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Zeros and poles of Dirichlet series

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Teoria dei numeri. — *Zeros and poles of Dirichlet series.* Nota di Enrico Bombieri e Alberto Perelli, presentata (*) dal Socio E. Bombieri.

Abstract. — Under certain mild analytic assumptions one obtains a lower bound, essentially of order r, for the number of zeros and poles of a Dirichlet series in a disk of radius r. A more precise result is also obtained under more restrictive assumptions but still applying to a large class of Dirichlet series.

Key words: General Dirichlet series; Almost-periodic functions; Nevanlinna theory.

Riassunto. — *Zeri e poli delle serie di Dirichlet.* Sotto ipotesi molto generali di tipo analitico si dimostra una stima dal basso, essenzialmente di ordine r, per il numero di zeri e poli di una serie di Dirichlet in un cerchio di raggio r. Un risultato più preciso si ottiene sotto ipotesi più restrittive.

1. Results and proofs

For a meromorphic function $f(s)$ in the complex plane, we denote by $n(r, a; f)$ the number of solutions, counted with multiplicity, of the equation $f(s) = a$ in the disk $|s| \le r$, and write as usual

$$\begin{aligned} N(r, a; f) &= \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t}\, dt + n(0, a; f) \log r, \\ m(r, a; f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| d\theta, \\ m(r, \infty; f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \\ T(r, f) &= N(r, \infty; f) + m(r, \infty; f). \end{aligned}$$

The order $\rho(f)$ of $f(s)$ is given by

$$\rho(f) := \overline{\lim_{r \to +\infty}} \frac{\log T(r, f)}{\log r}.$$

By Nevanlinna's first theorem, we have $N(r, a; f) + m(r, a; f) = T(r, f) + O(1)$ for every fixed a. In particular,

$$T(r, f) \ge N(r, a; f) - O(1). \tag{1}$$

An analytic function $f(s)$ of the complex variable s is said to be uniformly almost periodic (briefly, u.a.p.) in a strip $b < \Re(s) < c$ (b and c may be $\pm\infty$) if for every $\varepsilon > 0$ the set of real numbers τ such that

$$|f(s + i\tau) - f(s)| < \varepsilon \quad \text{for} \quad b < \Re(s) < c$$

(*) Nella seduta del 9 febbraio 2001.

is relatively dense, in other words if for every $\varepsilon > 0$ there is an $l > 0$ such that every interval of length l contains such a number τ.

It is well known (see for instance [1, Ch. III, Th. 6, Cor.]) that the sum of an exponential series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad \lambda_n \in \mathbb{R}$$

uniformly convergent in the strip $b < \Re(s) < c$ is u.a.p. there. An immediate consequence of almost periodicity and uniform convergence in a strip is that if the equation $f(s) = a$ is soluble in the strip, then it will have infinitely many solutions, and their imaginary parts will form a relatively dense set; in particular $N(r, a; f) \gg r$. This is a well-known application of Rouché's Theorem (see for example [3, 6. Theorem]), which we repeat for reader's convenience. Let s_0 be a zero of $f(s) - a$ in the strip. Then there exists an $\eta_0 > 0$ such that the circle $C = \{s : |s - s_0| = \eta_0\}$ is contained in the strip and $f(s) \neq a$ there. Take ε to be the minimum of $|f(s) - a|$ along C. By u.a.p., there is l such that every interval of length l contains τ such that $|f(s + i\tau) - f(s)| < \varepsilon$ along C. By Rouché's Theorem, we deduce that $f(s) - a$ and $f(s + i\tau) - a$ have the same number of zeros inside the circle C, proving what we want.

Thus by (1) if $f(s)$ is non-constant and u.a.p. in a strip then

$$T(r, f) \gg r \quad \text{and} \quad \rho(f) \geq 1 \tag{2}$$

as $r \to +\infty$.

We prove the following theorem.

Theorem 1. *Let $f(s) = \sum a_n e^{\lambda_n s}$, $\lambda_n \in \mathbb{R}$, be the sum of an exponential series uniformly convergent in a half-plane $\Re(s) > b$, admitting an analytic continuation in the whole complex plane as a non-constant meromorphic function of finite order. Suppose also that $f(s)$ tends to a non-zero finite limit as $\Re(s) \to +\infty$. Then for any fixed $\gamma < 1$ we have*

$$\overline{\lim_{r \to +\infty}} \frac{N(r, 0; f) + N(r, \infty; f)}{r^{\gamma}} > 0.$$

Proof. We may assume that $f(s) \to 1$ as $\Re(s) \to +\infty$. Since $f(s)$ has finite order, we can write

$$f(s) = \frac{A(s)}{B(s)} e^{h(s)},$$

where $A(s)$ and $B(s)$ are the Weierstrass products associated to the zeros and poles of $f(s)$, and where $h(s)$ is a polynomial. The degree of $h(s)$ and the orders of the entire functions $A(s)$ and $B(s)$ do not exceed the order of $f(s)$.

Let $\omega > 0$ and let τ be the operator

$$\tau f(s) = f(s)/f(s + \omega).$$

Then if q is an integer greater than the degree of $h(s)$ we have

$$f_q(s) := \tau^q f(s) = \frac{\tau^q A(s)}{\tau^q B(s)}$$

because $h(s)$ has degree at most $q-1$ and hence its finite difference of order q vanishes. In particular,

$$\rho(f_q) \le \max(\rho(A)\,,\,\rho(B)). \tag{3}$$

Next, we verify that $f_q(s)$ is not constant. Otherwise we would have $f_q(s) = 1$ identically and since $f_q(s) = f_{q-1}(s)/f_{q-1}(s+\omega)$ the function $f_{q-1}(s)$ would be periodic, with period ω. But $f_{q-1}(s)$ is bounded for $\Re(s)$ sufficiently large, hence $f_{q-1}(s)$ would be bounded everywhere and it would be a constant by Liouville's theorem. Since $f(s) \to 1$ as $\Re(s) \to \infty$, we would get $f_{q-1}(s) = 1$. By descending induction, we would find that $f(s)$ is a constant, which was excluded.

Note also that $f_q(s)$ is again u.a.p. in some right half-plane. Therefore, by (2) and (3) we obtain

$$1 \le \max(\rho(A)\,,\,\rho(B)).$$

On the other hand, if $N(r,0;f) + N(r,\infty;f) \ll r^{\gamma+\varepsilon}$ for any fixed $\varepsilon > 0$, we have $\max(\rho(A)\,,\,\rho(B)) \le \gamma$. Hence $\gamma \ge 1$, proving what we want. □

Remark. A more difficult argument, which we leave to the interested reader, yields the stronger result that on the hypotheses of the theorem the sum $\sum 1/(1+|\rho|)$, taken over all zeros and poles of $f(s)$ counting multiplicities, is divergent. A proof can be obtained using the rather delicate Cartan's Lemma, see [5, I.8.Th.11].

It remains an open question whether the conclusion of the theorem holds with $\gamma = 1$, which would be best possible. One can prove

Theorem 2. *In addition to the hypotheses of Theorem* 1, *suppose that* $f(s)$ *is u.a.p. in some half plane* $\Re(s) < c$ *and* $f(s)$ *tends to a non-zero finite limit as* $\Re(s) \to -\infty$. *Then we have*

$$\overline{\lim_{r\to+\infty}}\ \frac{N(r,0;f) + N(r,\infty;f)}{r} > 0.$$

Proof. Let $k \ge 0$. The function $g(s) = f(k+s)f(k-s)$ is meromorphic, of order at most the order of $f(s)$, and is u.a.p. in some right half-plane. Let again $g_q(s) = \tau^q g(s)$, where q is larger than the degree of $h(s)$. By (2), we have $T(r,g_q) \gg r$ and $\rho(g_q) \ge 1$ provided $g_q(s)$ is not constant, and we can satisfy this condition by choosing k and ω appropriately.

On the other hand, $g_q(s)$ is even; therefore, we have $g_q(s) = \psi(s^2)$ for some meromorphic function $\psi(s)$. Since $T(r,g_q) = T(r^2\,,\,\psi)$, we have

$$\rho(\psi) = \rho(g_q)/2.$$

Note also that $N(r,a;g_q) = N(r^2\,,\,a;\psi)$.

If $\rho(g_q) > 1$, we verify as in (3) that $\rho(g_q) \le \max(\rho(A)\,,\,\rho(B))$ and we end the proof as we did for Theorem 1.

If instead $\rho(g_q) = 1$, we obtain $\rho(\psi) = \frac{1}{2}$. By a theorem of R. Nevanlinna (see for instance [4, Ch. 4, Th. 4.5]) we deduce

$$\overline{\lim_{r\to+\infty}} \frac{N(r,0;\psi)+N(r,\infty;\psi)}{T(r,\psi)} \geq \frac{1}{2}. \tag{4}$$

By (2), we have $T(r^2,\psi)=T(r,g_q)\gg r$; hence using

$$N(r^2,0;\psi)+N(r^2,\infty;\psi)=N(r,0;g_q)+N(r,\infty;g_q)\leq$$
$$\leq 2^{q+1}\big[N(r+k+q|\omega|,0;f)+N(r+k+q|\omega|,\infty;f)\big]$$

and (4) we get Theorem 2. □

2. Concluding remarks

A typical example of function $f(s)$ as in Theorem 2 is the quotient of two L-functions $F(s)$ and $G(s)$ satisfying the same functional equation. In this case, Theorem 2 provides a lower bound for the cardinality $D_{F,G}(T)$ of the symmetric difference of the non-trivial zeros up to T, counted with multiplicity, of such L-functions. In particular, it follows from Theorem 2 that under the above condition

$$D_{F,G}(T)=\Omega(T). \tag{5}$$

Observe that (5) is obtained using only the function-theoretic properties of $F(s)$ and $G(s)$, disregarding their arithmetical aspects. This is, in fact, our viewpoint in Theorems 1 and 2. We recall that (5) has been proved by Murty and Murty [6] for any two distinct L-functions $F(s)$ and $G(s)$ in the framework of the Selberg class [8]. However, the Selberg class deals only with Dirichlet series satisfying a functional equation of standard type and certain additional arithmetic conditions, and these conditions are much more restrictive than those which have been considered here. The better lower bound

$$D_{F,G}(T)\gg T\log T$$

is expected to hold in the Selberg class, which would be best possible.

We conclude by remarking that our results do not imply any lower bound for the cardinality $D(F,G;T)$ of the asymmetric difference of the non-trivial zeros up to T, i.e. the excess of zeros of $F(s)$ over those of $G(s)$, counted with multiplicity. In fact, from our hypotheses we cannot exclude, for example, that $F(s)$ divides $G(s)$. The problem of the asymmetric difference of zeros is studied in [2], where the best possible lower bound

$$D(F,G;T)\gg T\log T$$

is obtained for $F(s)$ and $G(s)$ in a rather general class of L-functions, under some natural conditions needed to exclude divisibility phenomena and an additional technical hypothesis on the density of the off-line zeros. This problem has also been recently investigated in [7] for certain concrete families of L-functions, with the aim of proving that $D(F,G;T)\to\infty$. However, most results in [7] can be obtained as special cases of a general result showing that $D(F, G; T) \to \infty$ for pairs of L-functions in the Selberg class satisfying functional equations of the same degree. A proof of such a result can be obtained by a straightforward analysis of the integral representation of the n-th coefficient of the Dirichlet series $G(s)/F(s)$, using a formula akin to Landau's well-known formula for the Von Mangoldt function $\Lambda(n)$.

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