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## On the unique extension problem for functionals of the calculus of variations

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**Calcolo delle variazioni.** — *On the unique extension problem for functionals of the calculus of variations.* Nota di Luciano Carbone e Riccardo De Arcangelis, presentata (*) dal Socio M. Miranda.

Abstract. — By drawing inspiration from the treatment of the non parametric area problem, an abstract functional is considered, defined for every open set in a given class of open subsets of $\mathbb{R}^n$ and every function in $C^\infty(\mathbb{R}^n)$, and verifying suitable assumptions of measure theoretic type, of invariance, convexity, and lower semicontinuity. The problem is discussed of the possibility of extending it, and of the uniqueness of such extension, to a functional verifying analogous properties, but defined in wider families of open sets and less smooth functions. A suitable extension is constructed, and minimal sufficient conditions for its uniqueness are proposed. The results are applied to some examples in Calculus of Variations.

Key words: Extension of functionals; Uniqueness; Lower semicontinuous envelopes; Inner regular envelopes.

Riassunto. — *Sul problema dell'estensione unica per funzionali del Calcolo delle variazioni.* Traendo ispirazione dalla trattazione del problema dell'area non parametrica, si considera un funzionale astratto, definito per ogni aperto in un assegnato insieme di sottoinsiemi aperti di $\mathbb{R}^n$ ed ogni funzione in $C^\infty(\mathbb{R}^n)$, verificante opportune ipotesi di tipo mensurale, di invarianza, convessità e semicontinuità inferiore, e si discute il problema della sua estendibilità, e dell'unicità di tale estensione, ad un funzionale verificante proprietà analoghe, ma definito su famiglie più ampie di aperti e di funzioni meno regolari. Si costruisce un'opportuna estensione e si forniscono condizioni sufficienti minimali per la sua unicità. I risultati sono applicati a diversi esempi in Calcolo delle variazioni.

## 1. Introduction and presentation of the main results

Starting from the well celebrated example of H.A. Schwarz (in 1880) and G. Peano (in 1882), the problem of the definition of the concept of area of a surface and of the study of its properties, both in the parametric and non-parametric cases, and possibly also in the non continuous framework, interested many important mathematicians.

The researches developed produced a great amount of fruitful ideas and techniques. We refer to the book of Cesari (cf. [8]) for a survey and a bibliography up to 1956, and to [14, 17, 19, 20, 26, 29, and the references quoted therein], even if it must be pointed out that researches on the subject are still in progress.

To analyse the problem, various kinds of approaches were proposed, among which also some of axiomatic type in which conditions on an abstract functional, defined on sets of «generalized surfaces» and furnishing the value of the area on the smooth ones, were proposed in order to identify it as unique extension to non smooth surfaces. These last approaches were essentially based on the topological (*e.g.* lower semicontinuity) and the measure theoretic properties of the area functional.

(*) Nella seduta del 12 gennaio 2001.

The success of Caccioppoli and De Giorgi definition (cf. [4, 11, 12]), and the equivalence established between this and the Lebesgue one for continuous surfaces (cf. [27] for the equivalence result, see also [28] and [13] for a historical sketch), induced mathematicians to partially shelve the problem of area definition.

About this question it is useful to report an old Caccioppoli's idea on the intrinsic uniqueness of the different definitions (cf. [3]): «Per un'altra ragione ancora ho rinunziato ad analizzare qui i rapporti fra la mia definizione e quelle di Lebesgue e Radò: perché ritengo che l'identità che affermo non sia casuale, come potrebbe far credere una faticosa verifica diretta, ma dipenda da circostanze semplici ed assolutamente generali. Per una curva non v'ha che una definizione ammissibile di lunghezza: cioè il funzionale *lunghezza* ammette, a partire del campo delle poligonali, un *unico prolungamento per semicontinuità inferiore*. Un risultato analogo deve potersi stabilire per le superficie, previa opportuna definizione della *proprietà additiva* del funzionale *area*.

«Lo scetticismo che desta spesso la molteplicità delle definizioni di area sarebbe però infondato: per l'area *interna*, dotata cioè della semicontinuità inferiore, proprietà che ci si accorda oggi tutti a ritenere fondamentale, vi sarebbe un'unica determinazione *ammissibile*, cui condurrebbe ogni definizione che avesse un minimo di *verosimiglianza* geometrica, conferitole dalla proprietà additiva».

In this *Note*, we want to make some remarks in order to obtain uniqueness of the extension for classes of functionals, including the area one, in an axiomatic context. So, having in mind the non-parametric area case, and by studying in depth some remarks in [5], we enlarge the classical point of view by keeping into account also a vectorial property of the area functional: the convexity.

Then, we consider an abstract functional, say $F$, given on a collection of elementary smooth functions and open sets, and propose sets of conditions fulfilled by $F$ that select classes of functionals, defined on spaces of less smooth functions and open sets, in which $F$ possesses a unique extension. This (unique) extension turns out to be strongly linked to the relaxed functional of $F$ in the $L^1$-topology introduced, in the case of integral functionals, in [31] and [32], and represented in [22] (for an exposition on relaxation theory see also [2, 16, 29]). We point out that the idea of relaxation seems to have been introduced by Lebesgue for the non-parametric area functional in the framework of uniform convergence, by Fréchet in an abstract context, and essentially by Caccioppoli again in the surface area context but in the framework of $L^1$ convergence (for this case see also [21]).

To present the results precisely, we need to give some definitions.

Let us denote by $\mathcal{A}_0$ the set of the bounded open subsets of $\mathbb{R}^n$.

For every $A, B \in \mathcal{A}_0$ we write $A \subset\subset B$ if $\overline{A}$ is a compact subset of $B$.

For every $\mathcal{O} \subseteq \mathcal{A}_0$, every set $U$, $\Phi \colon \mathcal{O} \times U \to [0, +\infty]$, and $\mathcal{E} \subseteq \mathcal{O}$, we introduce the $\mathcal{E}$-inner regular envelope $\Phi_{\mathcal{E}_-}$ of $\Phi$ as the function defined by

$$\Phi_{\mathcal{E}_-} \colon (\Omega, u) \in \mathcal{A}_0 \times U \mapsto \begin{cases} 0 & \text{if } \{A \in \mathcal{E} : A \subset\subset \Omega\} = \emptyset \\ \sup\{\Phi(A, u) : A \in \mathcal{E}, A \subset\subset \Omega\} & \text{if } \{A \in \mathcal{E} : A \subset\subset \Omega\} \neq \emptyset, \end{cases}$$

and say that $\Phi$ is $\mathcal{E}$-inner regular, or simply inner regular when $\mathcal{E} = \mathcal{O}$, if

$$\Phi(\Omega, u) = \Phi_{\mathcal{E}\_}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{O} \times U.$$

For every function $u$ on $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ we set

$$T[x_0]u \colon x \in \mathbb{R}^n \mapsto u(x + x_0).$$

If $\mathcal{O}$ is such that $x_0 + \Omega \in \mathcal{O}$ whenever $x_0 \in \mathbb{R}^n$, $\Omega \in \mathcal{O}$, and $U$ is a set of functions on $\mathbb{R}^n$ such that

$$T[x_0]u \in U \text{ whenever } u \in U, \ x_0 \in \mathbb{R}^n, \tag{1.1}$$

we say that $\Phi$ is translation invariant if

$$\Phi(\Omega - x_0, T[x_0]u) = \Phi(\Omega, u) \text{ for every } \Omega \in \mathcal{O}, \ x_0 \in \mathbb{R}^n, \ u \in U.$$

For every topological vector space $U$, we say that $\Phi$ is convex if for every $\Omega \in \mathcal{O}$, $\Phi(\Omega, \cdot)$ is convex, and say that $\Phi$ is $U$-lower semicontinuous if for every $\Omega \in \mathcal{O}$, $\Phi(\Omega, \cdot)$ is $U$-lower semicontinuous.

We observe that the above notions are classical in the framework of area definition. Indeed the notion of inner regular envelope is of measure theoretic nature, the one of translation invariance is of geometric type (cf. [18, 24, 23]), and the one of lower semicontinuity is classical and well recognized when dealing with extension procedures (cf. [18]). We also point out that the notion of convexity is linked to energy and statistics type considerations: in fact the convexity property that we will exploit is essentially the feature of a functional to take values on averages of configurations smaller than the corresponding average of the ones on the single configurations (Jensen inequality).

Finally, we denote again by $L^1_{\rm loc}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$ the usual topologies of $L^1_{\rm loc}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n)$ that make them Fréchet spaces, *i.e.* metrizable complete locally convex topological vector spaces, and, for every $\mathcal{E}_0 \subseteq \mathcal{A}_0$ and $\Phi \colon \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to [0, +\infty]$, we denote by $\overline{\Phi}$ the relaxed functional of $\Phi$ in the $L^1_{\rm loc}(\mathbb{R}^n)$-topology defined by

$$\overline{\Phi} \colon (\Omega, u) \in \mathcal{E}_0 \times L^1_{\rm loc}(\mathbb{R}^n) \mapsto \inf \left\{ \liminf_{h \to +\infty} \Phi(\Omega, u_h) : \{u_h\} \subseteq C^\infty(\mathbb{R}^n), \, u_h \to u \text{ in } L^1_{\rm loc}(\mathbb{R}^n) \right\},$$

*i.e.* the greatest $L^1_{\rm loc}(\mathbb{R}^n)$-lower semicontinuous functional on $L^1_{\rm loc}(\mathbb{R}^n)$ less than or equal to $\Phi(\Omega, \cdot)$ on $C^\infty(\mathbb{R}^n)$.

We recall that, given $\Phi$ as above, $\overline{\Phi}$ is $L^1_{\rm loc}(\mathbb{R}^n)$-lower semicontinuous.

For every $\mathcal{D}$, $\mathcal{O} \subseteq \mathcal{A}_0$ we say that $\mathcal{D}$ is dense with respect to $\mathcal{O}$ if for every $A$, $B \in \mathcal{O}$ with $A \subset\subset B$ there exists $D \in \mathcal{D}$ such that $A \subset\subset D \subset\subset B$.

The extension uniqueness result is then the following.

Let $\mathcal{E}_0 \subseteq \mathcal{A}_0$ be dense with respect to $\mathcal{A}_0$ and satisfying

$$x_0 + \Omega \in \mathcal{E}_0 \text{ whenever } x_0 \in \mathbb{R}^n, \ \Omega \in \mathcal{E}_0, \tag{1.2}$$

and let $F \colon \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to [0, +\infty]$ be inner regular, translation invariant, convex, and $C^\infty(\mathbb{R}^n)$-lower semicontinuous. Then, for every $\mathcal{E} \subseteq \mathcal{A}_0$ satisfying

$$x_0 + \Omega \in \mathcal{E} \text{ whenever } x_0 \in \mathbb{R}^n, \ \Omega \in \mathcal{E}, \tag{1.3}$$

such that $\mathcal{E}_0 \subseteq \mathcal{E}$, and for every Hausdorff locally convex topological vector space $U$ satisfying (1.1), and

$$C^\infty(\mathbb{R}^n) \subseteq U \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^n)\,, \tag{1.4}$$

(1.5) $C^\infty(\mathbb{R}^n)$ is finer than the topology of $U$, the topology of $U$ is finer than $L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\text{for every } u \in U,\ \ y \in \mathbb{R}^n \mapsto T[y]u \in U \text{ is continuous}\,, \tag{1.6}$$

the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E} \times U$ is the only inner regular, translation invariant, convex, $U$-lower semicontinuous functional from $\mathcal{E} \times U$ to $[0\,, +\infty]$ that agrees with $F$ on $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$ (cf. Theorem 4.6).

We point out that the above result can be no more true if the convexity assumption is dropped, as already observed in [1] and [5].

Roughly speaking, we can say that convexity gives the necessary uniformity to ensure essentially that the relaxed functional of $F$ in the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-topology is the unique semicontinuous extension of $F$.

As corollary, we deduce a unique extension result for the integral functional

$$F\colon (\Omega\,, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \mapsto \int_\Omega f(u\,, \nabla u\,, \nabla^2 u\,, \dots\,, \nabla^k u)\,dx\,,$$

where $k \in \mathbb{N}$, $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \ldots \times \mathbb{R}^{n^k} \to [0\,, +\infty]$ is convex and lower semicontinuous, and $\nabla^k$ is the $k$-th partial derivatives operator (cf. Proposition 6.1).

We also give a result on the uniqueness and the representation of the extension of the integral functional

$$F\colon (\Omega\,, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \mapsto \int_\Omega f(\nabla u)\,dx\,,$$

where $f\colon \mathbb{R}^n \to [0\,, +\infty]$ is convex and lower semicontinuous. Indeed, we prove that the functional

$$\widetilde{F}\colon (\Omega\,, u) \in \mathcal{A}_0 \times BV_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right) dD^s u$$

is the only inner regular, translation invariant, convex, $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous extension of $F$ to $\mathcal{A}_0 \times BV_{\mathrm{loc}}(\mathbb{R}^n)$, where $BV_{\mathrm{loc}}(\mathbb{R}^n)$, $f^\infty$, and the other symbols are defined in section 2.2 (cf. Proposition 6.2).

Eventually, we deduce that the functional

$$\widehat{A}\colon (\Omega\,, u) \in \mathcal{A}_0 \times L^1_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \begin{cases} \displaystyle\int_\Omega \sqrt{1 + |\nabla u|^2}\,dx + |D^s u|(\Omega) & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \setminus BV(\Omega) \end{cases}$$

is the only inner regular, translation invariant, convex, $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous functional on $\mathcal{A}_0 \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ equal to the non-parametric area functional on elementary open sets and functions (cf. Corollary 6.3).

We point out that the above results are still valid if we replace the inner regularity assumption with the requirement that for every $u \in C^\infty(\mathbb{R}^n)$, $F(\cdot\,,u)$ is the restriction of a Borel measure. In this case the extension is unique in the smaller class of the functionals that, for every $u \in U$, are restrictions of Borel measures.

For close results, but from a different point of view, and with different aims, cf. also [10, Chapter 23].

## 2. Preliminaries

### 2.1. *Some remarks on the definition of a functional on functions and on their equivalence classes.*

In the following we will consider functions and equivalence classes of functions that we will need to compare. To do this it is necessary to make some simple considerations.

First of all we recall that $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a space of equivalence classes of functions defined on $\mathbb{R}^n$, being two such functions equivalent if they agree everywhere on $\mathbb{R}^n$ except possibly for a set of Lebesgue zero measure, and that, as usual, its elements are thought as functions defined almost everywhere in $\mathbb{R}^n$. Thus, when considering a subspace $W$ of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we will regard its elements as equivalence classes of summable functions on $\mathbb{R}^n$, or to functions defined almost everywhere in $\mathbb{R}^n$.

In particular this holds when $W = C^\infty(\mathbb{R}^n)$.

On the other side, $C^\infty(\mathbb{R}^n)$, especially if endowed with the $C^\infty(\mathbb{R}^n)$-topology, is naturally a space of functions defined everywhere in $\mathbb{R}^n$, therefore a way to identify its elements with their equivalence classes, and to introduce the corresponding topology on this set, is needed.

To do this, let us denote, for the moment and for sake of clearness, by $C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$ the set of the $C^\infty$-functions on $\mathbb{R}^n$, and by $C^\infty_{\mathrm{cls}}(\mathbb{R}^n)$ the one of the equivalence classes of the elements of $C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$. Then it is obvious that for every $\mathbf{u} \in C^\infty_{\mathrm{cls}}(\mathbb{R}^n)$ there exists a unique $\mathcal{J}\mathbf{u} \in C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$ such that $\mathcal{J}\mathbf{u} \in \mathbf{u}$.

By virtue of this, the map $\mathcal{J} : \mathbf{u} \in C^\infty_{\mathrm{cls}}(\mathbb{R}^n) \mapsto \mathcal{J}\mathbf{u} \in C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$ turns out to be well defined, linear, and one to one. Consequently $\{\mathcal{J}^{-1}(A) : A \text{ open set in } C^\infty(\mathbb{R}^n)\}$ turns out to be a topology on $C^\infty_{\mathrm{cls}}(\mathbb{R}^n)$ that makes it a Fréchet space, and $\mathcal{J}$ an isomorphism between topological vector spaces.

Then, given $F : C^\infty_{\mathrm{fct}}(\mathbb{R}^n) \to [0\,,+\infty]$, we identify it with the functional $F_{\mathrm{cls}} = F \circ \mathcal{J}$ defined on $C^\infty_{\mathrm{cls}}(\mathbb{R}^n)$ preserving its vectorial and topological properties, and keep to denote $F_{\mathrm{cls}}$ by $F$.

So, given $u \in C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$, we allow $F$ to act directly on all the functions in $\mathcal{J}^{-1}u$, by defining $F(v) = F(u)$ for every $v \in \mathcal{J}^{-1}u$. In this sense, we can say that if $u \in C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that $v = u$ a.e. in $\mathbb{R}^n$, then $F(v) = F(u)$.

Obviously, now $C^\infty_{\mathrm{fct}}(\mathbb{R}^n)$ and $C^\infty_{\mathrm{cls}}(\mathbb{R}^n)$ can be identified and denoted by $C^\infty(\mathbb{R}^n)$.

This standard identification procedure is fundamental: it allows to translate problems defined on regular classes of functions into «regular» Lebesgue equivalence classes.

We also point out that in some situations such identification procedure is impracticable. For example, the classical total variation functional can produce different values when evaluated on two functions, one of which possibly smooth, differing just in one point.

2.2. *Notations and recalls.*

Let $\Omega$ be an open subset of $\mathbb{R}^n$. By $BV(\Omega)$ we denote the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Borel measures with bounded total variations in $\Omega$.

For every $u \in BV(\Omega)$, we denote the $\mathbb{R}^n$-valued measure gradient of $u$ by $Du$, and the total variation of $Du$ by $|Du|$. Moreover, according to Lebesgue decomposition theorem, we have $Du(E) = \int_E \nabla u dx + D^s u(E)$ for every Borel subset $E$ of $\Omega$, where $\nabla u$ is the density of the absolutely continuous part of $Du$, and $D^s u$ is the singular part of $Du$, both with respect to Lebesgue measure. We also denote by $\frac{dD^s u}{d|D^s u|}$ the Radon-Nikodym derivative of $D^s u$ with respect to its total variation $|D^s u|$.

By $BV_{\text{loc}}(\mathbb{R}^n)$ we denote the set of the functions in $L^1_{\text{loc}}(\mathbb{R}^n)$ that are in $BV(\Omega)$ for every $\Omega \in \mathcal{A}_0$.

We refer, for example, to [20] and [34] for a survey on $BV$ spaces, here we only recall that $BV_{\text{loc}}(\mathbb{R}^n)$ is a Fréchet space.

It is well known that, for every $f : \mathbb{R}^n \to [0, +\infty]$ convex and $z_0 \in \mathbb{R}^n$ such that $f(z_0) < +\infty$, the limit $\lim_{t \to +\infty} \frac{f(z_0+tz)-f(z_0)}{t}$ exists for every $z \in \mathbb{R}^n$, therefore we define the recession function of $f$ by

$$f^\infty : z \in \mathbb{R}^n \mapsto \lim_{t \to +\infty} \frac{f(z_0 + tz) - f(z_0)}{t}.$$

We recall that the definition of $f^\infty$ does not depend on $z_0$ when it varies in the set where $f$ is finite.

2.3. *Integral of functions with values in locally convex topological vector spaces.*

For any subset $E$ of $\mathbb{R}^n$ we denote by $|E|$ the Lebesgue measure of $E$.

In the sequel we will make use of the notion of integral of a function with values in a topological vector space (cf. [30]).

Let $W$ be a Hausdorff locally convex topological vector space, and $\{p_\theta\}_{\theta \in \mathcal{T}}$ be a family of seminorms defining the topology of $W$.

Definition 2.1. *Let $E$ be a Lebesgue measurable subset of $\mathbb{R}^n$, and $f : E \to W$. We say that $f$ is $W$-integrable on $E$ if for every Lebesgue measurable subset $S$ of $E$, $u(S) \in W$ can be found such that for every $\theta \in \mathcal{T}$ and $\eta > 0$ there exists a subdivision $\{B_{S,\theta,\eta,j}\}_{j \in \mathbb{N}}$ of $E$ into measurable, pairwise disjoint sets whose union is $E$, and $J_{S,\theta,\eta} \subseteq \mathbb{N}$ finite such that, whenever $J \subseteq \mathbb{N}$ is finite and contains $J_{S,\theta,\eta}$, it results*

$$\sup \left\{ p_\theta \left( \sum_{j \in J} f(y_j) |S \cap B_{S,\theta,\eta,j}| - u(S) \right) : y_j \in B_{S,\theta,\eta,j} \text{ for every } j \in J \right\} < \eta .$$

*The vector $u(S)$ is the value of the integral of $f$ on $S$, and is denoted by $(W) \int_S f(y) dy$.*

Remark 2.2. It is clear that, if $V$ is another Hausdorff locally convex topological vector space containing $W$ and having a topology less fine than the one of $W$, if $E$ is a Lebesgue measurable subset of $\mathbb{R}^n$, and if $f: E \to W$ is $W$-integrable on $E$, then $f$ turns out to be also $V$-integrable on $E$, and

$$(V)\int_E f(y)dy = (W)\int_E f(y)dy.$$

The following result is proved in [30, Corollary 5.2].

Theorem 2.3. *Let $W$ be a Hausdorff locally convex topological vector space, $E$ be Lebesgue measurable, $f: E \to W$ be $W$-integrable on $E$, and $L \in W'$. Then $\langle L, f\rangle$ is Lebesgue summable on $E$, and*

$$\int_E \langle L, f(y)\rangle dy = \left\langle L, (W)\int_E f(y)dy \right\rangle .$$

We now recall the definition of regularization of a function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$, $\rho \in C^\infty(\mathbb{R}^n)$ be such that $\rho \geq 0$, $\mathrm{spt}(\rho) \subseteq B_1$, and $\int_{\mathbb{R}^n} \rho(y)dy = 1$. For every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $\varepsilon > 0$ we define the regularization $u_\varepsilon$ of $u$ as

$$u_\varepsilon : x \in \mathbb{R}^n \mapsto \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right) u(y)dy. \tag{2.1}$$

It is well known that $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, whenever $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $\varepsilon > 0$.

We now study the convergence properties of the regularizations of a function with values in a locally convex topological vector subspace $W$ of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 2.4. *Let $W$ be a Hausdorff locally convex topological vector space satisfying* (1.1), (1.4)-(1.6) *with $U = W$. Let $\rho$ be the function appearing in* (2.1), *then, for every $u \in W$ and $\varepsilon > 0$, $\rho(\cdot)T[\varepsilon\cdot]u$ is $W$-integrable on $\mathbb{R}^n$, and*

$$\left((W)\int_{\mathbb{R}^n} \rho(y)T[\varepsilon y]u dy\right)(x) = u_\varepsilon(x) \text{ for a.e. } x \text{ in } \mathbb{R}^n ,$$

*$u_\varepsilon$ being defined in* (2.1).

Proof. Let us denote by $\widehat{W}$ the completion of $W$, then $\widehat{W}$ is a complete Hausdorff locally convex topological vector space such that, by the right-hand side of (1.4) and the right-hand side of (1.5) with $U = W$, $\widehat{W} \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $u \in W$, $\varepsilon > 0$, then (cf. [9, Proposition 3.1]) $\rho(\cdot)T[\varepsilon\cdot]u$ is $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-integrable on $\mathbb{R}^n$, and

$$\left((L^1_{\mathrm{loc}}(\mathbb{R}^n))\int_{\mathbb{R}^n} \rho(y)T[\varepsilon y]u dy\right)(x) = u_\varepsilon(x) \text{ for a.e. } x \text{ in } \mathbb{R}^n. \tag{2.2}$$

Moreover, we also recall that (cf. for example [9, Proposition 2.4]), being $\widehat{W}$ a sequentially complete Hausdorff locally convex topological vector space, and, by (1.6) with $U = W$, $\rho(\cdot)T[\varepsilon\cdot]u$ continuous with compact support, $\rho(\cdot)T[\varepsilon\cdot]u$ turns out to be $\widehat{W}$-integrable on $\mathbb{R}^n$.

On the other side, by the right-hand side of (1.4) and the right-hand side of (1.5) with $U = W$, and by Remark 2.2, we have that

$$(\widehat{W})\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy = \big(L^1_{\mathrm{loc}}(\mathbb{R}^n)\big)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy, \tag{2.3}$$

from which, together with (2.2), we deduce that $(\widehat{W})\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy \in C^\infty(\mathbb{R}^n)$.

By virtue of this, and by the left-hand side of (1.4) with $U = W$, we get that $\rho(\cdot)\, T[\varepsilon\cdot]u$ is actually $W$-integrable on $\mathbb{R}^n$, and that

$$(W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy = \big(\widehat{W}\big)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy,$$

from which, together with (2.3) and (2.2), the thesis follows. □

Proposition 2.5. *Let $W$ be a Hausdorff locally convex topological vector space satisfying* (1.1), (1.4)-(1.6) *with $U = W$. For every $u \in W$ and $\varepsilon > 0$ let $u_\varepsilon$ be defined by* (2.1), *then for every $u \in W$, $\{u_\varepsilon\}_{\varepsilon>0}$ converges to $u$ in $W$ as $\varepsilon$ tends to* 0.

Proof. Let $u \in W$, then Proposition 2.4 yields that for every $\varepsilon > 0$, $\rho(\cdot)\, T[\varepsilon\cdot]u$ is $W$-integrable on $\mathbb{R}^n$.

Let $\{p_\theta\}_{\theta\in\mathcal{T}}$ be a family of seminorms defining the topology of $W$, $\theta \in \mathcal{T}$, $\eta > 0$, and let $\varepsilon > 0$. Then, since $(W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy$ belongs to the closure in $W$ of the convex hull of $\{T[\varepsilon y]u : y \in B_1\}$ (cf. [9, Proposition 3.2]), we can find $\lambda_1, \ldots, \lambda_m \in [0, 1]$ with $\sum_{j=1}^m \lambda_j = 1$, and $y_1, \ldots, y_m \in B_1$ such that

$$p_\theta\left((W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy - \sum_{j=1}^m \lambda_j T[\varepsilon y_j]u\right) < \frac{\eta}{2}. \tag{2.4}$$

Moreover, by (1.6) with $U = W$, there exists $\varepsilon_{\theta,\eta} > 0$ such that

$$\sup\{p_\theta(T[\varepsilon y]u - u) : y \in B_1\} < \frac{\eta}{2} \text{ for every } \varepsilon \in ]0, \varepsilon_{\theta,\eta}[. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we conclude that

$$p_\theta\left((W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy - u\right) \le$$
$$\le p_\theta\left((W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy - \sum_{j=1}^m \lambda_j T[\varepsilon y_j]u\right) + p_\theta\left(\sum_{j=1}^m \lambda_j T[\varepsilon y_j]u - \sum_{j=1}^m \lambda_j u\right) <$$
$$< \frac{\eta}{2} + \sum_{j=1}^m \lambda_j p_\theta(T[\varepsilon y_j]u - u) < \eta \text{ for every } \varepsilon \in ]0, \varepsilon_{\theta,\eta}[,$$

that is the convergence in $W$ of $\{(W)\int_{\mathbb{R}^n} \rho(y)\, T[\varepsilon y]u dy\}_{\varepsilon>0}$ to $u$ as $\varepsilon$ goes to 0.

By virtue of this, and by Proposition 2.4, the thesis follows. □

Eventually, we prove the following Jensen type inequality in the framework of locally convex topological vector spaces.

As usual, for every $f\colon \mathbb{R}^n \to [0, +\infty]$, the symbol $\int_{*\mathbb{R}^n} f(y)\,dy$ denotes the lower integral of $f$ on $\mathbb{R}^n$.

Theorem 2.6. *Let $W$ be a Hausdorff locally convex topological vector space, $\Phi\colon W \to [0, +\infty]$ be $W$-lower semicontinuous, and $\rho\colon \mathbb{R}^n \to [0, +\infty[$ be Lebesgue measurable and such that $\int_{\mathbb{R}^n} \rho(y)\,dy = 1$. Then $\Phi$ is convex if and only if for every $w\colon \mathbb{R}^n \to W$ such that $\rho w$ is $W$-integrable on $\mathbb{R}^n$ it results*

$$\Phi\left((W)\int_{\mathbb{R}^n} w(y)\rho(y)\,dy\right) \le \int_{*\mathbb{R}^n} \Phi(w(y))\rho(y)\,dy. \tag{2.6}$$

Proof. Let us assume first that $\Phi$ is convex. In this case the proof is based on well known separation arguments, and follows, for example, the outlines of the proof of [10, Lemma 23.2].

Let $w\colon \mathbb{R}^n \to W$ be such that $\rho w$ is $W$-integrable on $\mathbb{R}^n$. By the convexity and the $W$-lower semicontinuity of $\Phi$, and by separation arguments in locally convex topological vector spaces (cf. for example [33, Chapter 18; 16, Proposition 3.1: p. 14]), it follows that for every $t < \Phi((W)\int_{\mathbb{R}^n} w(y)\rho(y)\,dy)$ there exists $L \in W'$ and $c \in \mathbb{R}$ such that

$$t < \left\langle L, (W)\int_{\mathbb{R}^n} w(y)\rho(y)\,dy \right\rangle + c, \quad \langle L, v\rangle + c \le \Phi(v) \text{ for every } v \in W. \tag{2.7}$$

Consequently, by the right-hand side of (2.7) we deduce that

$$\langle L, w(y)\rangle + c \le \Phi(w(y)) \text{ for every } y \in \mathbb{R}^n,$$

from which, once observed that the $W$-integrability of $\rho w$ and Theorem 2.3 imply the summability of $\langle L, w(\cdot)\rho(\cdot)\rangle$, we have

$$\int_{\mathbb{R}^n} \langle L, w(y)\rangle \rho(y)\,dy + c \le \int_{*\mathbb{R}^n} \Phi(w(y))\rho(y)\,dy. \tag{2.8}$$

In conclusion, by the left-hand side of (2.7), Theorem 2.3, and (2.8), it turns out that

$$t < \left\langle L, (W)\int_{\mathbb{R}^n} w(y)\rho(y)\,dy \right\rangle + c = \int_{\mathbb{R}^n} \langle L, w(y)\rangle \rho(y)\,dy + c \le \int_{*\mathbb{R}^n} \Phi(w(y))\rho(y)\,dy,$$

from which (2.6) follows.

Let us assume now that (2.6) holds for every $w\colon \mathbb{R}^n \to W$ such that $\rho w$ is $W$-integrable on $\mathbb{R}^n$. Let $w_1, w_2 \in W$, $t \in [0, 1]$, $E \subseteq \mathbb{R}^n$ be Lebesgue measurable and such that $\int_E \rho(y)\,dy = t$, and let $w = \chi_E w_1 + \chi_{\mathbb{R}^n \setminus E} w_2$, then $\rho w$ turns out to be $W$-integrable on $\mathbb{R}^n$, $\Phi(w(\cdot))\rho(\cdot)$ Lebesgue measurable, and, by (2.6),

$$\Phi\left(tw_1 + (1-t)w_2\right) = \Phi\left((W)\int_{\mathbb{R}^n} w(y)\rho(y)\,dy\right) \le \int_{\mathbb{R}^n} \Phi(w(y))\rho(y)\,dy = t\Phi(w_1) + (1-t)\Phi(w_2),$$

that is the convexity of $\Phi$. $\square$

## 3. Some measure theoretic results

Let $\mathcal{O} \subseteq \mathcal{A}_0$.

Given $\mathcal{P} \subseteq \mathcal{A}_0$, we say that $\mathcal{P}$ is perfect with respect to $\mathcal{O}$ if for every $\Omega \in \mathcal{P}$, $A \in \mathcal{O}$ with $A \subset\subset \Omega$ there exists $P \in \mathcal{P}$ such that $A \subset\subset P \subset\subset \Omega$.

It is clear that if $\mathcal{D}$ is dense with respect to $\mathcal{O}$, and $\mathcal{D} \subseteq \mathcal{O}$, then $\mathcal{D}$ is also perfect with respect to $\mathcal{O}$.

Let now $\alpha \colon \mathcal{O} \to [0, +\infty]$.

We say that $\alpha$ is increasing if

$$\alpha(\Omega_1) \leq \alpha(\Omega_2) \text{ for every } \Omega_1, \ \Omega_2 \in \mathcal{O} \text{ such that } \Omega_1 \subseteq \Omega_2.$$

For every $\mathcal{E} \subseteq \mathcal{O}$, we define the $\mathcal{E}$-inner regular envelope $\alpha_{\mathcal{E}-}$ of $\alpha$ as the function defined by

$$\alpha_{\mathcal{E}-} \colon \Omega \in \mathcal{A}_0 \mapsto \begin{cases} 0 & \text{if } \{A \in \mathcal{E} : A \subset\subset \Omega\} = \emptyset \\ \sup\{\alpha(A) : A \in \mathcal{E}, \ A \subset\subset \Omega\} & \text{if } \{A \in \mathcal{E} : A \subset\subset \Omega\} \neq \emptyset, \end{cases}$$

and say that $\alpha$ is $\mathcal{E}$-inner regular, or simply inner regular when $\mathcal{E} = \mathcal{O}$, if

$$\alpha(\Omega) = \alpha_{\mathcal{E}-}(\Omega) \text{ for every } \Omega \in \mathcal{O}.$$

It is clear that, given $\mathcal{E} \subseteq \mathcal{O}$, $\alpha_{\mathcal{E}-}$ is increasing.

Proposition 3.1. *Let $\mathcal{O} \subseteq \mathcal{A}_0$, and $\alpha \colon \mathcal{O} \to [0, +\infty]$. Then*

*a) if $\mathcal{P} \subseteq \mathcal{A}_0$ is perfect with respect to $\mathcal{O}$,*

$$\left(\alpha_{\mathcal{O}-}\right)_{\mathcal{P}-}(\Omega) = \alpha_{\mathcal{O}-}(\Omega) \text{ for every } \Omega \in \mathcal{P},$$

*b) if $\alpha$ is increasing, $\mathcal{D} \subseteq \mathcal{O}$, and $\mathcal{D}$ is dense with respect to $\mathcal{O}$,*

$$\alpha_{\mathcal{O}-}(\Omega) = \alpha_{\mathcal{D}-}(\Omega) \text{ for every } \Omega \in \mathcal{O}.$$

Proof. Let us prove *a*).

Being $\alpha_{\mathcal{O}-}$ increasing, it is clear that

$$\left(\alpha_{\mathcal{O}-}\right)_{\mathcal{P}-}(\Omega) \leq \alpha_{\mathcal{O}-}(\Omega) \text{ for every } \Omega \in \mathcal{A}_0. \tag{3.1}$$

On the other side, let $\Omega \in \mathcal{P}$, and $A \in \mathcal{O}$ with $A \subset\subset \Omega$, then, being $\mathcal{P}$ perfect with respect to $\mathcal{O}$, there exists $B \in \mathcal{P}$ such that $A \subset\subset B \subset\subset \Omega$. Therefore we have

$$\alpha(A) \leq \alpha_{\mathcal{O}-}(B) \leq \left(\alpha_{\mathcal{O}-}\right)_{\mathcal{P}-}(\Omega) \text{ for every } \Omega \in \mathcal{P},$$

from which, together with (3.1), condition *a*) follows.

Let us prove *b*).

Since $\mathcal{D}$ is dense with respect to $\mathcal{O}$, and $\alpha$ is increasing, it is easy to deduce that

$$\alpha(A) \leq \alpha_{\mathcal{D}-}(\Omega) \text{ for every } \Omega, \ A \in \mathcal{O} \text{ with } A \subset\subset \Omega,$$

from which it follows that

$$\alpha_{\mathcal{O}-}(\Omega) \leq \alpha_{\mathcal{D}-}(\Omega) \text{ for every } \Omega \in \mathcal{O}. \tag{3.2}$$

By (3.2), being $\mathcal{D} \subseteq \mathcal{O}$ and consequently

$$\alpha_{\mathcal{D}-}(\Omega) \leq \alpha_{\mathcal{O}-}(\Omega) \text{ for every } \Omega \in \mathcal{A}_0 ,$$

condition *b*) follows. □

Let $\mathcal{O} \subseteq \mathcal{A}_0$.

We introduce the following condition

$$\Omega \setminus \overline{A} \in \mathcal{O} \text{ for every } \Omega ,\ A \in \mathcal{O} \text{ such that } A \subset\subset \Omega. \tag{3.3}$$

Moreover, given $\{A_n\} \subseteq \mathcal{O}$, and $\Omega \in \mathcal{O}$ such that $A_n \subseteq \Omega$ for every $n \in \mathbb{N}$, we say that $\{A_n\}$ is well increasing to $\Omega$ if $A_n \subset\subset A_{n+1}$ for every $n \in \mathbb{N}$, and $\cup_{n=1}^{\infty} A_n = \Omega$. We say that $\{A_n\}$ is well decreasing to the empty set with respect to $\Omega$ if $\{\Omega \setminus A_n\}$ is well increasing to $\Omega$.

Let now $\alpha \colon \mathcal{O} \to [0\,, +\,\infty]$. We recall the notions of superadditivity and subadditivity (cf. [15]), and introduce some variants of them that will be useful in the sequel.

We say that $\alpha$ is superadditive if

$$\alpha(\Omega_1) + \alpha(\Omega_2) \leq \alpha(\Omega) \text{ for every } \Omega ,\ \Omega_1 ,\ \Omega_2 \in \mathcal{O} \text{ with } \Omega_1 \cup \Omega_2 \subseteq \Omega \text{ and } \Omega_1 \cap \Omega_2 = \emptyset ,$$

subadditive if

$$\alpha(\Omega) \leq \alpha(\Omega_1) + \alpha(\Omega_2) \text{ for every } \Omega ,\ \Omega_1 ,\ \Omega_2 \in \mathcal{O} \text{ such that } \Omega \subseteq \Omega_1 \cup \Omega_2 ,$$

boundary superadditive if

$$\alpha(A) + \alpha(\Omega \setminus \overline{B}) \leq \alpha(\Omega) \text{ for every } \Omega ,\ A ,\ B \in \mathcal{O} \text{ such that } A \subset\subset B \subset\subset \Omega ,$$

boundary subadditive if

$$\alpha(\Omega) \leq \alpha(B) + \alpha(\Omega \setminus \overline{A}) \text{ for every } \Omega ,\ A ,\ B \in \mathcal{O} \text{ such that } A \subset\subset B \subset\subset \Omega.$$

PROPOSITION 3.2. *Let* $\mathcal{O} \subseteq \mathcal{A}_0$ *satisfy* (3.3), *and* $\alpha \colon \mathcal{O} \to [0\,, +\,\infty]$. *Assume that* $\alpha$ *is inner regular, and boundary superadditive. Then*

*i*) *for every* $\Omega \in \mathcal{O}$ *for which* $\alpha(\Omega) <+\,\infty$, $\alpha$ *is vanishing along the sequences in* $\mathcal{O}$ *that are well decreasing to the empty set with respect to* $\Omega$,

*ii*) *for every* $\Omega \in \mathcal{O}$ *for which* $\alpha(\Omega) =+\,\infty$, $\alpha$ *is diverging along the sequences in* $\mathcal{O}$ *that are well increasing to* $\Omega$.

*Conversely, assume that* $\mathcal{O}$ *is perfect with respect to* $\mathcal{A}_0$, *that* $\alpha$ *is increasing, boundary subadditive, and that i*) *and ii*) *hold. Then* $\alpha$ *is inner regular.*

PROOF. We prove the first part of the thesis.

Let $\Omega \in \mathcal{O}$ be such that $\alpha(\Omega) <+\,\infty$, and let $\{A_n\}$ be a sequence in $\mathcal{O}$ well decreasing to the empty set with respect to $\Omega$, then by (3.3), and the boundary superadditivity of $\alpha$ it follows that

$$\alpha(A_{n+1}) \leq \alpha(\Omega) - \alpha(\Omega \setminus \overline{A_n}) ,$$

from which, together with the inner regularity of $\alpha$, *i*) follows.

Moreover, the inner regularity of $\alpha$ implies condition *ii*), and the thesis.

Let us now prove the second part of the thesis.

Since $\alpha$ is increasing, it follows that

$$\alpha_{\mathcal{O}-}(\Omega) \leq \alpha(\Omega) \text{ for every } \Omega \in \mathcal{O}. \tag{3.4}$$

Let now $\Omega \in \mathcal{O}$, and assume for the moment that $\alpha(\Omega) <+\infty$. Let $K \in \mathcal{A}_0$ with $K \subset\subset \Omega$, then, being $\mathcal{O}$ perfect with respect to $\mathcal{A}_0$, there exists $A$, $B \in \mathcal{O}$ such that $K \subset\subset A \subset\subset B \subset\subset \Omega$.

By virtue of this, (3.3), the boundary subadditivity of $\alpha$, and being $\alpha$ increasing, we conclude that

$$\alpha(\Omega) \leq \alpha(B) + \alpha(\Omega \setminus \overline{A}) \leq \alpha_{\mathcal{O}-}(\Omega) + \alpha(\Omega \setminus \overline{A}),$$

from which, together with assumption *i*), the opposite inequality to (3.4) and the inner regularity of $\alpha$ at $\Omega$ when $\alpha(\Omega) <+\infty$ follow.

In conclusion, being by assumption *ii*) $\alpha$ inner regular at $\Omega$ also when $\alpha(\Omega) =+\infty$, the inner regularity of $\alpha$ follows. □

Eventually, we prove a variant of the De Giorgi-Letta extension result in our setting.

THEOREM 3.3. *Let $\mathcal{O} \subseteq \mathcal{A}_0$ be dense with respect to $\mathcal{A}_0$, and $\alpha\colon \mathcal{O} \to [0, +\infty]$ be increasing. Then $\alpha$ is the restriction to $\mathcal{O}$ of a Borel measure if and only if $\alpha$ is inner regular, superadditive, and subadditive.*

PROOF. It is clear that if $\alpha$ is the restriction to $\mathcal{O}$ of a Borel measure, then it is inner regular, superadditive, and subadditive.

Conversely, it is easy to verify that $\alpha_{\mathcal{O}-}$ is inner regular, and that the density of $\mathcal{O}$ with respect to $\mathcal{A}_0$ and the assumptions on $\alpha$ imply that $\alpha_{\mathcal{O}-}$ is superadditive and subadditive.

If $\alpha_{\mathcal{O}-}(\emptyset) \neq 0$, then, by using the superadditivity and subadditivity properties of $\alpha_{\mathcal{O}-}$, it must necessarily result $\alpha_{\mathcal{O}-}(\emptyset) =+\infty$, and $\alpha$ turns out to be the restriction to $\mathcal{O}$ of the Borel measure that is identically equal to $+\infty$.

If $\alpha_{\mathcal{O}-}(\emptyset) = 0$, the De Giorgi-Letta extension theorem (cf. [15, Théorème 5.6]) applied to $\alpha_{\mathcal{O}-}$ yields that $\alpha_{\mathcal{O}-}$ is the restriction to $\mathcal{A}_0$ of a Borel measure from which, together with the inner regularity of $\alpha$, the thesis follows. □

## 4. THE UNIQUE EXTENSION RESULT FOR INNER REGULAR FUNCTIONALS

In the present section we prove the main results of this paper on the existence and the uniqueness of the extension of a given functional defined on elementary functions and open sets, under inner regularity assumptions.

4.1. *An approximation result.*

In this subsection we prove a general approximation in energy result by smooth functions. This will be done by using a regularization via convolution argument, an idea already exploited in [32] and, in more general contexts, *e.g.* in [16, 25, 7, 10, 5].

Theorem 4.1. *Let $\mathcal{O} \subseteq \mathcal{A}_0$ be such that $x_0 + \Omega \in \mathcal{O}$ whenever $x_0 \in \mathbb{R}^n$, $\Omega \in \mathcal{O}$, $W$ be a Hausdorff locally convex topological vector space satisfying* (1.1), (1.4)-(1.6) *with $U = W$, and $\Phi : \mathcal{O} \times W \to [0, +\infty]$ be translation invariant, convex, and $W$-lower semicontinuous. Then*

$$\Phi(A, u_\varepsilon) \leq \Phi_{\mathcal{O}-}(\Omega, u) \text{ for every } \Omega \in \mathcal{A}_0,\ A \in \mathcal{O} \text{ with } A \subset\subset \Omega,\ \varepsilon \in ]0, \operatorname{dist}(A, \partial\Omega)[,\ u \in W.$$

Proof. Let $\Omega$, $A$, $\varepsilon$, $u$ be as above, and $\rho$ as in (2.1), then by Proposition 2.4 we get that $(W)\int_{\mathbb{R}^n} \rho(y)T[\varepsilon y]u dy$ exists in $W$.

By Theorem 2.6 applied to $\Phi(A, \cdot)$, once observed that (1.6) and the $U$-lower semicontinuity of $\Phi$ imply the lower semicontinuity and, consequently, the measurability of $\Phi(A, T[\varepsilon\cdot]u)$, we deduce that

$$\Phi\left(A, (W)\int_{\mathbb{R}^n} \rho(y)T[\varepsilon y]u dy\right) \leq \int_{\mathbb{R}^n} \Phi(A, T[\varepsilon y]u)\rho(y) dy. \tag{4.1}$$

On the other side, being $\Phi$ translation invariant, by (4.1) it follows that

$$\Phi\left(A, (W)\int_{\mathbb{R}^n} \rho(y)T[\varepsilon y]u dy\right) \leq \int_{\operatorname{spt}(\rho)} \Phi\left(A + \varepsilon y, u\right)\rho(y) dy \leq$$

$$\leq \int_{B_1} \Phi_{\mathcal{O}-}(\Omega, u)\rho(y) dy = \Phi_{\mathcal{O}-}(\Omega, u),$$

from which, together with Proposition 2.4, the thesis follows. □

4.2. *The uniqueness result.*

In the present subsection we prove the results about the uniqueness of the extension.

Proposition 4.2. *Let $\mathcal{E}_0 \subseteq \mathcal{A}_0$ satisfy* (1.2), *$U$ be a Hausdorff locally convex topological vector space satisfying* (1.1), (1.4)-(1.6), *and $G$, $H : \mathcal{E}_0 \times U \to [0, +\infty]$. Assume that $H$ is translation invariant, convex, that $G$ and $H$ are $U$-lower semicontinuous, and that*

$$G(\Omega, u) \leq H(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n). \tag{4.2}$$

*Then*

$$G_{\mathcal{E}_0-}(\Omega, u) \leq H_{\mathcal{E}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{A}_0 \times U.$$

Proof. The thesis is clearly true if $\{A \in \mathcal{E}_0 : A \subset\subset \Omega\} = \emptyset$.

Otherwise, let $(\Omega, u) \in \mathcal{A}_0 \times U$, then by (4.2), and Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}_0$, $W = U$, $\Phi = H$, we get

$$G(A, u_\varepsilon) \leq H(A, u_\varepsilon) \leq H_{\mathcal{E}_0-}(\Omega, u) \text{ for every } A \in \mathcal{E}_0 \text{ with } A \subset\subset \Omega,\ \varepsilon \in ]0, \operatorname{dist}(A, \partial\Omega)[,$$

from which, together with the $U$-lower semicontinuity of $G$, and Proposition 2.5, the thesis follows. □

THEOREM 4.3. *Let $\mathcal{E} \subseteq \mathcal{A}_0$, $\mathcal{E}_0 \subseteq \mathcal{E}$ be dense with respect to $\mathcal{E}$, and satisfying* (1.2). *Let $U$ be a Hausdorff locally convex topological vector space satisfying* (1.1), (1.4)-(1.6), *and $G, H: \mathcal{E} \times U \to [0, +\infty]$. Assume that G and H are inner regular, that their restrictions to $\mathcal{E}_0 \times U$ are translation invariant, convex, $U$-lower semicontinuous, and that*

$$G(\Omega, u) = H(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n).$$

*Then*

$$G(\Omega, u) = H(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times U.$$

PROOF. By a double application of Proposition 4.2 to the restrictions of $G$ and $H$ to $\mathcal{E}_0 \times U$, we infer that

$$G_{\mathcal{E}_0-}(\Omega, u) = H_{\mathcal{E}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{A}_0 \times U. \tag{4.3}$$

On the other hand, by *b*) of Proposition 3.1 we immediately deduce that

$$G_{\mathcal{E}-}(\Omega, u) = G_{\mathcal{E}_0-}(\Omega, u), \quad H_{\mathcal{E}-}(\Omega, u) = H_{\mathcal{E}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times U,$$

from which, together with the inner regularity of $G$ and $H$, and (4.3), the thesis follows. □

### 4.3. *The existence and uniqueness result.*

We now deal with the existence of the extension.

LEMMA 4.4. *Let $\mathcal{E}_0 \subseteq \mathcal{A}_0$ satisfy* (1.2), *and $F: \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to [0, +\infty]$ be translation invariant, convex, and $C^\infty(\mathbb{R}^n)$-lower semicontinuous. Then $F_{\mathcal{E}_0-}$ is $L^1_{loc}(\mathbb{R}^n)$-lower semicontinuous.*

PROOF. Let $\Omega \in \mathcal{A}_0$, $u \in C^\infty(\mathbb{R}^n)$, $\{u_h\} \subseteq C^\infty(\mathbb{R}^n)$ be such that $u_h \to u$ in $L^1_{loc}(\mathbb{R}^n)$. It is clear that, if $\{A \in \mathcal{E}_0 : A \subset\subset \Omega\} = \emptyset$, then

$$F_{\mathcal{E}_0-}\Omega, u) = 0 = \liminf_{h\to+\infty} F_{\mathcal{E}_0-}(\Omega, u_h).$$

Otherwise, for every $h \in \mathbb{N}$, $\varepsilon > 0$, let $u_{h,\varepsilon}$ be the regularization of $u_h$.

Let $A \in \mathcal{E}_0$ be such that $A \subset\subset \Omega$, and $\varepsilon \in ]0, \operatorname{dist}(A, \partial\Omega)[$, then $u_{h,\varepsilon} \to u_\varepsilon$ in $C^\infty(\mathbb{R}^n)$. By Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}_0$, $W = C^\infty(\mathbb{R}^n)$, $\Phi = F$, and by the $C^\infty(\mathbb{R}^n)$-lower semicontinuity of $F$, we get

$$F(A, u_\varepsilon) \leq \liminf_{h\to+\infty} F(A, u_{h,\varepsilon}) \leq \liminf_{h\to+\infty} F_{\mathcal{E}_0-}(\Omega, u_h). \tag{4.4}$$

By (4.4), and again the $C^\infty(\mathbb{R}^n)$-lower semicontinuity of $F$, once observed that $u_\varepsilon \to u$ in $C^\infty(\mathbb{R}^n)$, we conclude that

$$F(A, u) \leq \liminf_{\varepsilon\to 0+} F(A, u_\varepsilon) \leq \liminf_{h\to+\infty} F_{\mathcal{E}_0-}(\Omega, u_h) \text{ for every } A \in \mathcal{E}_0 \text{ with } A \subset\subset \Omega,$$

from which the thesis follows. □

PROPOSITION 4.5. *Let* $\mathcal{E}_0 \subseteq \mathcal{A}_0$ *satisfy* (1.2), *and* $F\colon \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to [0, +\infty]$. *Assume that* $F$ *is inner regular, translation invariant, convex, and* $C^\infty(\mathbb{R}^n)$*-lower semicontinuous. Then* $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ *is translation invariant, convex, and agrees with* $F$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$, *for every locally convex topological vector space* $U$ *satisfying* (1.1), (1.4)-(1.6), *its restriction to* $\mathcal{A}_0 \times U$ *is* $U$*-lower semicontinuous, and, for every* $\mathcal{E} \subseteq \mathcal{A}_0$ *perfect with respect to* $\mathcal{A}_0$, *its restriction to* $\mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *is inner regular.*

PROOF. It is easy to verify that $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ is translation invariant and convex. Moreover, by Lemma 4.4, we have that

$$F_{\mathcal{E}_0-}(\Omega, u) = \overline{F_{\mathcal{E}_0-}}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{A}_0 \times C^\infty(\mathbb{R}^n),$$

from which, together with the remark that $\mathcal{A}_0$ is perfect with respect to $\mathcal{E}_0$, *a*) of Proposition 3.1, and the inner regularity of $F$, we deduce the identity of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ with $F$ on $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$.

Let now $U$ be as above, then, by using also the right-hand side of (1.5), it is easy to deduce that the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{A}_0 \times U$ is $U$-lower semicontinuous.

Finally, given $\mathcal{E}$ as above, *a*) of Proposition 3.1 yields

$$\left(\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}\right)_{\mathcal{E}-}(\Omega, u) = \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

from which the inner regularity of the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ follows. □

We can now prove the existence and uniqueness result.

THEOREM 4.6. *Let* $\mathcal{E}_0 \subseteq \mathcal{A}_0$ *satisfy* (1.2), *and* $F\colon \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to [0, +\infty]$. *Assume that* $F$ *is inner regular, translation invariant, convex, and* $C^\infty(\mathbb{R}^n)$*-lower semicontinuous. Then, for every* $\mathcal{E} \subseteq \mathcal{A}_0$ *perfect with respect to* $\mathcal{A}_0$, *having* $\mathcal{E}_0$ *as a dense subset, and satisfying* (1.3), *and for every locally convex topological vector space* $U$ *satisfying* (1.1), (1.4)-(1.6), *the restriction of* $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ *to* $\mathcal{E} \times U$ *is the only inner regular, translation invariant, convex,* $U$*-lower semicontinuous functional from* $\mathcal{E} \times U$ *to* $[0, +\infty]$ *that agrees with* $F$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$.

PROOF. Follows from Proposition 4.5, and Theorem 4.3. □

## 5. THE UNIQUE EXTENSION RESULT FOR MEASURE LIKE FUNCTIONALS

In Theorem 4.6 a central role is played by inner regularity assumptions. In the theorems below we propose some results in the same order of ideas of Theorem 4.6, but under groups of assumptions implying inner regularity conditions, and determining again closed classes of functionals in which carry out the extension processes.

For every $\mathcal{O} \subseteq \mathcal{A}_0$, every set $U$, and $\Phi\colon \mathcal{O} \times U \to [0, +\infty]$, we say that $\Phi$ is increasing, boundary superadditive, boundary subadditive if so is $\Phi(\cdot, u)$ for every $u \in U$. We also say that $\Phi$ is a Borel measure if for every $u \in U$, $\Phi(\cdot, u)$ is the restriction to $\mathcal{O}$ of a Borel measure.

PROPOSITION 5.1. *Let* $\mathcal{E}_0 \subseteq \mathcal{A}_0$ *satisfy* (1.2), (3.3) *with* $\mathcal{O} = \mathcal{E}_0$, *and* $F: \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \to \to [0, +\infty]$. *Assume that* $F$ *is increasing, translation invariant, convex,* $C^\infty(\mathbb{R}^n)$*-lower semicontinuous, boundary superadditive, boundary subadditive, and satisfying the following conditions*

*i) for every* $(\Omega, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$ *such that* $F(\Omega, u) < +\infty$, $F$ *is vanishing along the sequences in* $\mathcal{E}_0$ *that are well decreasing to the empty set with respect to* $\Omega$,

*ii) for every* $(\Omega, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$ *such that* $F(\Omega, u) = +\infty$, $F$ *is diverging along the sequences in* $\mathcal{E}_0$ *that are well increasing to* $\Omega$.

*Then, for every* $\mathcal{E} \subseteq \mathcal{A}_0$ *perfect with respect to* $\mathcal{A}_0$, *having* $\mathcal{E}_0$ *as a dense subset, and satisfying* (1.3) *and* (3.3) *with* $\mathcal{O} = \mathcal{E}$, *and for every locally convex topological vector space* $U$ *satisfying* (1.1), (1.4)-(1.6), *the restriction of* $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ *to* $\mathcal{E} \times U$ *is the only functional from* $\mathcal{E} \times U$ *to* $[0, +\infty]$ *that*

*a) is equal to* $F$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$,

*b) is increasing, translation invariant, convex,* $U$*-lower semicontinuous, boundary superadditive, boundary subadditive,*

*c) vanishes along the sequences in* $\mathcal{E}$ *that are well decreasing to the empty set with respect to* $\Omega$, *for every* $(\Omega, u) \in \mathcal{E} \times U$ *in which it is finite,*

*d) diverges along the sequences in* $\mathcal{E}$ *that are well increasing to* $\Omega$, *for every* $(\Omega, u) \in \mathcal{E} \times U$ *in which it is not finite.*

PROOF. Let $\mathcal{E}$, $U$ be as above.

It is clear that $\mathcal{E}_0$ too is perfect with respect to $\mathcal{A}_0$, therefore by Proposition 3.2, the inner regularity of $F$ follows.

By virtue of this, and of the assumptions on $F$, Theorem 4.6 applies and we conclude that the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E} \times U$ is the only inner regular, translation invariant, convex, $U$-lower semicontinuous functional from $\mathcal{E} \times U$ to $[0, +\infty]$ that is equal to $F$ on $\mathcal{E} \times C^\infty(\mathbb{R}^n)$.

We now prove some additional properties of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$.

Obviously $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ is increasing.

Let us prove that the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E} \times U$ is boundary superadditive.

Let $\Omega$, $A$, $B \in \mathcal{E}$, with $A \subset\subset B \subset\subset \Omega$, $u \in U$, and, by using the properties of $\mathcal{E}_0$ and $\mathcal{E}$, let $\Omega'$, $B' \in \mathcal{E}_0$, be such that $B \subset\subset B' \subset\subset \Omega' \subset\subset \Omega$. Then by *a)* of Proposition 3.1, Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}$, $W = U$, and $\Phi = (\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ restricted to $\mathcal{E} \times U$, by the properties of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the inner regularity and the boundary superadditivity of $F$, and by (3.3) with $\mathcal{O} = \mathcal{E}_0$ we get that

$$\begin{aligned}(5.1)\qquad \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) &= \left(\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}\right)_{\mathcal{E}-}(\Omega, u) \geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega', u_\varepsilon) = F_{\mathcal{E}_0-}(\Omega', u_\varepsilon) = \\ &= F(\Omega', u_\varepsilon) \geq F(A, u_\varepsilon) + F(\Omega' \setminus \overline{B'}, u_\varepsilon) \geq F_{\mathcal{E}_0-}(A, u_\varepsilon) + F_{\mathcal{E}_0-}(\Omega' \setminus \overline{B'}, u_\varepsilon)\end{aligned}$$

for every $\varepsilon > 0$ sufficiently small.

By (5.1), and Proposition 2.5 we conclude that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega\,,u) \geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(A\,,u) + \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'\setminus\overline{B'}\,,u)$$
$$\text{for every } \Omega'\,,\ B'\in\mathcal{E}_0 \text{ with } B\subset\subset B'\subset\subset\Omega'\subset\subset\Omega\,,$$

from which, together with the density of $\mathcal{E}_0$ with respect to $\mathcal{E}$, the boundary superadditivity of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ follows as $\Omega'$ increases to $\Omega$ and $B'$ decreases to $B$.

Let us prove now that the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E}\times U$ is boundary subadditive.

Let $\Omega$, $A$, $B\in\mathcal{E}$, with $A\subset\subset B\subset\subset\Omega$, $u\in U$, and, by the density of $\mathcal{E}_0$ with respect to $\mathcal{E}$, let $\Omega'$, $A'$, $B'\in\mathcal{E}_0$, be such that $A\subset\subset A'\subset\subset B'\subset\subset B\subset\subset\Omega'\subset\subset\Omega$. Then, by the same arguments used above, we get that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(B,u)=\left(\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}\right)_{\mathcal{E}-}(B,u)\geq\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(B',u_\varepsilon)=F(B',u_\varepsilon) \tag{5.2}$$
$$\text{for every } \varepsilon>0 \text{ sufficiently small.}$$

Analogously, by (3.3) with $\mathcal{O}=\mathcal{E}_0$ we also deduce that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega\setminus\overline{A}\,,u)\geq F(\Omega'\setminus\overline{A'}\,,u_\varepsilon) \text{ for every } \varepsilon>0 \text{ sufficiently small.} \tag{5.3}$$

Therefore by (5.2), (5.3), and the boundary subadditivity of $F$ we conclude that

$$F(\Omega'\,,u_\varepsilon)\leq F(B'\,,u_\varepsilon)+F(\Omega'\setminus\overline{A'}\,,u_\varepsilon)\leq\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(B\,,u)+\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega\setminus\overline{A}\,,u)$$
$$\text{for every } \Omega'\in\mathcal{E}_0 \text{ with } \Omega'\subset\subset\Omega\,,\ \varepsilon>0 \text{ sufficiently small}\,,$$

from which, together with Proposition 2.5, we obtain as $\varepsilon$ decreases to 0 that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'\,,u)\leq\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(B\,,u)+\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega\setminus\overline{A}\,,u) \tag{5.4}$$
$$\text{for every } \Omega'\in\mathcal{E}_0 \text{ with } \Omega'\subset\subset\Omega.$$

By (5.4), and the density of $\mathcal{E}_0$ with respect to $\mathcal{E}$ the boundary subadditivity $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ follows as $\Omega'$ increases to $\Omega$.

Finally, by Proposition 3.2, the vanishing of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ along the sequences in $\mathcal{E}$ that are well decreasing to the empty set with respect to $\Omega$ for every $(\Omega\,,u)\in\mathcal{E}\times U$ for which $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}(\Omega\,,u)<+\infty$, and the diverging of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ along the sequences in $\mathcal{E}$ that are well increasing to $\Omega$ for every $(\Omega\,,u)\in\mathcal{E}\times U$ for which $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}(\Omega\,,u)=+\infty$ too follow.

In conclusion, since by Proposition 3.2 every functional satisfying *a)*-*d)* is inner regular, also the uniqueness part of the thesis follows. □

By Proposition 5.1 we deduce the following result.

Proposition 5.2. *Let* $\mathcal{E}_0\subseteq\mathcal{A}_0$ *be dense respect to* $\mathcal{A}_0$ *and satisfy* (1.2), (3.3) *with* $\mathcal{O}=\mathcal{E}_0$, *and let* $F\colon\mathcal{E}_0\times C^\infty(\mathbb{R}^n)\to[0\,,+\infty]$. *Assume that* $F$ *is translation invariant, convex,* $C^\infty(\mathbb{R}^n)$*-lower semicontinuous, and a Borel measure. Then, for every* $\mathcal{E}\subseteq\mathcal{A}_0$ *with*

*$\mathcal{E}_0 \subseteq \mathcal{E}$ and satisfying* (1.3) *and* (3.3) *with* $\mathcal{O} = \mathcal{E}$, *and for every locally convex topological vector space* $U$ *satisfying* (1.1), (1.4)-(1.6), *the restriction of* $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ *to* $\mathcal{E} \times U$ *is the only translation invariant, convex,* $U$*-lower semicontinuous functional from* $\mathcal{E} \times U$ *to* $[0, +\infty]$ *that is equal to* $F$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$, *and is a Borel measure.*

Proof. The thesis follows from Proposition 5.1, once we prove that $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ is a Borel measure, since every translation invariant, convex, $U$-lower semicontinuous functional from $\mathcal{E} \times U$ to $[0, +\infty]$ equal to $F$ on $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$, and that is a Borel measure, actually fulfills also conditions *a*)-*d*) of Proposition 5.1.

We prove that the conditions of Theorem 3.3 with $\mathcal{O} = \mathcal{E}$ are fulfilled.

Let us start with the superadditivity condition.

Let $u \in U$, $\Omega$, $\Omega_1$, $\Omega_2 \in \mathcal{E}$ with $\Omega_1 \cup \Omega_2 \subseteq \Omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$, and let $\Omega'_1$, $\Omega'_2 \in \mathcal{E}_0$ be such that $\Omega'_1 \subset\subset \Omega_1$, $\Omega'_2 \subset\subset \Omega_2$. By using the properties of $\mathcal{E}_0$ and $\mathcal{E}$, let $\Omega' \in \mathcal{E}_0$ satisfying $\Omega' \subset\subset \Omega$, $\Omega'_1 \subset\subset \Omega' \cap \Omega_1$, and $\Omega'_2 \subset\subset \Omega' \cap \Omega_2$. Then by Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}$, $W = U$, and $\Phi = (\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the inner regularity of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the properties of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the measure theoretic properties of $F$, and (3.3) we get that

$$
\begin{aligned}
\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega', u_\varepsilon) = F(\Omega', u_\varepsilon) \geq F(\Omega'_1, u_\varepsilon) + F(\Omega'_2, u_\varepsilon) \geq \\
\geq F_{\mathcal{E}_0-}(\Omega'_1, u_\varepsilon) + F_{\mathcal{E}_0-}(\Omega'_2, u_\varepsilon) \text{ for every } \varepsilon > 0 \text{ sufficiently small.}
\end{aligned}
\tag{5.5}
$$

By (5.5), and Proposition 2.5 we conclude that

$$
\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'_1, u) + \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'_2, u)
$$
$$
\text{for every } \Omega'_1, \ \Omega'_2 \in \mathcal{E}_0 \text{ with } \Omega'_1 \subset\subset \Omega_1, \ \Omega'_2 \subset\subset \Omega_2,
$$

from which, using again the properties of $\mathcal{E}_0$ and $\mathcal{E}$, and Proposition 3.1, it follows that

$$
\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_1, u) + \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_2, u)
$$
$$
\text{for every } \Omega, \ \Omega_1, \ \Omega_2 \in \mathcal{E} \text{ with } \Omega_1 \cup \Omega_2 \subseteq \Omega \text{ and } \Omega_1 \cap \Omega_2 = \emptyset, \ u \in U.
$$

We now prove the subadditivity condition.

Let $u \in U$, $\Omega$, $\Omega_1$, $\Omega_2 \in \mathcal{E}$ with $\Omega \subseteq \Omega_1 \cup \Omega_2$, and let $\Omega' \in \mathcal{E}_0$ be such that $\Omega' \subset\subset \Omega$. By the properties of $\mathcal{E}_0$ and $\mathcal{E}$, let $\Omega'_1$, $\Omega'_2 \in \mathcal{E}_0$ with $\Omega'_1 \subset\subset \Omega_1$, $\Omega'_2 \subset\subset \Omega_2$, and $\Omega' \subseteq \Omega'_1 \cup \Omega'_1$. Then by Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}$, $W = U$, and $\Phi = (\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the inner regularity of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the properties of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$, the measure theoretic properties of $F$, and (3.3) we get that

$$
\begin{aligned}
\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_1, u) + \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_2, u) &\geq \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'_1, u_\varepsilon) + \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'_2, u_\varepsilon) = \\
&= F(\Omega'_1, u_\varepsilon) + F(\Omega'_2, u_\varepsilon) \geq F(\Omega', u_\varepsilon) \geq \\
&\geq F_{\mathcal{E}_0-}(\Omega', u_\varepsilon)
\end{aligned}
\tag{5.6}
$$
$$
\text{for every } \varepsilon > 0 \text{ sufficiently small.}
$$

By (5.6), and Proposition 2.5 we conclude that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_1\,,\,u)+\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_2\,,\,u)\geq\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega'\,,\,u)$$
$$\text{for every } \Omega'\in\mathcal{E}_0 \text{ with } \Omega'\subset\subset\Omega\,,$$

from which it follows that

$$\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega\,,\,u)\leq\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_1\,,\,u)+\left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega_2\,,\,u)$$
$$\text{for every } \Omega\,,\Omega_1\,,\Omega_2\in\mathcal{E} \text{ with } \Omega\subseteq\Omega_1\cup\Omega_2\,,\ u\in U.$$

By the above conditions, and the inner regularity of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ the thesis follows by using Theorem 3.3. □

## 6. Some applications

In the present section we apply the results of the previous ones to some integral functionals of the Calculus of Variations.

Proposition 6.1. *Let $\mathcal{E}_0\subseteq\mathcal{A}_0$ satisfy (1.2), $k\in\mathbb{N}$, $f\colon\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n^2}\times\ldots\times\mathbb{R}^{n^k}\to$ $\to[0\,,+\infty]$ be convex, lower semicontinuous, and $F\colon(\Omega\,,\,u)\in\mathcal{E}_0\times C^\infty(\mathbb{R}^n)\mapsto$ $\mapsto\int_\Omega f(u\,,\nabla u\,,\nabla^2 u\,,\,\ldots\,,\nabla^k u)dx$. Then, for every $\mathcal{E}\subseteq\mathcal{A}_0$ perfect with respect to $\mathcal{A}_0$, having $\mathcal{E}_0$ as a dense subset, and satisfying (1.3), and for every locally convex topological vector space $U$ satisfying (1.1), (1.4)-(1.6), the restriction of $(\overline{F_{\mathcal{E}_0-}})_{\mathcal{A}_0-}$ to $\mathcal{E}\times U$ is the only inner regular (respectively measure, provided (3.3) with $\mathcal{O}=\mathcal{E}_0$ and $\mathcal{O}=\mathcal{E}$ are fulfilled), translation invariant, convex, $U$-lower semicontinuous functional from $\mathcal{E}\times U$ to $[0\,,+\infty]$ that agrees with $F$ on $\mathcal{E}_0\times C^\infty(\mathbb{R}^n)$.*

Proof. Follows trivially from Theorem 4.6 (respectively by Proposition 5.2). □

Proposition 6.2. *Let $\mathcal{E}_0\subseteq\mathcal{A}_0$ satisfy (1.2), $f\colon\mathbb{R}^n\to[0\,,+\infty]$ be convex, lower semicontinuous, and let $F\colon(\Omega\,,\,u)\in\mathcal{E}_0\times C^\infty(\mathbb{R}^n)\mapsto\int_\Omega f(\nabla u)dx$. Then, for every $\mathcal{E}\subseteq\mathcal{A}_0$ perfect with respect to $\mathcal{A}_0$, having $\mathcal{E}_0$ as a dense subset, and satisfying (1.3), the functional*

$$\widetilde{F}\colon(\Omega\,,\,u)\in\mathcal{E}\times BV_{\mathrm{loc}}(\mathbb{R}^n)\mapsto\int_\Omega f(\nabla u)dx+\int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right)d|D^s u|$$

*is the only inner regular (respectively measure, provided (3.3) with $\mathcal{O}=\mathcal{E}_0$ and $\mathcal{O}=\mathcal{E}$ are fulfilled), translation invariant, convex, $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous functional from $\mathcal{E}\times BV_{\mathrm{loc}}(\mathbb{R}^n)$ to $[0\,,+\infty]$ equal to $F$ on $\mathcal{E}_0\times C^\infty(\mathbb{R}^n)$.*

*If, in addition, $f$ satisfies*

$$|z|\leq f(z)\ \text{ for every } z\in\mathbb{R}^n\,, \tag{6.1}$$

*then, for every $\mathcal{E}\subseteq\mathcal{A}_0$ perfect with respect to $\mathcal{A}_0$, having $\mathcal{E}_0$ as a dense subset, and satisfying (1.3), the functional*

$$\widehat{F}\colon(\Omega,u)\in\mathcal{E}\times L^1_{\mathrm{loc}}(\mathbb{R}^n)\mapsto\begin{cases}\displaystyle\int_\Omega f(\nabla u)dx+\int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}\right)d|D^s u| & \text{if } u\in BV(\Omega)\\ +\infty & \text{if } u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\setminus BV(\Omega)\end{cases}$$

*is the only inner regular* (*respectively measure, provided* (3.3) *with* $\mathcal{O} = \mathcal{E}_0$ *and* $\mathcal{O} = \mathcal{E}$ *are fulfilled*), *translation invariant, convex,* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$*-lower semicontinuous functional from* $\mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *to* $[0, +\infty]$ *equal to* $F$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$.

PROOF. We prove only the part of the thesis relative to $\widehat{F}$, and under inner regularity assumptions, the other ones being analogous.

In this case the thesis follows from Theorem 4.6, once we prove that

$$(6.2) \qquad \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) = \widehat{F}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times BV_{\mathrm{loc}}(\mathbb{R}^n).$$

To do this let us first prove that $\widehat{F}$ is $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous.

Let $(\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\{u_h\} \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and let us assume that the limit $\lim_{h\to+\infty} \widehat{F}(\Omega, u_h)$ exists and is finite. By virtue of this we infer that $u_h \in BV(\Omega)$ for every $h \in \mathbb{N}$ and, by using (6.1) and the sequential weak*-compactness of the bounded subsets of $BV(\Omega)$, that $u \in BV(\Omega)$.

The proof of the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuity of $\widehat{F}$ is thus reduced to the one of the $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuity of its restriction to $\mathcal{E} \times BV(\Omega)$, and this holds for example by [6, Proposition 1.7].

The $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuity of $\widehat{F}$ implies that

$$\widehat{F}(A, u) \le \overline{F_{\mathcal{E}_0-}}(B, u) \le \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u)$$
$$\text{for every } \Omega,\ B \in \mathcal{A}_0,\ A \in \mathcal{E}_0 \text{ with } A \subset\subset B \subset\subset \Omega,\ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

from which, being $\mathcal{E}_0$ dense with respect to $\mathcal{E}$, we conclude that

$$(6.3) \qquad \widehat{F}(\Omega, u) = \widehat{F}_{\mathcal{E}_0-}(\Omega, u) \le \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Conversely, by Theorem 4.1 applied with $\mathcal{O} = \mathcal{E}_0$, $W = BV_{\mathrm{loc}}(\mathbb{R}^n)$, and $\Phi$ equal to $\widehat{F}$ we get that

$$\widehat{F}(\Omega, u) \ge \widehat{F}_{\mathcal{E}_0-}(\Omega, u) \ge \widehat{F}(A, u_\varepsilon) = F(A, u_\varepsilon) = F_{\mathcal{E}_0-}(A, u_\varepsilon)$$
$$\text{for every } \Omega \in \mathcal{E}, A \in \mathcal{E}_0 \text{ with } A \subset\subset \Omega, \varepsilon \in ]0, \mathrm{dist}(A, \partial\Omega)[, u \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

from which it follows that

$$(6.4) \quad \widehat{F}(\Omega, u) \ge \overline{F_{\mathcal{E}_0-}}(A, u) \text{ for every } \Omega \in \mathcal{E},\ A \in \mathcal{E}_0 \text{ with } A \subset\subset \Omega,\ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

By (6.4), and *b*) of Proposition 3.1 we get that

$$\widehat{F}(\Omega, u) \ge \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{E}_0-}(\Omega, u) = \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{E}-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

from which, being $\mathcal{E}$ perfect with respect to $\mathcal{A}_0$, we conclude that

$$(6.5) \qquad \widehat{F}(\Omega, u) \ge \left(\overline{F_{\mathcal{E}_0-}}\right)_{\mathcal{A}_0-}(\Omega, u) \text{ for every } (\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

By (6.3) and (6.5), equality (6.2) and the thesis follow. □

COROLLARY 6.3. *Let* $\mathcal{E}_0 \subseteq \mathcal{A}_0$ *satisfy* (1.2), *and* $A\colon (\Omega, u) \in \mathcal{E}_0 \times C^\infty(\mathbb{R}^n) \mapsto$ $\mapsto \int_\Omega \sqrt{1 + |\nabla u|^2}\,dx$. *Then, for every* $\mathcal{E} \subseteq \mathcal{A}_0$ *perfect with respect to* $\mathcal{A}_0$, *having* $\mathcal{E}_0$ *as a dense subset, and satisfying* (1.3) *the functional*

$$\widehat{A}\colon (\Omega, u) \in \mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n) \mapsto \begin{cases} \displaystyle\int_\Omega \sqrt{1 + |\nabla u|^2}\,dx + |D^s u|(\Omega) & \text{if } u \in BV(\Omega) \\ +\infty & \text{if } u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \setminus BV(\Omega) \end{cases}$$

*is the only inner regular* (*respectively measure, provided* (3.3) *with* $\mathcal{O} = \mathcal{E}_0$ *and* $\mathcal{O} = \mathcal{E}$ *are fulfilled*), *translation invariant, convex,* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$*-lower semicontinuous functional from* $\mathcal{E} \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *to* $[0, +\infty]$ *equal to* $A$ *on* $\mathcal{E}_0 \times C^\infty(\mathbb{R}^n)$.

PROOF. Follows from Proposition 6.2. □

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