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Fisica matematica. — *On a temperature-dependent Hele-Shaw flow in one dimension.*
Nota di Antonio Fasano e Laura Pezza, presentata (*) dal Socio M. Primicerio.

Abstract. — A model is presented for a Hele-Shaw flow with variable temperature in one space dimension. The problem to be solved is a free boundary problem for a parabolic equation with a non-linear and non-local free boundary condition. Existence and uniqueness are proved.

Key words: Hele-Shaw flows; Free boundary problems; Nonlocal conditions.

Riassunto. — *Su un flusso di Hele-Shaw unidimensionale dipendente dalla temperatura.* Si presenta un modello per un flusso di Hele-Shaw con temperatura variabile in una dimensione spaziale. Il problema da risolvere è un problema a frontiera libera per un'equazione parabolica con una condizione al contorno non lineare e non locale. Si dimostrano esistenza e unicità.

1. Introduction

It is well known that a Hele-Shaw cell consists of two parallel plates with a viscous incompressible fluid flowing in the gap. When the gap is sufficiently narrow and surface tension is negligible the pressure at the fluid-air interface satisfies the conditions

$$p = 0\,, \tag{1.1}$$

$$-\nabla p \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n} \tag{1.2}$$

($\mathbf{n}$ external normal unit vector, $\mathbf{v}$ velocity of the interface, all variables nondimensional and suitably normalized). Moreover the pressure is a harmonic function in the flow region

$$\Delta p = 0. \tag{1.3}$$

For the derivation of the equations above see [3].

Some basic references about the Hele-Shaw flow are [8], the original paper, [2, 6, 10, 12-15], each corresponding to a significant progress in the study of the problem. In the typical Hele-Shaw cell the liquid is injected or sucked through a circular hole in one of the plates and the flow is 2-D.

One of the basic assumptions leading to (1.1)-(1.3) is that the fluid viscosity is constant. Such a condition is satisfied if the temperature is everywhere constant.

Recently, several generalizations have been considered. For instance non-planar Hele-Shaw cells [4], or the case of non-Newtonian fluids [1].

In this paper we want to analyze a nonisothermal quasi-steady flow taking into account the temperature dependence of viscosity. Thus we have a thermal conduction-convection problem coupled with a Hele-Shaw flow with variable viscosity.

(*) Nella seduta del 15 dicembre 2000.

In the next section we shall derive the governing equations justifying the quasi-steady assumption and the rest of the paper is devoted to the proof of existence and uniqueness of a classical solution to the 1-D problem. The one-dimensional geometry is obtained by injecting the fluid through a rectilinear slot and averaging all quantities in the transversal direction. The problem is then trivial if the viscosity is constant (and it coincides with the so-called Green-Ampt model for fluid injection into porous media [7]). In the temperature dependent case however, besides the coupling between the equations for the flow and the heat transfer, the front advancement is described by an integrodifferential equation involving the temperature in a nonlocal way.

Our approach differs from the one adopted by other authors (see [16, 17, 9]), whose studies refer specifically to the flow of magmas for which the Péclet number is considerably large, making transverse heat convection not negligible. Indeed we select physical situations in which this contribution to the heat flux is not important and transverse motion can be neglected altogether (typical values compatible with this picture can be a gap of 1 *mm*, characteristic length in the motion direction 1 *m*, thermal diffusivity larger than $10^{-6} m^2 sec^{-1}$, the ratio between the driving pressure gradient and viscosity not larger than $1\, m^{-1} sec^{-1}$).

Also we note that *e.g.* in [16] the boundary temperature is prescribed, while in our case it is not specified and heat exchange with the exterior through the plates is governed by a linear radiation law. After an averaging procedure in the transversal direction the boundary heat exchange rate appears as a source term in the diffusion-convection equation. Assuming that the heat exchange coefficient through the plates is small enough, the thermal problem is mathematically not trivial (in the sense that we must deal with the full parabolic equation in the presence of a free boundary).

REMARK 1.1. It is well known that Hele-Shaw flows with suction are unstable. Here we are mainly interested in the injection case (the injection pressure larger than the pressure at the moving boundary). We will make some remarks on the general case in the last section.

2. Derivation of the model

As we said, we consider a physical situation in which we can neglect the velocity component in the direction normal to the plates, so we deal with a one-dimensional flow.

We choose a coordinate x in the direction of the flow, $x = 0$ being the inflow line, and a coordinate z such that the plates coincide with the planes $z = \pm h$. We introduce a characteristic length L ($x = L$ representing the boundary of the cell) such that $h/L \ll 1$.

Since the fluid is incompressible, the only scalar component of the velocity field does not depend on x : $v = v(z, t)$. The flow region is bounded between $x = 0$ and the unknown moving front $x = f(z, t)$.

Neglecting gravity, it is immediately seen that the flow is described by the equations

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\eta(\theta)\frac{\partial v}{\partial z}\right), \tag{2.1}$$

$$0 = -\frac{\partial p}{\partial z} + \eta'(\theta)\frac{\partial \theta}{\partial x}\frac{\partial v}{\partial z}, \tag{2.2}$$

where ρ is the density (here considered constant). Consistently with the latter assumption on ρ we have disgregated possible transverse forces due to thermally induced buoyancy. As a matter of fact, the model is formulated for the case in which the relative temperature excursion in the z-direction is small enough, in the sense that will be specified later. It has to be stressed that the viscosity is a known differentiable function $\eta(\theta)$ of the temperature θ and therefore it depends on space and time through the (unknown) thermal field. In the sequel we will introduce various assumptions on the data for the thermal field guaranteeing that during the whole process the fluid does not undergo any crystallization or glassification, so that (2.1), (2.2) can safely be used as flow equations.

Now we introduce a reference pressure p_0 and a reference viscosity η_0 (the viscosity corresponding to some reference temperature θ_0) and we define the dimensionless variables

$$\widetilde{x} = x/L, \qquad \widetilde{z} = z/h, \qquad \widetilde{p} = p/p_0, \quad \widetilde{\theta} = \theta/\theta_0, \quad \widetilde{\eta}(\widetilde{\theta}) = \eta(\theta)/\eta(\theta_0)$$

$$\widetilde{t} = t/t_0, \qquad \widetilde{v} = v/v_0, \qquad \widetilde{f}(\widetilde{z}, t) = (1/L)f(h\widetilde{z}, t_0\widetilde{t}),$$

where $v_0 = \frac{p_0 h^2}{L\eta_0}$, $t_0 = \frac{L}{v_0}$.

In the new variables equations (2.1), (2.2) become

$$R\frac{\partial \widetilde{v}}{\partial \widetilde{t}} = -\frac{\partial \widetilde{p}}{\partial \widetilde{x}} + \frac{\partial}{\partial \widetilde{z}}\left(\widetilde{\eta}(\widetilde{\theta})\frac{\partial \widetilde{v}}{\partial \widetilde{z}}\right), \tag{2.3}$$

$$\frac{\partial \widetilde{p}}{\partial \widetilde{z}} = \left(\frac{h}{L}\right)^2 \frac{\partial \widetilde{v}}{\partial \widetilde{z}}\frac{d\widetilde{\eta}}{d\widetilde{\theta}}\frac{\partial \widetilde{\theta}}{\partial \widetilde{x}}, \tag{2.4}$$

where $R = \frac{\rho v_0 L}{\eta_0}\left(\frac{h}{L}\right)^2$ is the Reynolds number.

Our basic assumption is that $\frac{h}{L}$ is so small that we are allowed to replace (2.3), (2.4) by

$$\frac{\partial \widetilde{p}}{\partial \widetilde{x}} = \frac{\partial}{\partial \widetilde{z}}\left(\widetilde{\eta}(\widetilde{\theta})\frac{\partial \widetilde{v}}{\partial \widetilde{z}}\right), \tag{2.5}$$

$$\frac{\partial \widetilde{p}}{\partial \widetilde{z}} = 0, \tag{2.6}$$

so that the flow can be considered quasi-steady. Note that in (2.5) the dependence on $\widetilde{x}$ is contained in the viscosity (through the thermal field that we have not yet described).

At this point we make another simplifying assumption, neglecting the possible dependence of $\widetilde{\eta}$ (*i.e.* of $\widetilde{\theta}$) on $\widetilde{z}$, which is quite sensible in view of the previous hypothesis and will be clearly justified later. We define the average velocity

$$V(\widetilde{t}\,) = \int_0^1 \widetilde{v}(\widetilde{z}\,,\,\widetilde{t}\,)\,d\widetilde{z}\,, \tag{2.7}$$

and, recalling (2.6), we separate the spatial variables in (2.5):

$$\frac{\partial^2 \widetilde{v}}{\partial \widetilde{z}^2} = \frac{1}{\widetilde{\eta}}\frac{\partial \widetilde{p}}{\partial \widetilde{x}}\,, \tag{2.8}$$

so that both sides depend on $\widetilde{t}$ only. The boundary conditions

$$\widetilde{v}(1\,,\,\widetilde{t}\,) = 0\,, \qquad \left.\frac{\partial \widetilde{v}}{\partial \widetilde{z}}\right|_{\widetilde{z}=0} = 0 \tag{2.9}$$

lead to

$$\widetilde{v}(\widetilde{z}\,,\,\widetilde{t}\,) = -\frac{1}{2\widetilde{\eta}}\frac{\partial \widetilde{p}}{\partial \widetilde{x}}(1-\widetilde{z}^2)\,, \tag{2.10}$$

from which we deduce the equation for $V(\widetilde{t}\,)$

$$V(\widetilde{t}\,) = -\frac{1}{3\widetilde{\eta}(\widetilde{\theta}\,)}\frac{\partial \widetilde{p}}{\partial \widetilde{x}}. \tag{2.11}$$

REMARK 2.1. Clearly the average velocity (2.7) can be defined only when the transverse cross section is completely filled by the fluid. Therefore the use of (2.11) close to the free boundary is an approximation requiring that the latter is flat enough.

The pressure is given at the injection slot and at the moving boundary $\widetilde{x} = \widetilde{f}\,(\widetilde{z}\,,\,\widetilde{t}\,)$ and it can be rescaled so that these conditions take the form

$$\widetilde{p}(0\,,\,\widetilde{t}\,) = \widetilde{p}_i(\widetilde{t}\,) > 0\,, \tag{2.12}$$

$$\widetilde{p}(\widetilde{f}\,,\,\widetilde{t}\,) = 0. \tag{2.13}$$

The condition $\widetilde{p}_i > 0$ is consistent with Remark 1.1 and will be removed later.

In addition, the points of the moving boundary move with the fluid velocity and therefore, putting

$$F(\widetilde{t}\,) = \int_0^1 \widetilde{f}\,(\widetilde{z}\,,\,\widetilde{t}\,)\,d\widetilde{z} \tag{2.14}$$

we impose that $\dot{F}(\widetilde{t}\,) = V(\widetilde{t}\,)$ (see Remark 2.1). Thus from (2.11), (2.12), (2.13) we obtain the remaining free boundary condition in the following integro-differential form

$$\dot{F}(\widetilde{t}\,) = \frac{\widetilde{p}_i(\widetilde{t}\,)}{3\displaystyle\int_0^{F(\widetilde{t})} \widetilde{\eta}(\widetilde{\theta}\,(\widetilde{x}\,,\,\widetilde{t}\,))\,d\widetilde{x}}\,, \tag{2.15}$$

which in fact summarizes the whole flow problem, together with the initial condition

$$F(0) = F_0 > 0. \tag{2.16}$$

Let us come to the thermal field equation which in the original variables is

$$c\frac{\partial \theta}{\partial t} + cv\frac{\partial \theta}{\partial x} = k\Delta\theta\,, \tag{2.17}$$

where the heat capacity c and thermal conductivity k are taken constant, and we have neglected the heat generated by internal friction in the flow.

For θ we have the initial condition

$$\theta(x, z, 0) = \Theta(x, z)\,, \qquad 0 < x < f(z, 0) \tag{2.18}$$

and the boundary conditions

$$\theta(0, z, t) = \theta_i(t) \tag{2.19}$$

$$\frac{\partial \theta}{\partial z} = -\lambda_1(\theta - \theta_1)\,, \qquad z = h\,, \tag{2.20}$$

$$\frac{\partial \theta}{\partial z} = 0\,, \qquad z = 0\,, \tag{2.21}$$

$$\left(\frac{\partial \theta}{\partial x} - \frac{\partial \theta}{\partial z}\frac{\partial f}{\partial z}\right)\left[1 + \left(\frac{\partial f}{\partial z}\right)^2\right]^{-1/2} = -\lambda_2(\theta - \theta_2)\,, \quad x = f(z, t)\,, \tag{2.22}$$

λ_1, λ_2 being positive constants whose order of magnitude will be specified soon (Remark 2.2). The conditions on the functions $\Theta(x, z)$, $\theta_i(t)$ and on the constants θ_1, θ_2 are mainly dictated by the requirement that the rheological behaviour of the system is compatible with equations (2.1), (2.2). It is sufficient to suppose that $\Theta, \theta_i, \theta_1, \theta_2$ are all larger than some critical temperature θ^*, above which the system can be reasonably considered as a Newtonian fluid.

Now we use the rescaled temperature $\widetilde{\theta} = \theta/\theta_0$ and, defining the coefficient $\gamma = \frac{k}{cLv_0} = \frac{k\eta_0}{cp_0h^2}$, we write (2.17)-(2.22) in nondimensional form

$$\frac{\partial \widetilde{\theta}}{\partial \widetilde{t}} + \widetilde{v}\frac{\partial \widetilde{\theta}}{\partial \widetilde{x}} = \gamma\frac{\partial^2 \widetilde{\theta}}{\partial \widetilde{x}^2} + \gamma\left(\frac{L}{h}\right)^2\frac{\partial^2 \widetilde{\theta}}{\partial \widetilde{z}^2}\,, \tag{2.23}$$

$$\widetilde{\theta}(\widetilde{x}, \widetilde{z}, 0) = \widetilde{\Theta}(\widetilde{x}, \widetilde{t})\,, \qquad 0 < \widetilde{x} < \widetilde{f}(\widetilde{z}, 0)\,, \tag{2.24}$$

$$\widetilde{\theta}(0, \widetilde{z}, \widetilde{t}) = \widetilde{\theta}_i(\widetilde{t})\,, \tag{2.25}$$

$$\frac{\partial \widetilde{\theta}}{\partial \widetilde{z}} = 0\,, \qquad \text{for } \widetilde{z} = 0\,, \tag{2.26}$$

$$\frac{\partial \widetilde{\theta}}{\partial \widetilde{z}} = -\lambda_1 h(\widetilde{\theta} - \widetilde{\theta}_1)\,, \qquad \text{for } \widetilde{z} = 1\,, \tag{2.27}$$

$$\frac{\partial \widetilde{\theta}}{\partial \widetilde{x}} = -\lambda_2 L(\widetilde{\theta} - \widetilde{\theta}_2)\,, \qquad \text{on the moving boundary.} \tag{2.28}$$

Here we have assumed that the moving boundary has a sufficiently small curvature so that $\left|\frac{\partial f}{\partial z}\right| \ll 1$, and we can replace the l.h.s. of (2.22) simply by $\frac{\partial \theta}{\partial x}$. The definitions of $\widetilde{\Theta}$, $\widetilde{\theta}_i$, $\widetilde{\theta}_1$, $\widetilde{\theta}_2$ are obvious.

At this point we define

$$U(\widetilde{x}, \widetilde{t}) = \int_0^1 \widetilde{\theta}(\widetilde{x}, \widetilde{z}, \widetilde{t}) d\widetilde{z} \tag{2.29}$$

and we integrate (2.23) w.r.t. $\widetilde{z}$. Assuming that $\frac{\partial \widetilde{v}}{\partial \widetilde{z}}$, $\frac{\partial \widetilde{\theta}}{\partial \widetilde{z}}$ are so small that we can use the approximations $\widetilde{v} = V$, $\widetilde{\theta}(\widetilde{x}, 1, \widetilde{t}) = U(\widetilde{x}, \widetilde{t})$ when needed, we obtain the system

$$\frac{\partial U}{\partial \widetilde{t}} + \dot{F}(\widetilde{t}) \frac{\partial U}{\partial \widetilde{x}} = \gamma \frac{\partial^2 U}{\partial \widetilde{x}^2} - \gamma_1 (U - U_1), \quad 0 < \widetilde{x} < F(\widetilde{t}), \ \ \widetilde{t} > 0 \tag{2.30}$$

$$U(\widetilde{x}, 0) = U_0(\widetilde{x}) \qquad 0 < \widetilde{x} < \widetilde{F}(0), \tag{2.31}$$

$$U(0, \widetilde{t}) = U_i(\widetilde{t}), \qquad \widetilde{t} > 0, \tag{2.32}$$

$$\frac{\partial U}{\partial \widetilde{x}} = -\gamma_2 (U - U_2), \qquad \widetilde{x} = F(\widetilde{t}), \ \ \widetilde{t} > 0, \tag{2.33}$$

where $\gamma_1 = \lambda_1 \gamma \frac{L^2}{h}$, $\gamma_2 = \lambda_2 L$, $U_1 = \widetilde{\theta}_1$, $U_2 = \widetilde{\theta}_2 = U_i = \widetilde{\theta}_i$.

REMARK 2.2. We are interested in the case in which $\gamma = O(1)$, *i.e.* the diffusion time over distances of order L is comparable to t_0. This also implies that in the transversal direction temperature is basically uniform as required. Also we assume that γ_1 is not too large, which implies that λ_1 is of order $\frac{h}{L^2}$ (*i.e.* heat transmission across the plates is small). Taking $\gamma_1 \gg 1$ would imply $U \simeq U_1$, making the thermal problem trivial.

Thus we have formulated our free boundary problem consisting of (2.30)-(2.33) and of (2.15), which we rewrite in the form

$$\dot{F}(\widetilde{t}) = \frac{\widetilde{p}_i(\widetilde{t})}{3 \displaystyle\int_0^{F(\widetilde{t})} \widetilde{\eta}(U(\widetilde{x}, \widetilde{t}))\, d\widetilde{x}}, \quad \widetilde{t} > 0, \ \ F(0) = F_0, \tag{2.34}$$

keeping the same symbol $\widetilde{\eta}$ for the rescaled viscosity. Let us summarize the conditions under which the model above is meaningful:

(*i*) the Reynolds number is small,

(*ii*) the moving boundary $x = f(z, t)$ is flat enough (*i.e.* $\left|\dfrac{\partial f}{\partial z}\right| \ll 1$),

(*iii*) $\left|\dfrac{\partial \widetilde{v}}{\partial \widetilde{z}}\right|$, $\left|\dfrac{\partial \widetilde{\theta}}{\partial \widetilde{z}}\right|$ are small enough (in particular this allows to take the viscosity constant over each cross-section). As we said this is implied by $\gamma = 0(1)$.

We omit the precise definition of a classical solution (F, U) to (2.30)-(2.34), because it is obvious. From now on we drop the tilde from all the symbols in order to use a simpler notation.

3. Existence and uniqueness theorem

Let us recall that

(H1) the data U_0, U_i, U_1, U_2 are all larger than 1,

meaning that the physical system can be considered as a Newtonian fluid at the corresponding temperatures.

Moreover we assume that

(H2) $F(0) = F_0 > 0$,
(H3) $U_0 \in H^{1+2\alpha}([0, F_0])$ for some $\alpha \in (0, 1/2)$, U_i is Lipschitz continuous in $[0, T]$ for any $T > 0$,
(H4) the initial and boundary data for the temperature are compatible, *i.e.* $U_0(0) = U_i(0)$, $U_0'(F_0) = -\gamma_2(U_0(0) - U_2)$,
(H5) $p_i \in H^\alpha([0, T])$ for any $T > 0$, $p_i \geq 0$,
(H6) $\eta(U)$ is a decreasing Lipschitz continuous function for $U \geq 1$.

The Lipschitz norm of U_i and the Hölder norm of p_i in $[0, T]$ may depend on T.

Theorem 3.1. *Under the assumptions* (H1)-(H6) *problem* (2.30)-(2.34) *has a unique classical solution* (F, U) *which exists globally in time. Moreover* $U > 1$ *and* $\dot{F} \in H^\alpha$.

Proof. We use a fixed point technique. Let us introduce the set

$$\Sigma = \{\varphi \in C^1([0, T]) : \varphi(0) = F_0,\ 0 \leq \dot{\varphi} \leq B,\ |\dot{\varphi}(t'') - \dot{\varphi}(t')| \leq A(t'' - t')^\alpha,\ 0 \leq t' < t'' \leq T\},$$

where the constants A, B are to be chosen and α is the same as in (H3). For any $\varphi \in \Sigma$ the problem (2.30)-(2.33) with $F(t)$ replaced by $\varphi(t)$ has one unique solution $U(x, t)$ in $H^{2+2\alpha, 1+\alpha}(D_{\varphi, T})$, with $D_{\varphi, T} = \{(x, t) : 0 < t < T,\ 0 < x < \varphi(t)\}$, with $\frac{\partial U}{\partial x}$ continuous in $\overline{D}_{\varphi, T}$. Moreover, a standard application of the maximum principle and Hopf's lemma (in the parabolic version: see *e.g.* [5]) shows that $U > 1$ and $U \leq M(T)$, where $M(T) = \max(\|U_0\|, \|U_i\|_T, U_1, U_2)$. Here $\|U_0\|$ is the sup norm in $(0, F_0)$, $\|U_i\|_T$ the sup norm in $(0, T)$. In addition (see [11, Theorem 10.1: p. 204]) $U \in H^{\beta, \beta/2}(\overline{D}_{\varphi, T})$ for β arbitrarily close to 1 (and in particular independent of α), so that we can take $\beta \geq 2\alpha$. At this point we can differentiate (2.26) w.r.t. x and observe that $\frac{\partial U}{\partial x}$ satisfies a parabolic problem with initial data in $H^{2\alpha}$ and boundary data in H^α. Thus [11, Theorem 12.1: p. 22] $\frac{\partial U}{\partial x} \in H^{2\alpha, \alpha}(\overline{D}_{\varphi, T})$, with a Hölder norm depending on the data only.

Once we have calculated $U(x, t)$ we define

$$F(t) = F_0 + \int_0^t \frac{p_i(\tau)}{3 \displaystyle\int_0^{\varphi(\tau)} \eta(U(x, \tau))\, dx}\, d\tau \tag{3.1}$$

and we show that the mapping $\mathcal{T}\varphi = F$ has a fixed point in Σ by choosing the constants A, B in a suitable way.

We note that

$$0 < \eta_m = \eta(M) \le \eta \le \eta_1 = \eta(1) \tag{3.2}$$

and we set $P_i = \|p_i\|_T$. Clearly we have

$$0 \le \dot{F}(t) \le \frac{P_i}{3F_0\eta_m} \tag{3.3}$$

and consequently we choose $B = \frac{P_i}{3F_0\eta_m}$. Denoting by π_α the Hölder coefficient of $p_i(t)$ and by N_α the Hölder coefficient of U w.r.t. time, it is easy to check that

$$|\dot{F}(t') - \dot{F}(t'')|(t'' - t')^{-\alpha} \le \frac{1}{3F_0^2\eta_m^2}\left\{(F_0 + BT)(N_\alpha P_i H + \eta_1\pi_\alpha) + \eta_1 BT^{1-\alpha}\right\}, \tag{3.4}$$

where H is the Lipschitz coefficient of the function $\eta(U)$.

The right-hand side of this inequality defines the constant $A(T)$ such that $\mathcal{T}\Sigma \subset \Sigma$.

Now we want to show that $\mathcal{T}$ is continuous in the selected topology. More precisely we shall prove that there exists a constant $c > 0$ depending on T and on the data, such that

$$\|\dot{F}_1 - \dot{F}_2\|_T \le c\|\varphi_1 - \varphi_2\|_T, \tag{3.5}$$

for any $\varphi_1, \varphi_2 \in \Sigma$, where $F_i = \mathcal{T}\varphi_i$, $i = 1, 2$.

From (3.1) we immediately realize that

$$\|\dot{F}_1 - \dot{F}_2\|_T \le C_1\|\varphi_1 - \varphi_2\|_T + C_2\|U^{(1)} - U^{(2)}\|_T, \tag{3.6}$$

$U^{(1)}, U^{(2)}$ being the solutions of (2.30)-(2.33) corresponding to φ_1, φ_2 respectively and the norm $\|U^{(1)} - U^{(2)}\|_T$ is the sup norm over $D_{\varphi_1,T} \cap D_{\varphi_2,T} = D'_T$.

Therefore (3.5) will be proved if we show that

$$\|U^{(1)} - U^{(2)}\|_T \le K\|\varphi_1 - \varphi_2\|_T \tag{3.7}$$

for some positive constant K that can be found in terms of the data.

In order to study the difference $U^{(1)} - U^{(2)}$ in D'_T we first perform the transformation $\xi = x - F(t)$, $W(\xi, t) = U(\xi + F(t), t)$ in (2.30)-(2.33). Then we see that the function $Z(\xi, t) = W^{(1)}(\xi, t) - W^{(2)}(\xi, t)$ satisfies the problem

$$\frac{\partial Z}{\partial t} = \gamma\frac{\partial^2 Z}{\partial \xi^2} - \gamma_1 Z, \qquad -\psi(t) < \xi < 0,\ 0 < t < T \tag{3.8}$$

$$Z(\xi, 0) = 0, \qquad -F_0 < \xi < 0,\ 0 < t < T \tag{3.9}$$

$$Z(-\psi(t), t) = Z_i(t), \qquad 0 < t < T, \tag{3.10}$$

$$\frac{\partial Z}{\partial \xi} = -\gamma_2 Z, \qquad \xi = 0,\ 0 < t < T, \tag{3.11}$$

where $\psi(t) = \min(\varphi_1(t), \varphi_2(t))$, $Z_i(t) = \frac{\partial W^{(1)}}{\partial \xi}(\widetilde{\xi}(t), t)(\varphi_1(t) - \varphi_2(t))$ if $\varphi_1(t) \ge \varphi_2(t)$, while $Z_i(t) = \frac{\partial W^{(2)}}{\partial \xi}(\overline{\xi}(t), t)(\varphi_1(t) - \varphi_2(t))$ if $\varphi_1(t) \le \varphi_2(t)$, and $\overline{\xi}(t)$ is a suitable point between $\varphi_1(t)$ and $\varphi_2(t)$. We already know that $\left|\frac{\partial U}{\partial x}\right|$, and consequently $\left|\frac{\partial W}{\partial \xi}\right|$, is bounded in terms of the data and of the parameters defining the set Σ (which are now some known functions of the data).

At this point the inequality

$$\|Z\|_T \leq K_1 \|\varphi_1 - \varphi_2\|_T \tag{3.12}$$

is a trivial consequence of the maximum principle (K_1 being a constant depending on the data). Coming back to the original variables, we have

$$\| U^{(1)} - U^{(2)} \|_T \leq \|Z\|_T + K_2 \|\varphi_1 - \varphi_2\|_T \,, \tag{3.13}$$

where K_2 is nothing but an upper estimate for $\left|\frac{\partial U}{\partial x}\right|$, which, as we said, is uniform w.r.t. $\varphi \in \Sigma$.

We conclude that (3.7) follows from (3.12) and (3.13) with $K = K_1 + K_2$. At this point not only we have proved that the mapping $\mathcal{T}$ is continuous for any T, but also that for $T < \frac{1}{K}$ it is contractive. Hence the use of Schauder's and of Banach's theorems is the final step of the proof of Theorem 3.1. □

4. Continuous dependence

Consider the two sets of data $p_i^{(j)}(t)$, $F_0^{(j)}$, $U_0^{(j)}(x)$, $U_i^{(j)}(t)$, $U_1^{(j)}$, $U_2^{(j)}$, $j = 1, 2$, satisfying the assumptions of Theorem 3.1. Let $(F^{(j)}, U^{(j)})$ be the corresponding solutions. We have the following continuous dependence theorem.

Theorem 4.1. *For all $T > 0$ there exists a constant $C(T)$ such that*

$$\begin{aligned} \|F^{(1)} - F^{(2)}\|_{C^1([0,T])} \leq C(T)\Big\{ \|p_i^{(1)} - p_i^{(2)}\|_T + |F_0^{(1)} - F_0^{(2)}| + \| U_0^{(1)} - U_0^{(2)}\| + \\ + \| U_i^{(1)} - U_i^{(2)}\|_T + |U_1^{(1)} - U_1^{(2)}| + |U_2^{(1)} - U_2^{(2)}|\Big\}. \end{aligned} \tag{4.1}$$

The constant $C(T)$ depends on the quantities defining the class in which the data are selected but not on the choice of the data within such a class.

Proof. As in the last part of the proof of the previous theorem we introduce the functions $W^{(j)} = U^{(j)}(\xi + F^{(j)}(t), t)$, $Z(\xi, t) = W^{(1)}(\xi, t) - W^{(2)}(\xi, t)$, $\psi(t) = \min(F^{(1)}(t), F^{(2)}(t))$, $\Psi(t) = \max(F^{(1)}(t), F^{(2)}(t))$, and we consider the problem

$$\frac{\partial Z}{\partial t} = \gamma \frac{\partial^2 Z}{\partial \xi^2} - \gamma_1 Z + \gamma_1 (U_1^{(1)} - U_1^{(2)}) \,, \qquad -\psi(t) < \xi < 1 \,,\ 0 < t < T, \tag{4.2}$$

$$Z(\xi, 0) = U_0^{(1)}(\xi + F_0^{(1)}) - U_0^{(2)}(\xi + F_0^{(2)}) \,, \quad -\psi(0) < \xi < 0 \,, \tag{4.3}$$

$$Z(-\psi(t), t) = U_i^{(1)}(t) - U_i^{(2)}(t) + \frac{\partial W^{(i)}}{\partial \xi}(\overline{\xi}(t), t)[F^{(1)}(t) - F^{(2)}(t)] \,, \qquad 0 < t < T, \tag{4.4}$$

$$\frac{\partial Z}{\partial \xi}(0, t) = -\gamma_2 Z(0, t) + \gamma_2 (U_2^{(1)} - U_2^{(2)}) \,, \qquad 0 < t < T, \tag{4.5}$$

where in (4.4) $j = 1$ if $F^{(1)}(t) > F^{(2)}(t)$ and $j = 2$ otherwise.

We can split Z into the sum $Z_1 + Z_2$, where Z_1 satisfies (4.2)-(4.5) without the terms containing the differences $U_j^{(1)} - U_j^{(2)}$, $j = 1, 2$, while the problem for Z_2 contains such terms and has zero data in (4.3), (4.4). The norm $\|Z_1\|_T$ is easily estimated by means of the maximum principle. As to Z_2, first we eliminate the term $\gamma_1 Z_2$ in the differential equation taking the function $\widehat{Z}_2 = Z_2 e^{-\gamma_1 t}$. Then we note that

$$|\widehat{Z}_2| \leq \gamma_1 |U_1^{(1)} - U_1^{(2)}| T e^{\gamma_1 T} + Z_3 , \tag{4.6}$$

where Z_3 is the solution of the heat equation in the domain $-\infty < \xi < 0$, $0 < t < T$, with zero initial data and such that

$$\frac{\partial Z_3}{\partial \xi}(0, t) = \gamma_2 \|\widehat{Z}_2\|_t + \gamma_2 |U_2^{(1)} - U_2^{(2)}|. \tag{4.7}$$

Using the well known representation of Z_3, from (4.6) we obtain an integral inequality of Gronwall type with a kernel of Abel type. The conclusion is that

$$\begin{aligned} \| U^{(1)} - U^{(2)} \|_T \leq C_1(T) \Big\{ & \|p_i^{(1)} - p_i^{(2)}\|_T + |F_0^{(1)} - F_0^{(2)}| + \\ & + \| U_0^{(1)} - U_0^{(2)} \| + \| U_i^{(1)} - U_i^{(2)} \|_T + | U_1^{(1)} - U_1^{(2)} | + | U_2^{(1)} - U_2^{(2)} | \Big\}. \end{aligned} \tag{4.8}$$

This information is used in

$$\begin{aligned} \dot{F}^{(1)}(t) - \dot{F}^{(2)}(t) = & \left[3 \int_0^{F^{(1)}(t)} \eta(U^{(1)}(x, t))\, dx \int_0^{F^{(2)}(t)} \eta(U^{(2)}(x, t))\, dx \right]^{-1} \times \\ & \times \left\{ p_i^{(1)}(t) \int_0^{F^{(2)}(t)} \eta(U^{(2)}(x, t))\, dx - p_i^{(2)}(t) \int_0^{F^{(1)}(t)} \eta(U^{(1)}(x, t))\, dx \right\} \end{aligned}$$

in order to get an integral inequality to which Gronwall's theorem is once more applicable. The result is that $\|F^{(1)} - F^{(2)}\|_T$ is estimated by a sum like the r.h.s. of (4.1). Proving (4.1) is now trivial. □

5. Some extensions

We can allow γ_1, U_1 to depend on x, t, provided that they are Hölder continuous in time with their x-derivative in a Hölder class. Similarly, γ_2, U_2 may depend on time and be Hölder continuous. The other requirements on γ_1, γ_2 must be kept. It is possible to relax the condition U_1, $U_2 > 1$, but in that case we can solve the problem as long as we can say that the rescaled temperature U is greater than 1, otherwise the rheology of the system is affected. It is not difficult to find an a-priori estimate on the existence time T for which the inequality $U > 1$ is satisfied.

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