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An elementary theory of the oblique impact of rods

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Meccanica dei solidi. — *An elementary theory of the oblique impact of rods.* Nota di
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Abstract. — An extension is proposed of a classical approximate method for estimating the stress state in an elastic rod obliquely colliding against a rigid wall.

Key words: Elasticity; Impact; Rods.

Riassunto. — *Una teoria elementare dell'urto obliquo delle sbarre.* Si propone un'estensione di un classico metodo approssimato per valutare lo stato di sforzo di una sbarra elastica che urta obliquamente contro una parete rigida.

1. The classical approximation

The longitudinal impact of an elastic rod against a rigid wall has been described by F. Neumann [2] in terms of a system of second order partial differential equations, which, for the axial displacement, reduce to a single d'Alembert equation. This equation, however, is not easily solved, especially when, as frequently occurs, the cross-section of the rod is variable or the stress exceeds the elastic limit of the rod. Consequently, many contributors, including Tredgold [6], Cox [1], Saint-Venant and Flamant [4], have sought approximate methods of solution. In reviewing these various approaches, Pöschl [3] has remarked that the rod is usually regarded as a system possessing one degree of freedom and the aim is to seek the configuration of maximum compression when the rod first achieves instantaneous equilibrium.

A simple example of this treatement is offered by a light linear elastic bar ab of length l and constant cross-section impinging with normal impact velocity v_y on a rigid wall, as shown in fig. 1. We assume that a rigid solid of mass M is clamped at the end a. The bar ab has extensional rigidity EA and flexural rigidity EJ, where E is the elastic modulus of the material and A, J respectively denote the area and moment of inertia of the uniform cross-section. The mass of the bar is neglected compared with M.

Let us suppose that the moment of impact of the end b of the bar against the rigid wall is the instant $t = 0$. During subsequent contact and before rebounding, the system will behave as a simple harmonic oscillator whose motion is described by the equation

$$M\,\ddot{y} + H\,y = 0\ , \tag{1}$$

where $y(t)$ is the normal displacement of the point a, H is the coefficient EA/l of the axial restoring force, and a superimposed dot indicates differentiation with respect to time, t.

(*) Nella seduta del 13 novembre 2000.

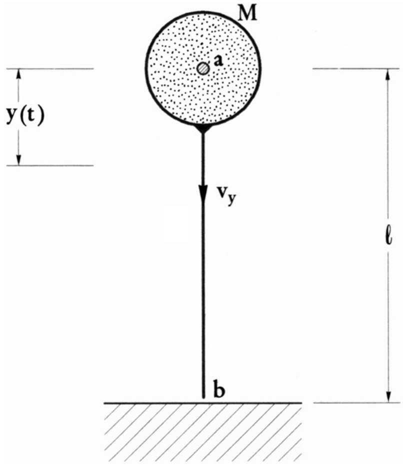

Fig. 1.

On setting $\lambda^2 = H/M$, and recalling that $y(0) = 0$, $\dot{y}(0) = v_y$, we obtain the solution to (1) in the form

$$y(t) = (v_y/\lambda) \sin \lambda t \ , \tag{2}$$

which is given in standard texts on elementary mechanics (cf. *e.g.* [5, § 23]). From (2) may be deduced the duration of contact, the maximum descent of the mass M, and the maximum axial force during the first half of the compression period $[0, \pi/2\lambda]$.

2. Oblique impact

The analysis of the previous section fails when the bar impacts obliquely on the rigid wall. In order to study this problem, we suppose that, immediately prior to impact, the bar with the clamped mass M is moving as a rigid body with constant velocity in a plane perpendicular to the wall and the longitudinal axis of the bar inclined at an angle α with the wall at the instant of first contact. We assume that $0 \leq \alpha \leq \pi/2$, but the analysis may easily be extended to $\pi/2 < \alpha \leq \pi$. The solid mass M has a moment of inertia Θ with respect to an axis perpendicular to the plane of motion (fig. 2). At the instant $t = 0$, the end b of the bar collides with the wall, and the subsequent motion after impact is to be determined from general principles of mechanics subject to certain additional assumptions to be stated. In particular, the rebound depends upon whether the surface of the wall is smooth or rough.

In either case, we assume that the duration of the rebound process is not instantaneous but nevertheless is sufficiently small for the bar to remain approximately the same length l as before the impact and inclined at an angle α to the wall. Simultaneously, compressive behaviour of the bar is permitted in accordance with standard approximations of the linear theory of elasticity. Rates of change of the displacements to first and higher orders are therefore not necessarily zero during the rebound. Of

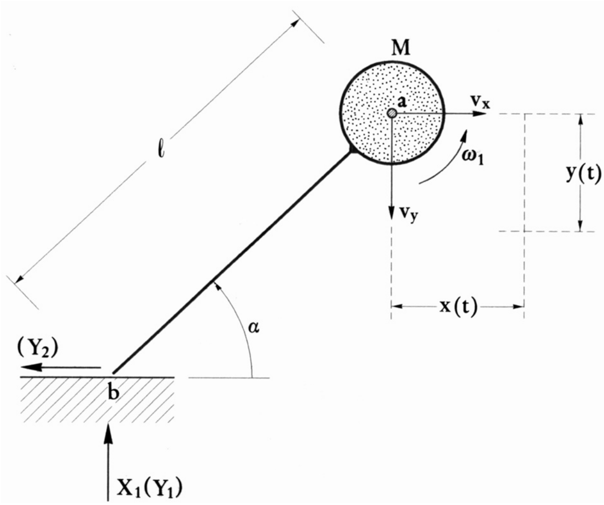

Fig. 2.

course, after impact, conversion of the rotational inertia of the clamped mass M causes the bar immediately to commence bending.

In the plane of motion we select a two-dimensional Cartesian system of coordinates with origin at the position of a on impact, and y-axis normal to the wall and directed towards it. The components of the uniform velocity prior to impact are given by (v_x , v_y), while, during the period subsequent to contact, the displacement components of the end a are $(x(t) , y(t))$ and the anti-clockwise rotational displacement of the clamped mass M is denoted by $\varphi(t)$.

We separately discuss two limiting cases.

a) *Smooth contact.*

The reaction on impact (fig. 2) consists only of a normal force X_1 as the wall is smooth. Linear momentum of the mass M is conserved in the x-direction and consequently the system maintains a uniform velocity v_x after the impact. On resolving in the y-direction and taking moments about the end a we obtain

(3) $$M\ddot{y} + X_1 = 0 \ ,$$

(4) $$\Theta\ddot{\varphi} + X_1 \, l \cos\alpha = 0 \ .$$

Elimination of X_1 between (3), (4) and integration leads to

(5) $$My \, l \cos\alpha = \Theta\varphi + Gt \ ,$$

where $G = Mv_y \, l \cos\alpha - \Theta\,\omega_1$, and the initial conditions $y(0) = \varphi(0) = 0$, $\dot{y}(0) = v_y$, $\dot{\varphi}(0) = \omega_1$ are used. Furthermore, elementary beam theory yields the relation

(6) $$X_1 = K(y + l\varphi\cos\alpha) \ ,$$

where

$$K = \left[\frac{l^3 \cos^2 \alpha}{3\, EJ} + \frac{l \sin^2 \alpha}{EA} \right]^{-1} . \tag{7}$$

On setting

$$y(t) = A_2 t + z(t) \ , \tag{8}$$

where

$$A_2 = \frac{Gl \cos \alpha}{M\, l^2 \cos^2 \alpha + \Theta} \ , \tag{9}$$

and on eliminating φ between (5) and (6), we obtain from (3)

$$z(t) = A_3 \sin \lambda_1 t \ , \tag{10}$$

where

$$\lambda_1 = \left[\frac{K(M\, l^2 \cos^2 \alpha + \Theta)}{M\Theta} \right]^{\frac{1}{2}} \ , \tag{11}$$

$$A_3 = \frac{\Theta(v_y + l \cos \alpha\, \omega_1)}{\lambda_1 (M\, l^2 \cos^2 \alpha + \Theta)} \ . \tag{12}$$

We deduce from (5) that

$$\varphi(t) = B_2 t + B_3 \sin \lambda_1 t \ , \tag{13}$$

with

$$B_2 = \frac{-G}{M\, l^2 \cos^2 \alpha + \Theta} \ , \quad B_3 = \frac{M\, l \cos \alpha\, A_3}{\Theta} \ , \tag{14}$$

while (3) yields

$$X_1 = (A_3\, \lambda_1^2\, M) \sin \lambda_1\, t \ . \tag{15}$$

Inspection of expressions (8), (13) reveals that the motion in the short interval of time after impact is the sum of a linear part and an oscillatory compressive and elongational part. The maximum reactive force X_1, attained at the end $t = \pi/2\lambda_1$ of the first half period of compression, is

$$X_1|_{\max} = A_3 \lambda_1^2\, M \ , \tag{16}$$

while the strain energy absorbed by the bar after the same interval is

$$W = \frac{1}{2} K^{-1} \left(X_1|_{\max} \right)^2 = \frac{1}{2} \frac{M\Theta(v_y + l \cos \alpha \omega_1)^2}{(Ml^2 \cos^2 \alpha + \Theta)} . \tag{17}$$

It follows from (15) that contact with the wall is broken by the rebounding bar at time $t = \pi/\lambda_1$. At this instant the end b of the bar must be moving instantaneously away from the wall, but on recalling that $l\omega_1 = -(v_x \sin \alpha + v_y \cos \alpha)$, we may deduce from (8) and (10) that provided $0 \leq \alpha \leq \pi/4$ the end a is moving towards the wall ($\dot{y}(\pi/\lambda_1) > 0$). The end a moves away from the wall ($\dot{y}(\pi/\lambda_1) < 0$) when $\pi/4 < \alpha \leq \pi/2$ and either l is sufficiently large or certain conditions are satisfied that relate Ml^2, Θ, v_x, and v_y. The conditions are easily derivable from (8) and (10) and are omitted. On loss of contact at $t = \pi/\lambda_1$, the end a is displaced by a distance $A_2\pi/\lambda_1$, although the maximum displacement occurs at $t = \pi/2\lambda_1$ and is given by $A_2\pi/2\lambda_1 + A_3$.

Note that for $t > \pi/\lambda_1$, the present analysis ceases to be appropriate and consequently attention is confined to the interval $[0, \pi/\lambda_1]$.

b) Rough contact.

It is now assumed that the rigid wall in the neighborhood of the instantaneous impact is rough with sufficiently small coefficient of friction μ to allow sliding of the base b after the impact. The value of μ ($0 \leq \mu \leq 1$) is determined subsequently. The reaction of the point of contact consists of a normal force Y_1 and a force Y_2 parallel to the wall. We continue to assume that, during the interval of compression and restitution (supposed sufficiently small), the length of the bar and its inclination to the wall may be approximated by their respective values just before impact, but that the first and higher order rates of change of the displacement components $(x(t), y(t))$ of the end a are not necessarily zero.

The equations of motion of the mass M normal and parallel to the wall become

$$M\ddot{y} + Y_1 = 0 , \tag{18}$$

$$M\ddot{x} + Y_2 = 0 , \tag{19}$$

where the frictional force Y_2 is assumed to be (cf. [5, § 22])

$$Y_2 = -\mu \frac{v}{|v|} Y_1 \quad \text{for } v \neq 0 , \qquad Y_2 = 0 \quad \text{for } v = 0 , \tag{20}$$

and v is the velocity of b along the wall, *i.e.* $v(t) = \dot{x}(t) + l \sin\alpha\, \dot{\varphi}(t)$.

During the first period of compression, we assume $v(t)$ does not alter sign, so that $Y_2 = -\mu Y_1$. Then the normal force Y_1 again may be determined in terms of the variables $y(t)$, $\varphi(t)$, and elementary beam theory yields the expression

$$Y_1 = L(y + l\cos\alpha\,\varphi) , \tag{21}$$

where

$$L = \left[\frac{l^3}{3EJ}(\cos^2\alpha + \mu\sin\alpha\,\cos\alpha) + \frac{l}{EA}(\sin^2\alpha - \mu\sin\alpha\,\cos\alpha)\right]^{-1} . \tag{22}$$

We henceforth introduce the hypothesis that the bar is sufficiently slender, which implies the inequality $l^2 \geq (3J/A)$ and hence $L \leq K$.

On taking moments about a, we obtain

$$\Theta\ddot{\varphi} + Y_1 l(\cos\alpha + \mu\sin\alpha) = 0 . \tag{23}$$

Equations (18), (19), (23), subject to the initial conditions $x(0) = y(0) = \varphi(0) = 0$, $\dot{x}(0) = v_x$, $\dot{y}(0) = v_y$, $\dot{\varphi}(0) = \omega_1$, may be solved in similar fashion to before and result in the expressions

$$x(t) = A_4 t + A_5 \sin \lambda_2 t \ , \tag{24}$$

$$y(t) = A_6 t + A_7 \sin \lambda_2 t \ , \tag{25}$$

$$\varphi(t) = B_4 t + B_5 \sin \lambda_2 t \ , \tag{26}$$

where

$$\lambda_2 = \sqrt{\frac{LN}{\Theta M}} \ , \tag{27}$$

$$N = \Theta + M\, l^2 \cos\alpha\, (\cos\alpha + \mu \sin\alpha) \ , \tag{28}$$

$$A_4 = v_x + \frac{\mu\,\Theta}{N}\,(v_y + l\,\cos\alpha\,\omega_1) \ , \quad A_5 = -\mu\, A_7 \ , \tag{29}$$

$$A_6 = \frac{l\,\cos\alpha}{N}\left[M\,l(\cos\alpha + \mu\,\sin\alpha)\,v_y - \Theta\,\omega_1\right] \ , \quad A_7 = \frac{\Theta}{\lambda_2 N}(v_y + l\,\cos\alpha\,\omega_1) \ , \tag{30}$$

$$B_4 = -\frac{\left[M\,l(\cos\alpha + \mu \sin\alpha)\,v_y - \Theta\,\omega_1\right]}{N} \ , \quad B_5 = \frac{Ml}{\Theta}(\cos\alpha + \mu \sin\alpha)\,A_7 \ . \tag{31}$$

From (25) and (18) we obtain

$$Y_1 = M\,\lambda_2^2 A_7 \sin \lambda_2 t \ , \tag{32}$$

which enables Y_2 to be determined from (20). The maximum value of Y_1 occurs at $t = \pi/2\lambda_2$ and is given by $M\lambda_2^2 A_7$. Contact is first broken at time $t = \pi/\lambda_2$ and accordingly, we consider behaviour only in the interval $[0\,,\pi/\lambda_2]$. Conditions may be derived from (25) under which the end a is moving either towards or away from the wall at the instant of broken contact. In particular, $\dot{y}(\pi/\lambda_2) < 0$ for $\pi/4 < \alpha \leq \pi/2$ and l sufficiently large. Expression (25) may be used to discuss the displacement of the end a.

The stress in the bar is due to the combined reactive forces $(Y_1\,,\,Y_2)$ which determine the stress resultants at each cross section of the bar. The strain energy absorbed by the bar at the end of the compression period ($t = \pi/2\lambda_2$) is now

$$W_\mu = \frac{1}{2}\left(Y_1|_{\max}\right)^2 \left[\frac{l^3}{3EJ}(\cos\alpha + \mu\,\sin\alpha)^2 + \frac{l}{EA}(-\sin\alpha + \mu\,\cos\alpha)^2\right] = \tag{33}$$

$$= \frac{1}{2}ML\Theta(v_y + \omega_1\,l\,\cos\alpha)^2\left[\Theta + M\,l^2\,\cos\alpha(\cos\alpha + \mu\,\sin\alpha)\right]^{-1}$$

$$\cdot\left[\frac{l^3}{3EJ}(\cos\alpha + \mu\,\sin\alpha)^2 + \frac{l}{EA}(-\sin\alpha + \mu\,\cos\alpha)^2\right] \ , \tag{34}$$

where L is given by (22).

The relative magnitudes of $W\mu$ and W depend upon μ and the mechanical properties of the system. For example, from (17) and (34) we conclude that $W_\mu \leq W$ provided the following conditions are satisfied

$$FEA\Theta \leq Ml^2 , \tag{35}$$

and

$$\mu[\Theta(FEA \sin^2 \alpha + 1) + Ml^2 \cos^2 \alpha] \leq \cos\alpha \sin\alpha (Ml^2 - FEA\Theta) , \tag{36}$$

where $F = (1/3EJA)(l^2 A - 3J)$. By a previous assumption we have

$$F \geq 0. \tag{37}$$

The maximum value of μ that permits sliding to occur at b after impact is determined from the condition that the velocity of b remains positive throughout the period of contact. From (24), (26) we obtain the restriction

$$0 < \dot{x}(t) + l \sin\alpha\, \dot{\varphi}(t) = (A_4 + l \sin\alpha\, B_4) + \lambda_2 \cos\lambda_2 t(A_5 + l \sin\alpha B_5) .$$

The minimum of the right-hand side occurs at $t = \pi/\lambda_2$ and hence μ must be such that

$$\lambda_2(A_5 + l \sin\alpha\, B_5) < A_4 + l \sin\alpha\, B_4 . \tag{38}$$

Different values of μ satisfy (38) depending upon the system's properties and the initial velocity (v_x , v_y). As illustration, let us suppose that

$$v_x \cos\alpha - v_y \sin\alpha > 0 \tag{39}$$

and

$$2\Theta < Ml^2(1 + \sin^2\alpha). \tag{40}$$

Then on setting $l\omega_1 = -(v_x \sin\alpha + v_y \cos\alpha)$, we conclude that (38) holds provided μ satisfies the condition

$$\mu < \cot\alpha[\Theta + Ml^2(1 + \sin^2\alpha)]/[Ml^2(1 + \sin^2\alpha) - 2\Theta]. \tag{41}$$

It remains to justify the assumption that the period of compression is small but finite. The respective measures for rough and smooth contact are proportional to λ_2^{-1} and λ_1^{-1}. But by (11) and (27), and subject to (37) we have that $\lambda_2^{-1} \leq \lambda_1^{-1}$. Under the additional condition (35), the period of compression decreases as μ increases to the maximum value given by (38).

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