding algebra $\mathcal{A}_F$ is the direct sum of the algebras $\mathbb{C}$, $\mathbb{H}$ (the quaternions), and $M_3(\mathbb{C})$ of 3×3 complex matrices.

The algebra $\mathcal{A}_F$ corresponds to a *finite* space where the standard model fermions and the Yukawa parameters (masses of fermions and mixing matrix of Kobayashi Maskawa) determine the spectral geometry in the following manner. The Hilbert space is finite-dimensional and admits the set of elementary fermions as a basis. For example for the first generation of quarks, this set is

$$u_L\,,\,u_R\,,\,d_L\,,\,d_R\,,\,\bar{u}_L\,,\,\bar{u}_R\,,\,\bar{d}_L\,,\,\bar{d}_R\,. \tag{85}$$

The algebra $\mathcal{A}_F$ admits a natural representation in $\mathcal{H}_F$ (see [53]) and the Yukawa coupling matrix Y determines the operator D.

The detailed structure of Y (and in particular the fact that color is not broken) allows to check the axioms of noncommutative geometry.

The next step consists in the computation of internal deformations of the product geometry $M \times F$ where M is a 4-dimensional Riemannian spin manifold. The computation gives the standard model gauge bosons $\gamma\,,\,W^{\pm}\,,Z$, the eight gluons and the Higgs fields φ with accurate quantum numbers.

Let us explain how internal deformations of the geometry arise in the general noncommutative case. Parallel to the normal subgroup $\mathrm{Int}\,\mathcal{A} \subset \mathrm{Aut}^{+}\,\mathcal{A}$ of inner automorphisms of $\mathcal{A}$,

$$\alpha(f) = ufu^* \qquad \forall\, f \in \mathcal{A} \tag{86}$$

where u is a unitary element of $\mathcal{A}$ (*i.e.* $uu^* = u^*u = 1$), there exists a natural foliation of the space of spectral geometries on $\mathcal{A}$ by equivalence classes of *inner deformations* of a given geometry. To understand how they arise we need to understand how to transfer a given spectral geometry to a Morita equivalent algebra. Given a spectral triple $(\mathcal{A}\,,\mathcal{H}\,,\,D)$ and the Morita equivalence [54] between $\mathcal{A}$ and an algebra $\mathcal{B}$ where

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \tag{87}$$

where $\mathcal{E}$ is a finite, projective, hermitian right $\mathcal{A}$-module, one gets a spectral triple on $\mathcal{B}$ by the choice of a *hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the rules [38]

$$\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da \qquad \forall\, \xi \in \mathcal{E}\ \ ,\ \ a \in \mathcal{A} \tag{88}$$

$$(\xi\,,\nabla\eta) - (\nabla\xi\,,\eta) = d(\xi\,,\eta) \qquad \forall\, \xi\,,\eta \in \mathcal{E} \tag{89}$$

where $da = [D,\,a]$ and where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$-bimodule of operators of the form

$$A = \Sigma\, a_i[D,\,b_i]\ \ ,\ \ a_i\,,\,b_i \in \mathcal{A}\,. \tag{90}$$

Any algebra $\mathcal{A}$ is Morita equivalent to itself (with $\mathcal{E} = \mathcal{A}$) and when one applies the above construction in the above context one gets the inner deformations of the spectral geometry.

Such a deformation is obtained by the following formula (with suitable signs depending on the dimension mod 8) without modifying neither the representation of $\mathcal{A}$ in $\mathcal{H}$ nor the anti-linear isometry J

$$D \to D + A + JAJ^{-1} \tag{91}$$

where $A = A^*$ is an arbitrary selfadjoint operator of the form (90). The action of the group $\mathrm{Int}(\mathcal{A})$ on the spectral geometries is simply the following gauge transformation of A

(92) $$\gamma_u(A) = u[D, u^*] + uAu^* .$$

The required unitary equivalence is implemented by the following representation of the unitary group of $\mathcal{A}$ in $\mathcal{H}$,

(93) $$u \to uJuJ^{-1} .$$

The transformation (91) is the identity in the usual Riemannian case. To get a nontrivial example it suffices to consider the product of a Riemannian triple by the unique spectral geometry on the finite-dimensional algebra $\mathcal{A}_F = M_N(\mathbb{C})$ of $N \times N$ matrices on $\mathbb{C}$, $N \geq 2$. One then has $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathrm{Int}(\mathcal{A}) = C^\infty(M, PSU(N))$ and inner deformations of the geometry are parameterized by the gauge potentials for the gauge theory of the group $SU(N)$. The space of pure states of the algebra $\mathcal{A}$, $P(\mathcal{A})$, is the product $P = M \times P_{N-1}(\mathbb{C})$ and the metric on $P(\mathcal{A})$ determined by the formula (69) depends on the gauge potential A. It coincide with the Carnot metric [55] on P defined by the horizontal distribution given by the connection associated to A. The group $\mathrm{Aut}^+(\mathcal{A})$ is the following semi-direct product

(94) $$\mathrm{Aut}^+(\mathcal{A}) = \mathcal{U} \rtimes \mathrm{Diff}^+(M)$$

of the local gauge transformation group $\mathrm{Int}(\mathcal{A})$ by the group of diffeomorphisms.

Now coming back to space-time the question that comes about is how do you recover the original action functional which contained both the Einstein-Hilbert term as well as the standard model? The answer is very simple: the Fermionic part of this action is there from the start and one recovers the bosonic part as follows. Both the Hilbert-Einstein action functional for the Riemannian metric, the Yang-Mills action for the vector potentials, the self interaction and the minimal coupling for the Higgs fields all appear with the correct signs in the asymptotic expansion for large Λ of the number $N(\Lambda)$ of eigenvalues of D which are $\leq \Lambda$ (cf. [56]),

(95) $$N(\Lambda) = \#\ \text{eigenvalues of } D \text{ in } [-\Lambda, \Lambda].$$

This step function $N(\Lambda)$ is the superposition of two terms,

$$N(\Lambda) = \langle N(\Lambda)\rangle + N_{\mathrm{osc}}(\Lambda).$$

The oscillatory part $N_{\mathrm{osc}}(\Lambda)$ is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system and does not concern us here. The average part $\langle N(\Lambda)\rangle$ is computed by a semiclassical approximation from local expressions involving the familiar heat equation expansion and delivers the correct terms. Other nonzero terms in the asymptotic expansion are cosmological, Weyl gravity and topological terms. We showed in [62] that if one studies natural presentations of the algebra generated by $\mathcal{A}$ and D one naturally gets only metrics with a fixed volume form so that the bothering cosmological term does not enter in the variational equations associated to the spectral action $\langle N(\Lambda)\rangle$. It is tempting to speculate that the phenomenological Lagrangian of physics, combining matter and gravity appears from the solution of an extremely simple operator theoretic equation along the lines described in [62]. As a starting point for such investigations see [57].

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