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Analisi matematica. — *A counterexample to Schauder estimates for elliptic operators with unbounded coefficients.* Nota (*) di Enrico Priola, presentata dal Socio G. Da Prato.

Abstract. — We consider a homogeneous elliptic Dirichlet problem involving an Ornstein-Uhlenbeck operator in a half space $\mathbb{R}^2_+$ of $\mathbb{R}^2$. We show that for a particular initial datum, which is Lipschitz continuous and bounded on $\mathbb{R}^2_+$, the second derivative of the classical solution is not uniformly continuous on $\mathbb{R}^2_+$. In particular this implies that the well known maximal Hölder-regularity results fail in general for Dirichlet problems in unbounded domains involving unbounded coefficients.

Key words: Optimal Hölder-regularity results; Dirichlet problems; The Ornstein-Uhlenbeck operator.

Riassunto. — *Un controesempio alle stime di Schauder per operatori ellittici con coefficienti illimitati.* Si considera un problema ellittico di Dirichlet in un semispazio $\mathbb{R}^2_+$ di $\mathbb{R}^2$. In esso compare un operatore di tipo Ornstein-Uhlenbeck. Si dimostra, con calcoli espliciti, che per un particolare dato iniziale lipschitziano la corrispondente soluzione classica non ha la derivata seconda uniformemente continua su $\mathbb{R}^2_+$. Questo risultato implica in particolare che le ben note stime di Schauder non valgono in generale per problemi di Dirichlet su domini illimitati se i coefficienti sono illimitati.

1. Introduction and preliminaries

The global Schauder estimates for the Laplacian in spaces of Hölder continuous and bounded functions is a well known topic in PDE's. These estimates have been fruitfully extended to linear and nonlinear elliptic equations, and elliptic boundary value problems in sufficiently smooth domains, assuming that the coefficients are bounded (see for instance [10, 11]). After the classical works [1, 2], only recently optimal Hölder-regularity results have been obtained for a large class of second order elliptic and parabolic equations, on the whole of $\mathbb{R}^n$, involving unbounded coefficients (see for instance [6-8, 15, 16]). Some of these results have been generalized to the infinite dimensional case, as in [5, 17]. These papers are motivated by applications to stochastic differential equations (cf. [9, 12, 20, 21]) and to financial mathematics (cf. [4]). However very little is known about global regularity results for elliptic problems with unbounded coefficients in unbounded domains different from $\mathbb{R}^n$; here we show that the maximal regularity results fail in general for such problems. Let us consider the following equation on $\mathbb{R}^n$

$$(1)\qquad \lambda\psi(z) - \mathcal{U}\psi(z) = \lambda\psi(z) - \frac{1}{2}\triangle\psi(z) - \sum_{i,j=1}^{n} B_{ij} z_i D_j \psi(z) = f(z), \quad z \in \mathbb{R}^n,$$

where $\lambda > 0$, $B = (B_{ij})$ is a nonzero matrix on $\mathbb{R}^n$. The operator $\mathcal{U}$ is the prototype of differential operators with unbounded coefficients; it is called the Ornstein-Uhlenbeck operator. In [8] Schauder estimates for equation (1) have been established. We recall

(*) Pervenuta in forma definitiva all'Accademia il 7 novembre 2000.

the theorem.

Let $f \in C_b^\theta(\mathbb{R}^n)$, i.e. f *is* θ*-Hölder continuous and bounded,* $\theta \in (0, 1)$. *Then equation* (1) *has a unique classical solution* $\psi \in C_b^{2+\theta}(\mathbb{R}^n)$ (i.e. ψ *has all the first and second partial derivatives which are* θ*-Hölder continuous and bounded on* $\mathbb{R}^n$) *and further* $\|\psi\|_{2+\theta} \le C\|f\|_\theta$, *where* $C = C(\theta, \lambda, B, n) > 0$.

The aim of this paper is to provide an example, which shows that the previous optimal regularity result fails in general for Dirichlet problems involving the operator $\mathcal{U}$ in unbounded domains. To this end we consider the canonical 2-dimensional open half space $\mathbb{R}^2_+$ and study the following Dirichlet problem

$$(2)\qquad \begin{cases} \psi(x, y) - \frac{1}{2}D_{xx}\psi(x, y) - \frac{1}{2}D_{yy}\psi(x, y) + yD_y\psi(x, y) = |\cos(y)|, & x > 0, \\ \psi(0, y) = 0, \qquad y \in \mathbb{R} \end{cases}$$

(note that the map: $(x, y) \mapsto |\cos(y)|$ is Lipschitz continuous and bounded on $\mathbb{R}^2_+$ and independent of the first variable). We prove the following statement.

THEOREM 1. *There exists a unique classical solution* ψ *to problem* (2) (cf. Lemma 2). *Moreover the second partial derivative* $D_{xx}\psi$ *is not uniformly continuous on* $\mathbb{R}^2_+$.

In particular this implies that the Schauder estimates do not hold for problem (2) (remark that Theorem 1 gives even more since $\cup_{\theta\in(0,1)}C_b^\theta(\mathbb{R}^2_+) \subset UC_b(\mathbb{R}^2_+)$ with a *strict* inclusion). To prove our theorem, we find an explicit formula for the solution ψ of (2) and perform direct computations on $D_{xx}\psi$. We point out that in [18] we provide additional conditions, under which the desidered optimal regularity results for (2) can be proved. We finish Section 2, by giving an interpretation of Theorem 1 from the point of view of semigroups theory, see Remark 4.

In the last section we show that the Schauder estimates for problem (2) fail to hold even in spaces of Hölder continuous functions having polynomial growth, see Theorem 5. To this end we study (2), replacing the initial datum $|\cos(y)|$ with $y^2|\cos(y)|$.

We fix notations and give some preliminaries. Let Ω be any open subset of $\mathbb{R}^n$, $n \in \mathbb{Z}_+$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We denote by $\mathcal{B}_b(\Omega)$ the Banach space of all real, Borel and bounded functions on Ω, endowed with the sup norm: $\|f\|_0 = \sup_{x\in\Omega}|f(x)|$, $f \in \mathcal{B}_b(\Omega)$. Moreover the space $UC_b(\Omega)$ stands for the Banach space of all real, uniformly continuous and bounded functions, endowed with the sup norm. Note that the uniform continuity of a map $f \in UC_b(\Omega)$ allows to consider values of f on $\partial\Omega$ and implies that $UC_b(\Omega) = UC_b(\overline{\Omega})$. The space $UC_b^k(\Omega)$, $k \in \mathbb{Z}_+$, is the set of all k-times differentiable functions f, whose partial derivatives, $D_\alpha f$, $\alpha \in \mathbb{Z}^n_+$, are uniformly continuous and bounded on Ω up to the order k. It is a Banach space endowed with the norm $\|f\|_k = \|f\|_0 + \sum_{|\alpha|\le k} \|D_\alpha f\|_0$, $f \in UC_b^k(\Omega)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Finally we define the space $C_b^\theta(\Omega)$, $\theta \in (0, 1)$, as the set of all functions $f \in UC_b(\Omega)$ such that

$$[f]_\theta = \sup_{z, w\in\Omega,\ z\ne w} |z - w|^{-\theta}\, |f(z) - f(w)| < \infty.$$

It is a Banach space endowed with the norm $\|f\|_\theta = \|f\|_0 + [f]_\theta$, $f \in \mathcal{C}_b^\theta(\Omega)$.

Let $m > 0$, $x \in \mathbb{R}$. We denote by $N(x, m)$, the *Gaussian measure* on $\mathbb{R}$ with mean x and covariance m; it has density $\frac{1}{\sqrt{2\pi m}}\, e^{-\frac{|x-y|^2}{2m}}$, with respect to the Lebesgue measure dy. Let us define the 1-dimensional *Ornstein-Uhlenbeck semigroup* U_t,

$$U_t f(x) = \int_{\mathbb{R}} f(e^{-t}x + v)\, N(0, q_t)\, dv, \quad f \in \mathcal{B}_b(\mathbb{R}),\ x \in \mathbb{R},\ t > 0, \tag{3}$$

$U_0 = I_{\mathcal{B}_b(\mathbb{R})}$, where $q_t = \frac{1-e^{-2t}}{2}$. It is not difficult to show that $U_t \in \mathcal{L}(UC_b(\mathbb{R}))$, $U_{t+s} = U_t\, U_s$, $t, s \geq 0$ and $\|U_t\|_{\mathcal{L}(UC_b(\mathbb{R}))} \leq 1$, $t \geq 0$ (if $(X, \|\cdot\|_X)$ is a Banach space, $\mathcal{L}(X)$ stands for the Banach space of all bounded linear operators from X into X, endowed with the norm: $\|T\|_{\mathcal{L}(X)} = \sup_{\|x\|_X \leq 1} \|Tx\|_X$, $T \in \mathcal{L}(X)$). One can also verify that the map: $(t, x) \mapsto U_t g(x)$ satisfies:

$$D_t\, U_t g(x) = \frac{1}{2} D_{xx} U_t g(x) - x\, D_x U_t g(x), \quad g \in \mathcal{B}_b(\mathbb{R}),\ x \in \mathbb{R},\ t > 0. \tag{4}$$

Moreover, for any $M > 0$, there results:

$$\lim_{s \to 0} \sup_{|y| \leq M} |U_{t+s} f(y) - U_t f(y)| = 0, \quad t \geq 0,\ f \in UC_b(\mathbb{R}). \tag{5}$$

2. Proof of the main result

Throughout this section, for simplicity of notation, we will set

$$f_o(y) = |\cos(y)|, \quad y \in \mathbb{R}.$$

We need to establish a preliminary lemma concerning the existence and uniqueness of classical solutions ψ for (2). It also provides an explicit expression for $D_{xx}\psi$. This lemma could be deduced following [18]. However, for the sake of completeness, we will give a direct proof, avoiding semigroups and interpolation theory techniques, which are necessary to treat the general case considered in [18].

Lemma 2. *Let us consider the following map*:

$$\psi(x, y) = \int_0^\infty e^{-t}\, \eta(t, x)\, U_t f_o(y)\, dt, \quad \textit{where} \quad \eta(t, x) = 2\int_0^x \frac{e^{-\frac{u^2}{2t}}}{\sqrt{2\pi t}}\, du, \tag{6}$$

$t > 0$, $(x, y) \in \mathbb{R}^2_+$. *The following statements hold*:

(i) *the map* $\psi \in UC_b^1(\mathbb{R}^2_+)$ *and there exist all the second partial derivatives of* ψ *in the classical sense on* $\mathbb{R}^2_+$; *the maps* $D_{xy}\psi$, $D_{yy}\psi \in UC_b(\mathbb{R}^2_+)$ *and* $D_{xx}\psi$ *is continuous and bounded on* $\mathbb{R}^2_+$.

(ii) ψ *solves* (2) *and it is the unique classical solution of* (2).

Proof. (i) It is straightforward to obtain the following estimates:

$$\|D_x \eta(t, \cdot)\|_0 = \sup_{x>0} \left| \frac{2\, e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \right| \leq \frac{c}{\sqrt{t}}, \quad \|D_y\, U_t f_o\|_0 \leq \|f_o'\|_0 \leq 1 \tag{7}$$

$t \geq 0$, where the derivative $f_o'(x) = -\frac{\sin(x)\cos(x)}{|\cos(x)|}$, $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Using (7), we can differentiate under the integral sign in (6) and get that there exist $D_x\psi$ and $D_y\psi$ in the classical sense on $\mathbb{R}^2_+$; in addition these derivatives are bounded on $\mathbb{R}^2_+$. Let us remark that the global estimates (7) allow us to prove also the uniform continuity of $D_x\psi$ and $D_y\psi$. Now note that

$$D_{yy}U_t f_o(y) = \frac{e^{-2t}}{q_t}\int_{\mathbb{R}} f_o'(e^{-t}y + v)v\, N(0\,, q_t)dv.$$

It follows that $\|D_{yy}U_t f_o\|_0 \leq c\big(1 + \frac{1}{\sqrt{t}}\big)$, $t > 0$. Using this estimate and the previous ones we get easily that there exists $D_{xy}\psi$ and $D_{yy}\psi$ on the whole $\mathbb{R}^2_+$. Moreover we deduce that $D_{xy}\psi$ and $D_{yy}\psi \in UC_b(\mathbb{R}^2_+)$. It remains to treat $D_{xx}\psi$. We have

$$D_{xx}\eta(t\,, x) \;=\; -\frac{2x}{t\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\;,\;\; x \geq 0\,,\;\; t > 0. \tag{8}$$

Let us consider the global estimate: $\sup_{x>0}|D_{xx}\eta(t, x)| = \frac{c}{t}$, $t > 0$. This is not useful in order to obtain the existence of $D_{xx}\psi$ (the map $\frac{1}{t}$ is not integrable on $[0\,, 1]$). Therefore we need to use the estimate

$$\sup_{x\geq\delta}|D_{xx}\eta(t, x)| \;\leq\; \frac{4}{e\sqrt{2\pi}}\,\frac{1}{\sqrt{t}}\,\frac{1}{\delta} \;\leq\; C\max\left(1\,,\; \frac{1}{\sqrt{t}\,\delta}\right)\,,\quad t > 0\,,\;\; \delta > 0. \tag{9}$$

Thanks to (9), one obtains that there exists $D_{xx}\psi$ on the whole of $\mathbb{R}^2_+$ and further

$$D_{xx}\psi(x\,, y) \;=\; -\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-t}\,\frac{x}{t\sqrt{t}}\,e^{-\frac{x^2}{2t}}\,U_t f_o(y)dt,\;\;(x\,, y) \in \mathbb{R}^2_+. \tag{10}$$

By (9), it follows the continuity of $D_{xx}\psi$ on $\mathbb{R}^2_+$ as well. To establish the boundedness of $D_{xx}\psi$, we change variable in (10): $\frac{x}{\sqrt{t}} = u$, $x > 0$. We find

$$\begin{aligned} |D_{xx}\psi(x\,, y)| \;&=\; \left|-\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\;e^{-\frac{x^2}{u^2}}\;U_{\frac{x^2}{u^2}}f_o(y)du\right| \leq \\ &\leq\; \frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\,du = 2\,, \qquad (x\,, y) \in \mathbb{R}^2_+. \end{aligned} \tag{11}$$

(*ii*) First note that $D_t\eta(t\,, x) = \frac{1}{2}D_{xx}\eta(t\,, x)$, x, $t > 0$. Then, using the previous estimates, formula (4) and an integration by parts, we get:

$$\begin{aligned} &\left[\frac{1}{2}D_{xx} + \frac{1}{2}D_{yy} - yD_y\right]\;\psi(x\,, y) = \\ &\quad= \int_0^\infty e^{-t}\,\frac{1}{2}D_{xx}\eta(t\,, x)\;U_t f_o(y)dt + \int_0^\infty e^{-t}\,\eta(t\,, x)\left[\frac{1}{2}D_{yy} - yD_y\right]\,U_t\,f_o(y)dt = \\ &\quad= \int_0^\infty e^{-t}\,D_t\Big(\eta(t\,, x)\;U_t f_o\,(y)\Big)\;dt = \\ &\quad= -\lim_{t\to 0+} e^{-t}\eta(t\,, x)U_t f_o(y) \;+\; \int_0^\infty e^{-t}\,\eta(t\,, x)U_t f_o(y)dt \;=\; -f_o(y) + \psi(x\,, y)\,, \end{aligned}$$

$(x, y) \in \mathbb{R}^2_+$. The uniqueness of ψ follows by a standard maximum principle, see for instance [11]. $\square$

We point out that, proceeding as in Lemma 4.2 of [18], one can actually show that $D_{xy}\psi$, $D_{yy}\psi \in C^\theta_b(\mathbb{R}^2_+)$, for any $\theta \in (0, 1)$. In the proof of Theorem 1, we will use the following result.

Lemma 3. *The following statement holds*:

$$\limsup_{x\to 0+} \sup_{y\in\mathbb{R}} \left| \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}} \, |\cos(e^{-\frac{x^2}{u^2}} y)| \, du \;-\; |\cos(y)| \right| \;\geq\; \frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2\pi}}. \tag{12}$$

Proof. First note that, applying the Dominated Convergence Theorem, it is easy to check that

$$\lim_{x\to 0+} \sup_{|y|\leq M} \left| \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}} \, |\cos(e^{-\frac{x^2}{u^2}} y)| du \;-\; |\cos(y)| \right| \;=\; 0\,, \quad M > 0.$$

Then assertion (12) is equivalent to the next one:

$$\limsup_{x\to 0+} \sup_{y\in\mathbb{R}} \left| \int_0^\infty e^{-\frac{u^2}{2}} \left[|\cos(e^{-\frac{x^2}{u^2}} y)| - |\cos(y)| \right] du \right| \;\geq\; \frac{\sqrt{3}-\sqrt{2}}{6}. \tag{13}$$

Let $a \in (0, 1)$ to be chosen later, we write

$$\begin{aligned}
\sup_{y\in\mathbb{R}} &\left| \int_0^\infty e^{-\frac{u^2}{2}} \left[|\cos(e^{-\frac{x^2}{u^2}} y)| - |\cos(y)| \right] du \right| \geq \\
&\geq \sup_{k\in\mathbb{Z}} \int_0^\infty e^{-\frac{u^2}{2}} \left| \cos\left(e^{-\frac{x^2}{u^2}} [4k+1] \frac{\pi}{2} \right) \right| du \geq \\
&\geq \sup_{k\in\mathbb{Z}} \int_a^1 e^{-\frac{u^2}{2}} \left| \cos\left(e^{-\frac{x^2}{u^2}} [4k+1] \frac{\pi}{2} \right) \right| du \geq \\
&\geq e^{-1/2}(1-a) \sup_{k\in\mathbb{Z}} \inf_{u\in[a,1]} \left| \cos\left(e^{-\frac{x^2}{u^2}} [4k+1] \frac{\pi}{2} \right) \right|, \quad x > 0.
\end{aligned}$$

We are going to prove that if $1 > a \geq \frac{\sqrt{2}}{\sqrt{3}}$, then it holds:

$$\limsup_{x\to 0+} \sup_{k\in\mathbb{Z}} \inf_{u\in[a,1]} \left| \cos\left(e^{-\frac{x^2}{u^2}} [4k+1] \frac{\pi}{2} \right) \right| \geq \frac{1}{2}. \tag{14}$$

This will imply (13). To verify (14), we construct a sequence (x_n), which tends to 0 and satisfies:

$$1 - \frac{1}{2(4n+1)} < e^{\frac{-x_n^2}{a^2}} < e^{-x_n^2} < 1 - \frac{1}{3(4n+1)}, \quad n \in \mathbb{Z}_+. \tag{15}$$

Since the map: $v \mapsto e^{-\frac{r}{v^2}}$ is increasing on $(0, \infty)$, for $r > 0$, in order to obtain (15) it is enough that (x_n) verifies:

$$a^2 \log\left(1 - \frac{1}{2(4n+1)}\right) < -x_n^2 < \log\left(1 - \frac{1}{3(4n+1)}\right). \tag{16}$$

Let us define $k_n = a^2 \log\big(1 - \frac{1}{2\,(4n+1)}\big)$ and $h_n = \log\big(1 - \frac{1}{3\,(4n+1)}\big)$, $n \in \mathbb{Z}_+$. Because $a^2 \geq \frac{2}{3}$, it is straightforward to check that

$$a^2 \log\Big(1 - \frac{r}{2}\Big) \;-\; \log\Big(1 - \frac{r}{3}\Big) \;<\; 0\,, \quad r \in [0\,,2).$$

It follows that (16) is satisfied by taking

$$x_n \;=\; \sqrt{-(h_n \,+\, k_n)/2}\,, \quad n \in \mathbb{Z}_+.$$

Now note that, for any $n \in \mathbb{Z}_+$, one has:

$$J_n = \sup_{k\in\mathbb{Z}} \inf_{u\in[a\,,1]} \Big|\cos\Big(e^{-\frac{x_n^2}{u^2}}\,[4k+1]\,\frac{\pi}{2}\Big)\Big| \;\geq\; \inf_{u\in[a\,,1]} \Big|\cos\Big(e^{-\frac{x_n^2}{u^2}}\,[4n+1]\,\frac{\pi}{2}\Big)\Big| \geq$$

$$\geq \inf_{s\in\big[e^{-\frac{x_n^2}{a^2}}\,(4n+1)\,\frac{\pi}{2}\;,\;e^{-x_n^2}\,(4n+1)\,\frac{\pi}{2}\big]} |\cos(s)|.$$

Using (15), we deduce

$$J_n \geq \inf_{s\in\big[(1-\frac{1}{2\,(4n+1)})\,(4n+1)\frac{\pi}{2}\;,\;(1-\frac{1}{3\,(4n+1)})\,(4n+1)\frac{\pi}{2}\big]} |\cos(s)| \geq \inf_{s\in\big[\frac{\pi}{4}\;,\;\frac{\pi}{3}\big]} |\cos(s)| \;=\; \frac{1}{2}\,, \quad n \in \mathbb{Z}_+.$$

Thus (14) is proved. The proof is complete. □

Now we are in position to prove the main result.

Proof of Theorem 1. Let ψ be the classical solution of (2), see (6). Thanks to (11), we know that

$$D_{xx}\psi(x\,,y) \;=\; -\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\; e^{-\frac{x^2}{u^2}}\; U_{\frac{x^2}{u^2}}\, f_o(y)\,du. \tag{17}$$

Let us remark that, by (5) and the Dominated Convergence Theorem, we infer:

$$\lim_{x\to 0+}\;\sup_{|y|\leq M}\; |D_{xx}\psi(x\,,y) \,+\, 2f_o(y)| =\; 0\,, \quad M > 0. \tag{18}$$

Thanks to (18) we obtain that $D_{xx}\psi$ has a unique continuous extension to $\overline{\mathbb{R}^2_+}$, which is equal to $-2f_o$ on the boundary of $\mathbb{R}^2_+$. In order to prove that $D_{xx}\psi$ is not uniformly continuous on $\mathbb{R}^2_+$, it is enough to show that

$$\limsup_{x\to 0+}\;\sup_{y\in\mathbb{R}}\; |D_{xx}\psi(x\,,y) \,+\, 2f_o(y)| \;>\; 0. \tag{19}$$

To verify (19) we proceed into two steps.

Step I. We introduce a map $\phi : \overline{\mathbb{R}^2_+} \to \mathbb{R}$,

$$\phi(x\,,y) \;=\; -\frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{u^2}{2}}\; f_o(\,e^{-\frac{x^2}{u^2}}\,y)\,du\,, \quad x \geq 0\,,\; y \in \mathbb{R}\,, \tag{20}$$

and prove that $\lim_{x\to 0+}\;\sup_{y\in\mathbb{R}}\; |D_{xx}\psi(x\,,y) \,-\, \phi(x\,,y)| \;=\; 0.$

Setting $c' = -\frac{4}{\sqrt{2\pi}}$, we have

$$D_{xx}\psi(x,y) - \phi(x,y) = G_1(x,y) + G_2(x,y) \quad \text{where}$$

$$G_1(x,y) = c' \int_0^\infty e^{-\frac{u^2}{2}} \left[e^{-\frac{x^2}{u^2}} - 1 \right] U_{\frac{x^2}{u^2}} f_o(y)\, du,$$

$$G_2(x,y) = c' \int_0^\infty e^{-\frac{u^2}{2}} \left[U_{\frac{x^2}{u^2}} f_o(y) - f_o(e^{-\frac{x^2}{u^2}} y) \right] du, \quad (x,y) \in \mathbb{R}^2_+.$$

As for G_1, using that U_t is a semigroup of contractions on $UC_b(\mathbb{R}^2_+)$, we readly infer $\lim_{x\to 0+} \sup_{y\in\mathbb{R}} |G_1(x,y)| = 0$. To treat G_2 we remark that

$$|U_t f_o(y) - f_o(e^{-t}y)| \le \int_{\mathbb{R}} |f_o(e^{-t}y + w) - f_o(e^{-t}y)|\, N(0, q_t)dw \le$$

$$\le \int_{\mathbb{R}} |w| N(0, q_t) dw \le \sqrt{q_t} = \sqrt{\frac{1-e^{-2t}}{2}}, \qquad t \ge 0,\ y \in \mathbb{R}.$$

It follows that

$$\sup_{y\in\mathbb{R}} |G_2(x,y)| \le c'' \int_0^\infty e^{-\frac{u^2}{2}} \sqrt{1 - e^{-\frac{2x^2}{u^2}}}\, du \ \to\ 0, \ \text{ as } \ x \to 0^+.$$

This proves the assertion.

Step II. We show that $\limsup_{x\to 0+} \sup_{y\in\mathbb{R}} |\phi(x,y) + 2f_o(y)| > 0.$

This will imply (19), combining Step I and the inequality

$$\limsup_{x\to 0+} \|\phi(x,\cdot) + 2f_o\|_0 \le$$
$$\le \limsup_{x\to 0+} \|\phi(x,\cdot) - D_{xx}\psi(x,\cdot)\|_0 + \limsup_{x\to 0+} \|D_{xx}\psi(x,\cdot) + 2f_o\|_0. \tag{21}$$

Our claim is equivalent to the next one:

$$\limsup_{x\to 0+} \sup_{y\in\mathbb{R}} \left| \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{u^2}{2}} |\cos(e^{-\frac{x^2}{u^2}} y)| du - |\cos(y)| \right| > 0. \tag{22}$$

This is proved in Lemma 3. The proof is complete. □

Remark 4. Let us introduce the following semigroup T_t of linear contractions on $UC_b(\mathbb{R})$,

$$T_t f(y) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t}{s\sqrt{s}} e^{-\frac{t^2}{4s}} e^{-s} U_s f(y) ds, \ \ y \in \mathbb{R}, \ \ f \in UC_b(\mathbb{R}), \ \ t > 0,$$

$T_0 = I$, where U_t is the Ornstein-Uhlenbeck semigroup, see (3). In [18] (see in particular the proof of Proposition 4.5) we have proved that T_t *is an analytic semigroup* on $UC_b(\mathbb{R})$ (we refer to [13] for the theory of analytic semigroups). According to the classical Bochner theory on subordination of semigroups, compare with [3, 19], we have called T_t the subordinated semigroup associated to the semigroup $e^{-t}U_t$.

By Lemma 3, we deduce an interesting property of T_t. Let ψ be the solution of (2). Then we have: $D_{xx}\psi(x,y) = -2\,T_{(\sqrt{2}x)} f_o(y)$, $x > 0$, $y \in \mathbb{R}$, where $f_o(y) = |\cos(y)|$, $y \in \mathbb{R}$. Thanks to (19) we obtain that

$$\limsup_{t\to 0+} \|T_t f_o - f_o\|_0 > 0. \tag{23}$$

It follows that T_t *is not strongly continuous* on $UC_b(\mathbb{R})$. Hence if A denotes the generator of the analytic semigroup T_t, then its domain $D(A)$ is not dense in $UC_b(\mathbb{R})$.

3. An extension

In [14] Schauder estimates for problem (1) have been established in spaces of Hölder continuous functions having polynomial and exponential growth. Here we show that even this optimal regularity result can not be extended to the Dirichlet problem (2).

Let us fix the weight function $p(z) = 1 + |z|^2$, $z \in \mathbb{R}^n$, and introduce the Banach space $UC_p^k(\Omega)$, $k \in \mathbb{Z}_+$ (here Ω is any open subset of $\mathbb{R}^n$). It consists of all functions $f : \Omega \to \mathbb{R}$ such that the map: $z \mapsto f(z)/p(z)$ belongs to $UC_b^k(\Omega)$ (recall that $UC_b^0(\Omega) = UC_b(\Omega)$) and it is endowed with the norm: $\|f\|_{k,p} = \|f/p\|_k$, $f \in UC_p^k(\Omega)$, see the notations in Section 1. Moreover we define the space $C_p^\theta(\Omega)$, $\theta \in (0,1)$, as the set of all functions $f \in UC_p(\Omega)$ such that $f/p \in C_b^\theta(\Omega)$. It is a Banach space endowed with the norm: $\|f\|_{\theta,p} = \|f/p\|_\theta$, $f \in C_b^\theta(\Omega)$.

Let us consider the following problem

$$\begin{cases} \psi(x,y) - \frac{1}{2}D_{xx}\psi(x,y) - \frac{1}{2}D_{yy}\psi(x,y) + yD_y\psi(x,y) = g_o(y) = y^2|\cos(y)|, \\ \psi(0,y) = 0, \qquad y \in \mathbb{R}, \quad x > 0 \end{cases} \tag{24}$$

(note that the map: $(x,y) \mapsto \frac{y^2|\cos(y)|}{1+y^2+x^2}$ is Lipschitz continuous and bounded on $\mathbb{R}^2_+$). We can prove the following theorem.

Theorem 5. *There exists a unique classical solution ψ to problem* (24). *Moreover the second partial derivative $D_{xx}\psi$ does not belong to $UC_p(\mathbb{R}^2_+)$.*

Proof. The proof is similar to the one of Theorem 1 with some changes. We proceed in several steps.

Step 1. We consider the semigroup U_t, see (3), acting on $UC_p(\mathbb{R})$.

Using standard properties of gaussian measures, we extend the semigroup U_t to the space $UC_p(\mathbb{R})$;

$$U_t f(x) = \int_{\mathbb{R}} f(e^{-t}x + v)\, N(0, q_t)\, dv, \quad f \in UC_p(\mathbb{R}), \; x \in \mathbb{R}, \; t > 0, \tag{25}$$

$U_0 = I$, $q_t = \frac{1-e^{-2t}}{2}$. It is not difficult to show that U_t is a semigroup of bounded linear operators on $UC_p(\mathbb{R})$. Moreover it holds: $\|U_t\|_{\mathcal{L}(UC_p(\mathbb{R}))} \le 4$, $t \ge 0$. Indeed one has, for any $f \in UC_p(\mathbb{R})$, $t \geq 0$, $y \in \mathbb{R}$,

$$\frac{|U_t f(y)|}{1+y^2} \leq \int_{\mathbb{R}} \frac{|f(e^{-t}y + w)|}{1 + (e^{-t}y + w)^2} \, \frac{1 + (e^{-t}y + w)^2}{1+y^2} N(0, q_t) dw \leq$$

$$\leq \|f\|_{0,p} \int_{\mathbb{R}} \frac{1 + e^{-2t}y^2 + w^2 + 2e^{-t}yw}{1+y^2} N(0, q_t) dw \leq 4 \|f\|_{0,p}.$$

Remark that the map $U_t g$ satisfies equation (4), for any $g \in UC_p(\mathbb{R})$.

STEP 2. One shows that the map ψ, defined in (6) (where f_o is replaced by g_o), is the unique classical solution of (24).

To check the claim one can argue as in the proof of Lemma 2. First note that there results:

$$\|D_y U_t g_o\|_{0,p} \leq 4\|g_o'\|_{0,p} \leq 8, \quad \|D_{yy} U_t g_o\|_{0,p} \leq c\left(1 + \frac{1}{\sqrt{t}}\right), \quad t > 0.$$

Then, using these estimates and the ones given in (7) and (9), one deduces that $\psi \in$ $\in UC_p^1(\mathbb{R}^2_+)$ and there exist all the second partial derivatives of ψ in the classical sense on $\mathbb{R}^2_+$; moreover the maps $D_{xy}\psi$, $D_{yy}\psi \in UC_p(\mathbb{R}^2_+)$. We also obtain that formula (10), concerning $D_{xx}\psi$, holds when f_o is replaced by g_o. Now to finish the proof it is enough to show that

$$\limsup_{x \to 0+} \sup_{y \in \mathbb{R}} \left| \frac{D_{xx}\psi(x, y)}{1 + y^2 + x^2} + \frac{2g_o(y)}{1+y^2} \right| > 0. \tag{26}$$

STEP 3. We consider the map ϕ, defined in (20) replacing f_o with g_o, and prove that $\lim_{x\to 0+} \sup_{y\in\mathbb{R}} \left| \frac{D_{xx}\psi(x,y) - \phi(x,y)}{1+y^2+x^2} \right| = 0$.

To this end we can proceed as in Step I of the proof of Theorem 1, using the estimate

$$\sup_{y\in\mathbb{R}} \left| \frac{U_t g_o(y) - g_o(e^{-t}y)}{1+y^2} \right| \leq$$

$$\leq \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-2t}y^2 \big| |\cos(e^{-t}y + w)| - |\cos(e^{-t}y)| \big| + w^2 + 2e^{-t}|yw|}{1+y^2} N(0, q_t) dw \leq$$

$$\leq 2(\sqrt{q_t} + q_t), \quad t \geq 0.$$

STEP 4. We prove that

$$\limsup_{x\to 0+} \sup_{y\in\mathbb{R}} \left| \frac{2}{\sqrt{2\pi}} \frac{\int_0^\infty e^{-\frac{u^2}{2}} e^{-\frac{2x^2}{u^2}} y^2 |\cos(e^{-\frac{x^2}{u^2}} y)| \, du}{1+y^2+x^2} - \frac{y^2|\cos(y)|}{1+y^2} \right| > 0. \tag{27}$$

Let $a \in [\frac{\sqrt{2}}{\sqrt{3}}\,,\,1)$, $x \in (0\,,\,1)$, we write:

$$\sup_{y\in\mathbb{R}} \Big| \frac{\int_0^\infty e^{-\frac{u^2}{2}}\, e^{-\frac{2x^2}{u^2}} y^2 |\cos(e^{-\frac{x^2}{u^2}} y)|\, du}{1+y^2+x^2} - \frac{y^2|\cos(y)|}{1+y^2} \Big| \geq$$

$$\geq \sup_{k\in\mathbb{Z}} \int_a^1 e^{-\frac{u^2}{2}}\, e^{-\frac{2x^2}{u^2}} \frac{[4k+1]^2\,\pi^2/4\;|\cos(e^{-\frac{x^2}{u^2}}\,[4k+1]\,\frac{\pi}{2})|\, du}{1\,+\,[4k+1]^2\,\pi^2/4\,+\,1} \geq$$

$$\geq \frac{1}{3} \sup_{k\in\mathbb{Z}} \int_a^1 e^{-\frac{u^2}{2}}\, e^{-\frac{2}{a^2}}\, |\cos\left(e^{-\frac{x^2}{u^2}}\,[4k+1]\,\frac{\pi}{2}\right)|\, du \geq$$

$$\geq \frac{e^{-1/2}(1-a)\, e^{-\frac{2}{a^2}}}{3} \sup_{k\in\mathbb{Z}} \inf_{u\in[a\,,1]} \Big|\cos\left(e^{-\frac{x^2}{a^2}}\,[4k+1]\,\frac{\pi}{2}\right)\Big|.$$

Now, appealing to (14), we obtain assertion (27).

From (27) it follows that (26) is verified replacing $D_{xx}\psi$ with ϕ. Using Step 3 and a simple inequality as in (21), we finally get (26). The proof is complete. □

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