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The apparent propagation velocity of a wave

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Meccanica dei solidi. — *The apparent propagation velocity of a wave.* Nota (*) del Socio PIERO VILLAGGIO.

ABSTRACT. — A one-dimensional model is proposed for determing the delay by which a wave reaching a certain point is effectively registered by a measuring instrument.

KEY WORDS: Waves; Reflection; Diffraction.

RIASSUNTO. — *La velocità di propagazione apparente di un'onda.* Si propone un modello monodimensionale per determinare il ritardo con cui un'onda che raggiunge un certo punto è effettivamente registrata da uno strumento di misura.

Among the different attempts spent for justifying the paradox of heat propagation with infinite velocity, G. Fichera [1] proposed an original answer to this *vexata questio*, by observing that, still accepting the classic Fourier's equation which predicts an infinite propagation velocity of the temperature, this must reach, at a given point not coinciding with the source, a certain value in order to be recorded by a thermometer placed at the point of measurement. The temperature of activation of the thermometer depends on the sensibility of the instrument, but it is, in any case, strictly positive, and, for reaching its value, a certain time is necessary. Thus this time of delay between the creation of a source and the first registration of an increment of temperature in another point is a measure of the apparent propagation velocity of the temperature. Fichera's idea has been recently applied by Manacorda [3] to three different solutions of heat equation, *i.e.* plane, spherical, and cylindrical propagations in an infinite medium. The result is that the apparent propagation velocity of temperature depends, not only on the thermal conductivity, but also on the shape of the wave front.

This seems a question strictly connected with Fourier's equation, which is parabolic, but it is not so. If, on the wave of Fichera's suggestion, we accept that a measuring instrument plays a conditioning part in the result of an experiment, then a similar argument can be applied to a purely hyperbolic equation like that of wave propagation in an elastic bar. Here mathematics predicts a well defined velocity for initial perturbations in the bar, but the presence of an elastic recorder necessarily increases the time of trasmission of a signal from a point to another of the bar.

In order to discuss a precise situation we consider the longitudinal motion of an elastic bar of infinite length spanned in the interval $-\infty < x < \infty$ of a system of rectangular coordinates (x, y) as shown in figure 1. The bar is stretched or contracted in such way that the planes x = constant move together in the x-direction. Let $u(x, t)$

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be the displacement in the x-direction at time t of the plane whose equilibrium position is at x. The axial force T is assumed to be proportional to the elongation $\frac{\partial u}{\partial x}$ according to the equation $T = EA\frac{\partial u}{\partial x}$, where E is Young's modulus and A the area of the cross-section. If ρ is the mass density and body forces in the x-direction are neglegible, the balance equation of motion is d'Alembert's equation (cf. [5])

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0\ , \tag{1}$$

where $c^2 = E/\rho$. Equation (1) requires two initial conditions, the displacement and the velocity at $t = 0$. For simplicity we assume the initial displacement to be equal to a given continuous function $2f(-\frac{x}{c})$ (where the choice of the variable $-\frac{x}{c}$ shortens some subsequent formulae), and the initial velocity to be zero. In this case the explicit solution of equation (1) is given by

$$u(x,t) = f\left(t - \frac{x}{c}\right) + f\left(-t - \frac{x}{c}\right)\ , \tag{2}$$

representing two waves traveling in opposite directions with velocities $\pm c$ without changing shapes.

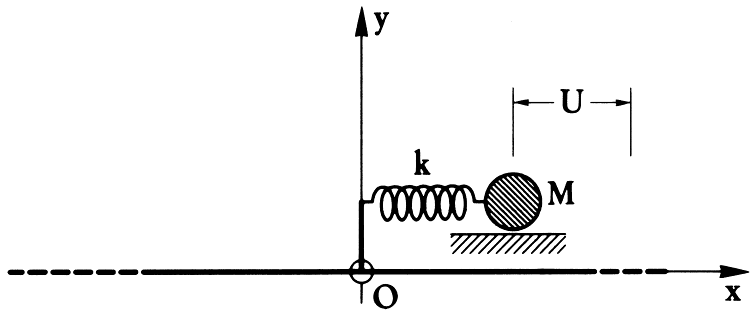

Fig. 1.

So far we have considered the free propagation of elastic waves in an inifinite bar. But the solution is different if a measuring instrument is placed at a given point, say $x = 0$, of the bar (fig. 1). Since the instrument is endowed with a proper elasticity κ and a mass M, it will necessarily modify the structure of waves crossing the section $x = 0$. More precisely, part of the wave is reflected and part transmitted. Following Lamb [2, Art. 176] we write, for the negative side,

$$u(x,t) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right)\ , \tag{3}$$

and for the positive side

$$u(x,t) = F\left(t - \frac{x}{c}\right)\ , \tag{4}$$

where the function f represents the original, prescribed, wave, and g, F the reflected and the transmitted portions, respectively. The continuity of the displacement at $x = 0$

requires the condition

$$f(t) + g(t) = F(t)\,. \tag{5}$$

We must also have at the same point a relation of balance between the jump of forces transmitted across the section $x = 0$:

$$EA\left[\frac{\partial u}{\partial x}(0\,,\,t)\right]_{-}^{+} + \kappa(U(t) - u(0\,,\,t)) = 0\;, \tag{6}$$

where $U(t)$ denotes the horizontal displacement of the mass M from its position of rest. Written in terms of the functions f, g, F, condition (6) becomes

$$EA\left(\frac{\partial F}{\partial x}(t) - \frac{\partial f}{\partial x}(t) - \frac{\partial g}{\partial x}(t)\right) + \kappa(U(t) - F(t)) = 0\,. \tag{7}$$

But, from (3) and (4), we also have the identities

$$\frac{\partial F}{\partial x} = -\frac{1}{c}\frac{\partial F}{\partial t}\;, \qquad \frac{\partial f}{\partial x} = -\frac{1}{c}\frac{\partial f}{\partial t}\;, \qquad \frac{\partial g}{\partial x} = \frac{1}{c}\frac{\partial g}{\partial t}\;, \tag{8), (9}$$

and, from (5), the equality

$$\frac{\partial g}{\partial t}(t) = \frac{\partial F}{\partial t}(t) - \frac{\partial f}{\partial t}(t)\,. \tag{10}$$

Then (7) assumes the form

$$\frac{\partial F}{\partial t}(t) = \frac{\partial f}{\partial t}(t) + \frac{\kappa\, c}{2\,EA}(U(t) - F(t))\;, \tag{11}$$

which is an ordinary differential equation in $F(t)$ provided that $U(t)$ is known ($f(t)$ is given). Let us suppose that, at a certain time $t = 0$, the cross-section at $x = 0$ is at rest, then we have the initial condition $F(0) = 0$, and hence the solution to (11) is

$$F(t) = \int_0^t \exp\left(-\frac{\kappa\, c}{2\,EA}(t - \bar{t}\,)\right)\left(\frac{\partial f}{\partial t}(\bar{t}\,) + \frac{\kappa\, c}{2\,EA}U(\bar{t}\,)\right) d\bar{t}\;\;. \tag{12}$$

In this equation $U(\bar{t}\,)$ is still unknown, but it is related to $u(0\,,\,t)$ by the equation of motion of the mass M:

$$\ddot{U}(t) + \omega^2(U(t) - u(0\,,\,t)) = 0\;, \tag{13}$$

where $\omega^2 = \kappa/M$. On assuming that M is at rest for $t = 0$, the initial conditions are $U(0) = \dot{U}(0) = 0$. Since, for $x = 0$, we know that $u(0\,,\,t) = F(t)$, combination of (12) with (13) yields an integro-differential equation for $U(t)$:

$$\ddot{U}(t) + \omega^2 U(t) = \omega^2 \int_0^t \exp\left(-\frac{\kappa\, c}{2\,EA}(t - \bar{t}\,)\right)\left(\frac{\partial f}{\partial t}(\bar{t}\,) + \frac{\kappa\, c}{2\,EA}U(\bar{t}\,)\right) d\bar{t}\;\;. \tag{14}$$

This equation can be formally integrated by Laplace transform (cf. [4, § 8]), but, in order to discuss the properties of solutions in detail, we consider the particular case in which the initial displacement is a function of the form

$$f(-\frac{x}{c}) = \begin{cases} 2U_0 = \text{constant} & \text{for } x < -\epsilon \ , \\ 2U_0 \dfrac{x}{\epsilon} & \text{for } -\epsilon < x < 0 \ , \\ 0 & \text{for } 0 < x \ , \end{cases} \tag{15}$$

where ϵ is a small quantity. Equation (15) corresponds to impressing the bar a uniform translation $2U_0$ from $-\infty$ to a section situated in the negative side at distance ϵ from the origin, leaving the bar indisturbed from 0 to infinity, and connecting the jump with a linear function. The graph of the piecewise function (15) is represented in figure 2. Since $f(-\frac{x}{c})$ is continuous but not continuously differentiable, the corresponding solution must be interpreted as a generalized solution to the problem (cf. [5, Section 2]).

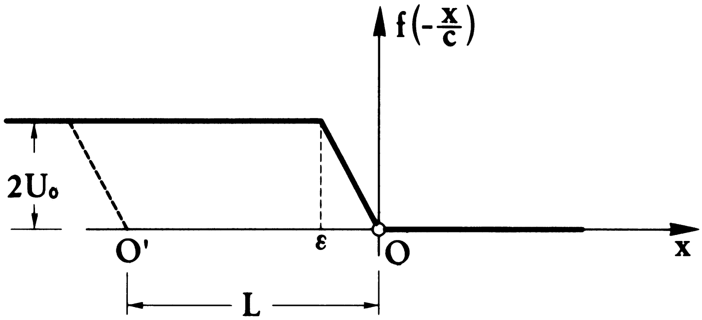

Fig. 2.

The initial displacement of figure 2 describes an incoming piecewise wave meeting the origin just at time t = 0, when it activates the motion of the mass M. If, instead of touching the origin, the head of the wave had occupied a place $0'$ at $x = -L$ (fig. 2), the motion of M would start at time $t = L/c$, when the signal reaches the origin. But let us consider the first case, in which the equation of the progressive wave (one half of the initial perturbation) is

$$f\left(t - \frac{x}{c}\right) = \begin{cases} U_0 & \text{for } 0 < t - \frac{x+\epsilon}{c} \ , \\ U_0 \frac{c}{\epsilon}\left(t - \frac{x}{c}\right) & \text{for } 0 < t - \frac{x}{c} < \frac{\epsilon}{c} \ , \\ 0 & \text{for } t - \frac{x}{c} < 0 \ , \end{cases}$$

and calculate its time derivative at $x = 0$:

$$\frac{\partial f}{\partial t}(t) = \begin{cases} U_0 \frac{c}{\epsilon} & \text{for } 0 < t < \frac{\epsilon}{c} \ , \\ 0 & \text{for } t > \frac{\epsilon}{c} \ . \end{cases} \tag{16}$$

Heaving determinated $\frac{\partial f}{\partial t}(t)$ we can also evaluate the integral

$$(17) \qquad \int_0^t e^{-\frac{\kappa c}{2EA}(t-\bar{t})}\frac{\partial f}{\partial t}(\bar{t})d\bar{t} \;=\begin{cases} \dfrac{U_0}{\epsilon}\,\dfrac{2\,EA}{\kappa}\left(1-e^{-\frac{\kappa c}{2\,EA}\,t}\right) & \text{for } 0<t<\frac{\epsilon}{c}\,,\\ \dfrac{U_0}{\epsilon}\,\dfrac{2\,EA}{\kappa}e^{-\frac{\kappa c}{2\,EA}\,t}\left(e^{\frac{\kappa}{2\,EA}\,\epsilon}-1\right) & \text{for } t>\frac{\epsilon}{c}\,.\end{cases}$$

Let us call $G(t,\epsilon)$ the (continuous) function at right hand side of (17). Then the integro-differential equation (14) can be written in the form

$$(18) \qquad \ddot{U}+\omega^2 U=\omega^2\,G(t,\epsilon)+\omega^2\,\frac{\kappa c}{2\,EA}\int_0^t \exp\left(-\frac{\kappa\,c}{2\,EA}(t-\bar{t})\right)\,U(\bar{t})d\bar{t}\;\;.$$

Integration of (18) can be done explicitly by Laplace transform, as Rothe and Szabó have shown in an exercise collected in their book [4, § 8]. For this purpose we take the Laplace transform of (18). On denoting the Laplace transform of $U(t)$ by $\widehat{U}(s)$ and applying some standard operational formulae, we arrive, after a lengthy but easy computation, at the result

$$(19) \quad (s^2+\omega^2)\widehat{U}(s)=\omega^2\,\frac{U_0}{\epsilon}\,\frac{\frac{2\,EA}{\kappa}}{s\left(s+\frac{\kappa c}{2\,EA}\right)}\,\left(1-e^{-s\frac{\kappa}{2\,EA}\epsilon}\right)+\omega^2\,\frac{\kappa c}{2\,EA}\,\frac{1}{s+\dfrac{\kappa c}{2\,EA}}\widehat{U}(s)\;,$$

which is an algebraic equation for $\widehat{U}(s)$. But a glance to the first term at right hand side of (19) shows that it admits a finite limit as ϵ tend to zero since

$$\lim_{\epsilon\to 0}\frac{1}{\frac{s\,\kappa\epsilon}{2\,EA}}\,\left(1-e^{-\frac{s\,\kappa\epsilon}{2\,EA}}\right)=1\,.$$

This case describes the propagation of a nucleus of strain initially concentrated at 0. Hence, in the limit, (19) assumes the surprisingly simple form

$$(s^2+\omega^2)\widehat{U}(s)=\omega^2\,\frac{U_0}{\left(s+\frac{\kappa c}{EA}\right)}+\omega^2\,\frac{\kappa c}{2\,EA}\,\frac{1}{s+\dfrac{\kappa c}{2\,EA}}\widehat{U}(s)\,.$$

By solving this equation with respect to $\widehat{U}(s)$ and then applying the inversion formula, we find the solution:

$$(20) \qquad U(t)=\omega^2\,U_0\,\left[\frac{1}{\omega^2}+\frac{1}{(s_1-s_2)}\left(\frac{e^{s_1 t}}{s_1}-\frac{e^{s_2 t}}{s_2}\right)\right]\;,$$

where s_1, s_2 (roots of the equation $s^2+\frac{\kappa c s}{2\,EA}+\omega^2=0$) have the values

$$(21) \qquad s_{1,2}=\frac{1}{2}\left(-\frac{\kappa c}{2\,EA}\mp\sqrt{\left(\frac{\kappa c}{2\,EA}\right)^2-4\omega^2}\right)\,.$$

These roots can be real or complex conjugate according to the sign of their discriminant. But the realistic case is that in which $4\omega^2\,(=4\,\frac{\kappa}{M})>(\frac{\kappa c}{2\,EA})^2$, because the mass M of the measuring instrument, is usually light. Under this assumption we can give (20) the trigonometric form

$$
\begin{aligned}
U(t) = U_0 \Bigg[1 - \frac{e^{-\frac{1}{2}(\frac{\kappa c}{2\,EA})t}}{\sqrt{4\omega^2 - \left(\frac{\kappa c}{2\,EA}\right)^2}} \left(\frac{\kappa c}{2\,EA} \sin\left(\frac{1}{2}\sqrt{4\omega^2 - \left(\frac{\kappa c}{2\,EA}\right)^2}\, t \right) + \right. \\
\left. + \sqrt{4\omega^2 - \left(\frac{\kappa c}{2\,EA}\right)^2} \cos\left(\frac{1}{2}\sqrt{4\omega^2 - \left(\frac{\kappa c}{2\,EA}\right)^2}\, t \right)\right)\Bigg] ,
\end{aligned}
\tag{22}
$$

and then describe the motion of the mass M after the first impact due to the incoming wave. In order to illustrate the qualitative properties of solution (22) we consider the following two limiting case.

(a) If $4\omega^2 \gg (\frac{\kappa c}{2\,EA})^2$ we take the limit of (22) as $\frac{\kappa c}{2\,EA}$ tends to zero. Thus (22) assumes the simple expression

$$U(t) = U_0(1 - \cos \omega t) , \tag{23}$$

which represents a simple harmonic motion with maximum elongation $2U_0$ from the initial position. Let us call U_s the «sensibility» of the measuring instrument, *i.e.* the minimum displacement of the mass M recorded by the instrument. Clearly U_s must be smaller than $2U_0$ in order for the passage of the wave to be registered. In this case the «time of activation» t_a of the instrument is the first root of the equation

$$U_s = U_0(1 - \cos \omega t_a) ,$$

namely

$$t_a = \frac{1}{\omega} \cos^{-1}\left(1 - \frac{U_s}{U_0}\right) . \tag{24}$$

(b) If $4\omega^2 \sim (\frac{\kappa c}{2\,EA})^2$, we consider the limit of (22) as $\frac{\kappa c}{2\,EA}$ tends to 2ω. The result is

$$U(t) = U_0 \left(1 - e^{-\omega t}(1 + \cos \omega t)\right) , \tag{25}$$

i.e. an exponential function tending asymptotically to U_0. In this case the time of activation t_a is the only root of the equation

$$U_s = U_0 \left(1 - e^{-\omega t_a}(1 + \cos \omega t_a)\right) .$$

Here U_s must be smaller than U_0 in order t_a to be real.

Let us now suppose that the head of the incoming wave is initially placed at a point $0'$ at $x = -L$ (fig. 2). Since it travels with velocity c it will reach the origin after a time L/c. But the measuring instrument will record the signal after a further time t_a.

Hence the apparent propagation velocity of the wave is not c but

$$c_{app} = \frac{L}{\left(\frac{L}{c} + t_a\right)} = \frac{c}{\left(1 + \frac{t_a c}{L}\right)} , \tag{27}$$

with an eventual remarkable discrepancy between the real and the observed value of the velocity.

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