Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Antonio Fasano, Vsevolod Solonnikov

Estimates of weighted Hölder norms of the solutions to a parabolic boundary value problem in an initially degenerate domain

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **13** (2002), n.1, p. 23–41.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2002_9_13_1_23_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2002.

Rend. Mat. Acc. Lincei
s. 9, v. 13:23-41 (2002)

Equazioni a derivate parziali. — *Estimates of weighted Hölder norms of the solutions to a parabolic boundary value problem in an initially degenerate domain.* Nota (*) di Antonio Fasano e Vsevolod Solonnikov, presentata dal Socio M. Primicerio.

Abstract. — A-priori estimates in weighted Hölder norms are obtained for the solutions of a one-dimensional boundary value problem for the heat equation in a domain degenerating at time $t = 0$ and with boundary data involving simultaneously the first order time derivative and the spatial gradient.

Key words: Heat equation; Boundary value problems of higher order; A-priori estimates in weighted Hölder spaces.

Riassunto. — *Stime della norma in spazi di Hölder con peso delle soluzioni di un problema al contorno per equazioni paraboliche in un domino inizialmente degenere.* Si ottengono stime a priori in opportune norme di Hölder con peso per le soluzioni di un problema unidimensionale per l'equazione del calore in un dominio mobile che degenera per $t = 0$ e con una condizione al contorno in cui compaiono simultaneamente le due derivate prime della funzione incognita.

Introduction

Boundary value problems for the heat equation in domains with a moving boundary are frequently encountered in applications, particularly when the moving boundary is a-priori unknown (phase change problems are typical examples). In some cases the domain can be initially degenerate, *i.e.* with zero thickness in the spatial direction. This is precisely what happens during the first stage of infiltration of porous materials. The analysis of a very peculiar flow problem through a porous medium with liquid absorbing granules during the unsaturated regime leads to an extremely complex free boundary problem for a parabolic equation with history-dependent coefficients that has been presented and partially solved in [2, 3] under some severe restrictions (see also the overview [4] and the extensions [1]). The basic model for the stage of unsaturated penetration in one-dimensional geometry (in a half space $x > 0$) is summarized as follows.

We introduce the main quantities:

p pressure,

Σ saturation, $\Sigma \in (0\,,\,1)$,

ϕ porosity,

$k(\phi\,,\,\Sigma)$ hydraulic conductivity,

$q_o(t)$ inflow rate at $x = 0$ as a function of time,

(*) Pervenuta in forma definitiva all'Accademia il 15 ottobre 2001.

$s(t)$ penetration depth of the fluid, $s(0) = 0$,

S_0 saturation threshold for penetration and adsorption,

$\theta(x)$ inverse function of $s(t)$, *i.e.* the time instant at which the penetration front $x = s(t)$ reaches the location x,

$V(x, t)$ volume of hydrophile granules per unit volume of the system, $V \leq V_{\max}$.

In the unsaturated regime the saturation is an increasing function of pressure $S(p)$ for $p < p_s$ (saturation pressure), owing to capillarity effects. The adsorption kinetics is described by

$$\frac{\partial V}{\partial t} = f(V_{\max} - V)(S - S_0)_+ ,$$

f being an increasing C^1 function such that $f(0) = 0$. Introducing the function

$$\Phi(V) = \int_{V_0}^{V} \frac{dy}{f(V_{\max} - y)} ,$$

V_0 being the initial value of V, and its inverse ψ, we can write

$$V(x, t) = \psi(\Theta)$$

with

$$\Theta(S, \theta, t) = \int_{\theta(x)}^{t} [S(x, \tau) - S_0]\, d\tau , \quad \forall\ t > \theta(x)$$

and consequently

$$\phi(x, t) = \phi_0 + V_0 - \psi(\Theta) ,$$

ϕ_0 being the initial porosity.

At this point we can write the flow equation in the domain $0 < x < s(t)$, $0 < t < t_S$ (t_S being the onset time of the saturation regime)

$$\phi S'(p) \frac{\partial p}{\partial t} + (1 - S)\psi'(\Theta)(S(p) - S_0) - \frac{\partial}{\partial x}\left(k(\phi , S) \frac{\partial p}{\partial x} \right) = 0$$

with boundary conditions

$$- k \frac{\partial p}{\partial x} = q_0(t) , \qquad x = 0 , \quad 0 < t < t_S ,$$

$$p(s(t) , t) = 0 ,$$

$$\dot{s}(t) = - \frac{k(\phi_0)}{\phi_0 S_0} \frac{\partial p}{\partial x} , \qquad x = s(t) , \ 0 < t < t_S .$$

Note that the last boundary condition, expressing the wetting front velocity, is precisely the Stefan condition. We stress the fact that Θ, appearing in the flow equation through k also in the main coefficient is a functional of $S(p)$ which depends also on the unknown function $\theta(x)$. Therefore the scheme above represents a free boundary problem that, despite the familiar conditions on the free boundary, is quite new and particularly difficult.

The first step in the study of the general case, which has been synthetized in [5] (a detailed presentation will appear in a forthcoming paper), is the transformation of the unknown domain $0 < x < s(t)\,, t > 0$ into the fixed domain $0 < z < l_0 t,\ t > 0$, l_0 being the initial slope $\dot{s}(0) = \frac{q(0)}{\Sigma_0 \epsilon_0}$ through the mapping $x = z + \eta(\frac{z}{t})[s(t) - l_0 t]$, where $\eta(\xi) = 0$ for $\xi < l_1\,, \eta(\xi) = 1$ for $\xi > l_2 (0 < l_1 < l_2 < l_0)$ and is monotone and smooth.

Next the problem is reformulated for the difference

$$r(z\,,\, t) = \widehat{p}(z\,,\, t) - p_0^1 \eta \left(\frac{z}{t}\right)(s(t) - l_0 t)\,,$$

where

$$p_0^1 = -\frac{q(0)}{k_0} \qquad \text{and} \qquad \widehat{p}(z\,,\, t) = p(x\,,\, t)\,.$$

Since for $z = l_0 t$ we have $r = -p_0^1(s(t) - l_0 t) \equiv \widetilde{r}(t)$, the Stefan boundary condition in the original problem can be read as a condition on a linear combination of the derivatives $\frac{d\widetilde{r}}{dt}$ and $\frac{\partial r}{\partial z}$ on the rectilinear boundary $z = l_0 t$.

In spite of the complexity of the partial differential equation, the key point in the proof of the existence theorem for the free boundary problem is the derivation of some a-priori estimates in suitable Hölder spaces in a domain of the type $0 < z < l_0 t$ for a problem for the heat equation with a Neumann boundary condition for $z = 0$ and a condition on $z = l_0 t$ consisting in assigning the value of a linear combination of the z-derivative and of the derivative of the unknown along the boundary, which, in view of the parabolicity of the equation, is effectively a condition of higher order in the space derivative.

This is a strong motivation for the present paper.

We observe that such estimates are by far not trivial and require original techniques, due to the concurrence of two difficulties: the degeneracy of the domain and the presence of a higher order boundary condition.

Therefore we thought that their derivation has a mathematical interest in itself and we decided to devote a self-contained paper to this subject.

2. Auxiliary estimates for a linear problem in a wedge

The problem to be studied is the following

$$u_t - u_{xx} = f(x\,,\, t)\,, \qquad \text{in } \Omega_T \tag{1}$$

$$u_x(0\,,\, t) = g(t)\,, \qquad 0 < t < T \tag{2}$$

$$\widetilde{u}_t(t) + b u_x(kt\,,\, t) = h(t)\,, \qquad 0 < t < T \tag{3}$$

where b and k are positive constants and

$$\widetilde{u}(t) = u(kt, t)\,. \tag{4}$$

In order to state our main result, let us define the weighted Hölder norms we are going to use.

Let

$$\Omega_{\tau_1,\tau_2} = \{(x, t) \in \Omega_T : 0 < \tau_1 < t < \tau_2 < T\}\ , \ \text{and}\ I_t = \{0 < x < kt\}$$

for $t \in (0, T)$. For any positive number l we consider the usual Hölder norms

$$[u]^{(l,l/2)}_{\Omega_{\tau_1,\tau_2}} = \sup_{t\in(\tau_1,\tau_2)} [u(\cdot\,, t)]^{(l)}_{I_t} + \sup_{t\in(\tau_1,\tau_2)} \sup_{h\in(0,\tau_2-t)} h^{-l/2+[l/2]} \left| \frac{\partial^{[l/2]} u(x, t+h)}{\partial t^{[l/2]}} - \frac{\partial^{[l/2]} u(x, t)}{\partial t^{[l/2]}} \right| ,$$

$$[u(\cdot\,, t)]^{(l)}_{I_t} = \sup_{x,y\in I_t} |x-y|^{-l+[l]} \left| \frac{\partial^{[l]} u(x, t)}{\partial x^{[l]}} - \frac{\partial^{[l]} u(x, t)}{\partial x^{[l]}} \right| ,$$

then we define $\widehat{C}^{\,l,l/2}(\Omega_T)$ as the space of the functions $u(x, t)$ such that the following norm is finite

$$\| u \|_{\widehat{C}^{\,l,l/2}(\Omega_T)} = \sup_{\tau\in(0,T)} [u]^{(l,l/2)}_{\Omega_{\tau/2,\tau}} + \sup_{t\in(0,T)} t^{-l} \sup_{I_t} |u(\cdot\,, t)|.$$

We also need weighted Hölder norms of functions defined on the interval $(0, T)$:

$$\| u \|_{\widehat{C}^{l}(0,T)} = \sup_{\tau\in(0,T)} \sup_{t\in(\tau/2,\tau)} \sup_{h\in(0,t-\tau)} h^{-l+[l]} \left| \frac{d^{[l]} u(t+h)}{dt^{[l]}} - \frac{d^{[l]} u(t)}{dt^{[l]}} \right| + \sup_{t\in(0,T)} t^{-2l} |u(t)|.$$

At this point we assume that the function f belongs to the weighted Hölder space $\widehat{C}^{\alpha,\alpha/2}(\Omega_T)$ and the data g, h belong to $\widehat{C}^{(1+\alpha)/2}(0, T)$.

We want to prove the following

THEOREM 1.1. *Let u be a solution of* (1)-(3). *Then*

$$\| u \|_{\widehat{C}^{2+\alpha,1+\alpha/2}(\Omega_T)} + \| \widetilde{u}_t \|_{\widehat{C}^{(1+\alpha)/2}(0,T)} \le$$

$$\le C(\| f \|_{\widehat{C}^{\alpha,\alpha/2}(\Omega_T)} + \| g \|_{\widehat{C}^{(1+\alpha)/2}(0,T)} + \| h \|_{\widehat{C}^{(1+\alpha)/2}(0,T)}).$$

The proof goes through two basic steps. First we work on an interval (ϵ, τ) excluding the origin and we obtain an estimate of the desired quantity involving the specified norms of f, g, h, and the L_2-norm of $u(\cdot\,, t)$ over intervals $(0, kt)$ (see (39)), multiplied by $t^{-5/2-\alpha}$. Then we show that also the latter norm is estimated in terms of the same weighted Hölder norms of f, g, h. An important tool will be the estimates for the solutions of a similar problem in a trapezoidal domain, whose proof is placed in the Appendix.

Theorem 1.1 can be slightly generalized, as we shall see in the sequel.

3. Estimates excluding the origin

For a fixed $\tau \in (0\,,\,T)$ we consider the interval $(\epsilon\,,\,\tau)$, with $\epsilon = \frac{\tau}{4}$ and we perform the following rescaling

$$U(x\,,\,t) = u(\epsilon x\,,\,\epsilon t) \tag{5}$$

$$F_\epsilon(x\,,\,t) = \epsilon^2 f(\epsilon x\,,\,\epsilon t) \;\;,\;\; G_\epsilon(t) = \epsilon g(\epsilon t) \;\;,\;\; H_\epsilon(t) = \epsilon h(\epsilon t) \tag{6}$$

so that (1)-(3) become

$$\epsilon U_t - U_{xx} = F_\epsilon\,, \qquad 0 < x < kt,\;\; 1 < t < 4\,, \tag{7}$$

$$U_x(0\,,\,t) = G_\epsilon\,, \qquad 1 < t < 4\,, \tag{8}$$

$$\widetilde{U}_t(t) + bU_x(kt\,,\,t) = H_\epsilon\,, \qquad 1 < t < 4 \tag{9}$$

with

$$\widetilde{U}(t) = U(kt\,,\,t). \tag{10}$$

From (7)-(9) we can derive an initial-boundary value problem in the same domain by introducing a cut-off function $\xi_\lambda(t)$ defined for each $\lambda \in [0\,,\,1]$ as follows

$$\xi_\lambda(t) = 1 \text{ for } t > 2 + \lambda\,, \quad \xi_\lambda(t) = 0 \text{ for } t < 2 + \frac{\lambda}{2}$$

$$|\xi'_\lambda| \le c\lambda^{-1}\,, \qquad |\xi''_\lambda| \le c\lambda^{-2}\,, \tag{11}$$

for some constant $c > 0$. We remark that $[\xi_\lambda]^{(\alpha)}_{(1\,,4)} \le c\lambda^{-\alpha}$, $[\xi'_\lambda]^{(\alpha)}_{(1\,,4)} \le c\lambda^{-1-\alpha}$ for $\alpha \in (0\,,\,1)$.

The product

$$v(x\,,\,t) = U(x\,,\,t)\xi_\lambda(t) \tag{12}$$

satisfies the following problem

$$\epsilon v_t - v_{xx} = F_\epsilon \xi_\lambda + \epsilon U \xi'_\lambda\,, \qquad 0 < x < kt,\;\; 1 < t < 4\,, \tag{13}$$

$$v(x\,,\,1) = 0\,, \qquad 0 < x < k\,, \tag{14}$$

$$v_x(0\,,\,t) = G_\epsilon \xi_\lambda\,, \qquad 1 < t < 4\,, \tag{15}$$

$$\widetilde{v}_t(t) + bv_x(kt\,,\,t) = H_\epsilon \xi_\lambda + \widetilde{U}(t)\xi'_\lambda\,, \quad 1 < t < 4\,, \tag{16}$$

to which the estimate (A.7) of the Appendix can be applied.

Thus, setting $J_\tau = \{x \in (0\,,\,k\tau)\}$, we obtain

$$\epsilon \sup_{\tau\in(2\,,4)} |v_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(2\,,4)} |v(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)} \leq$$

$$\leq C\Bigg\{ \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |F_\epsilon(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup |F_\epsilon| c\lambda^{-\frac{\alpha}{2}} + \epsilon\lambda^{-1} c \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |U(\cdot\,,\tau)|_{C^\alpha(J_\tau)} +$$

$$+ \frac{\epsilon}{\lambda} \sup |U| \lambda^{-\alpha/2} + \epsilon^{\frac{\alpha}{2}} \Bigg([\widetilde{F}_\epsilon]^{(\frac{\alpha}{2})}_{(2+\lambda/2\,,4)} + \sup |\widetilde{F}_\epsilon| c\lambda^{-\alpha/2} +$$

$$+ c\frac{\epsilon}{\lambda}[\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} + \frac{\epsilon}{\lambda} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)| c\lambda^{-\frac{\alpha}{2}} \Bigg) +$$

$$+ \sup_{\tau\in(1+\frac{\lambda}{2}\,,4)} |G_\epsilon(\tau)| + \epsilon^{\frac{1+\alpha}{2}} \Bigg([G_\epsilon]^{(\frac{1+\alpha}{2})}_{(1+\frac{\lambda}{2}\,,4)} + \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |G_\epsilon(\tau)| c\lambda^{-\frac{1+\alpha}{2}} \Bigg) +$$

$$+ \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |H_\epsilon(\tau)| + \frac{c}{\lambda} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)| + \epsilon^{\frac{\alpha}{2}} \Bigg([H_\epsilon]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} +$$

$$+ \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |H_\epsilon(\tau)| c\lambda^{-\frac{\alpha}{2}} + \frac{c}{\lambda}[\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} + \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)| c\lambda^{-1-\frac{\alpha}{2}} \Bigg)\Bigg\},$$

which can be simplified to

$$\begin{aligned}
&\epsilon \sup_{\tau\in(2\,,4)} |v_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(2\,,4)} |v(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)} \leq \\
&\leq C\Bigg\{ \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |F_\epsilon(\cdot\,,\tau)|_{C^\alpha(J_\tau)} \left(1 + \left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \\
&\quad + \epsilon^{\frac{\alpha}{2}} [\widetilde{F}_\epsilon]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} + \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |G_\epsilon(\tau)| \left(1 + \left(\frac{\epsilon}{\lambda}\right)^{\frac{1+\alpha}{2}}\right) + \\
&\quad + \epsilon^{\frac{1+\alpha}{2}} [G_\epsilon]^{(\frac{1+\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} + \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |H_\epsilon(\tau)| \left(1 + \left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \epsilon^{\frac{\alpha}{2}} [H_\epsilon]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} + \\
&\quad + \frac{\epsilon}{\lambda} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |U(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \frac{1}{\lambda} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)| \left(1 + \left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \\
&\qquad\qquad + \frac{1}{\lambda} [\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} \epsilon^{\frac{\alpha}{2}} \left(1 + \frac{\epsilon}{\lambda}\right)\Bigg\}.
\end{aligned} \tag{17}$$

Now we use the following elementary inequality

$$[\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} \leq \left(\sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}_\tau(\tau)| \right)^{\frac{\alpha}{2}} \left(2 \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)| \right)^{1-\frac{\alpha}{2}} \tag{18}$$

in connection with Young's inequality, to get

$$\lambda^{-1}\epsilon^{\frac{\alpha}{2}} [\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2}\,,4)} \leq \frac{\alpha}{2}\delta^{\frac{2}{\alpha}} \epsilon\lambda^{-1} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}_\tau(\tau)| + \left(1-\frac{\alpha}{2}\right)\delta^{-1/(1-\frac{\alpha}{2})}\lambda^{-1} \sup_{\tau\in(2+\frac{\lambda}{2}\,,4)} |\widetilde{U}(\tau)|$$

and choosing $\delta^{\frac{2}{\alpha}} = \lambda\delta_1$ for some $\delta_1 \ll 1$, we obtain

$$(19)\quad \lambda^{-1}\epsilon^{\frac{\alpha}{2}}[\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2},4)} \le \frac{\alpha}{2}\delta_1\epsilon \sup_{\tau\in(2+\frac{\lambda}{2},4)} |\widetilde{U}_\tau(\tau)| + 2\left(1-\frac{\alpha}{2}\right)\delta_1^{-\frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}}}\lambda^{-\frac{1}{1-\frac{\alpha}{2}}} \sup_{\tau\in(2+\frac{\lambda}{2},4)} |\widetilde{U}(\tau)|.$$

In addition we have

$$(20)\qquad |\widetilde{U}_\tau(\tau)| \le \sup_{x\in J_\tau} |U_\tau(x,\tau)| + k\sup_{x\in J_\tau}|U_x(x,\tau)|\,,$$

and we use the interpolation inequalities

$$[U(\cdot,\tau)]_{C^\alpha(J_\tau)} \le \delta_2^2[U]^{(2+\alpha)}_{J_\tau} + c\delta_2^{-\frac{1}{2}-\alpha}\| U\|_{L_2(J_\tau)}\,,$$

$$|U(x,\tau)| \le \delta_3^{2+\alpha}[U]^{(2+\alpha)}_{J_\tau} + c\delta_3^{-\frac{1}{2}}\| U\|_{L_2(J_\tau)}\,,$$

that, taking $\delta_2^2 = \delta_4\lambda$, $\delta_3^{2+\alpha} = \delta_4\lambda^{\frac{1}{1-\frac{\alpha}{2}}}$, for some $\delta_4 \ll 1$, can be rewritten in the form

$$(21)\qquad \lambda^{-1}[U(\cdot,\tau)]^{(\alpha)}_{J_\tau} \le \delta_4[U(\cdot,\tau)]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{5}{4}-\frac{\alpha}{2}}\| U\|_{L_2(J_\tau)}\,,$$

$$(22)\qquad \lambda^{-\frac{1}{1-\frac{\alpha}{2}}}\sup_{J_\tau}|U(x,\tau)| \le \delta_4[U(\cdot,\tau)]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{\frac{5}{4}+\frac{\alpha}{2}}{1-\alpha^2/4}}\| U\|_{L_2(J_\tau)}.$$

Finally we use the estimate

$$(23)\qquad \sup_{\tau\in(2+\frac{\lambda}{2},4)}\sup_{x\in J_\tau}|U_x(x,\tau)| \le C\left([U]^{(2+\alpha)}_{J_\tau} + \| U\|_{L_2(J_\tau)}\right)$$

in order to complete the estimation of all terms containing U on the r.h.s. of (17), namely:

$$(24)\qquad \frac{\epsilon}{\lambda}\sup_{\tau\in(2+\frac{\lambda}{2},4)}|U(\cdot,\tau)|_{C^\alpha(J_\tau)} \le \epsilon\left(\delta_4[U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{5}{4}-\frac{\alpha}{2}}\| U\|_{L_2(J_\tau)}\right)$$

(consequence of (21)),

$$(25)\qquad \lambda^{-1}\sup_{\tau\in(2+\frac{\lambda}{2},4)}|\widetilde{U}| \le \lambda^{\frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}}}\sup_{\tau\in(2+\frac{\lambda}{2},4)}\left[\delta_4[U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{\frac{5}{4}+\frac{\alpha^2}{4}}{1-\alpha^2/4}}\| U\|_{L_2(J_\tau)}\right]$$

(following from (22)),

$$(26)\qquad \begin{aligned}\lambda^{-1}\epsilon^{\frac{\alpha}{2}}[\widetilde{U}]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2},4)} \le \Bigg\{&\delta_1\epsilon\left[\sup_{\tau\in(2+\frac{\lambda}{2},4)}|U_\tau(\cdot,\tau)| + kC\sup_{\tau\in(2+\frac{\lambda}{2},4)}[U]^{(2+\alpha)}_{J_\tau} + \| U\|_{L_2(J_\tau)}\right] +\\ &+ c(\delta_1)\sup_{\tau\in(2+\frac{\lambda}{2},4)}\left[\delta_4[U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{\frac{5}{4}+\frac{\alpha}{2}}{1-\alpha^2/4}}\| U\|_{L_2(J_\tau)}\right]\Bigg\}\end{aligned}$$

(use first (19) and then (20), (23) and (22)).

At this point, we recall the definitions (11), (12) of ξ_λ and of v, and we deduce from (17) the basic inequality

$$\begin{aligned}
&\epsilon \sup_{\tau\in(2+\lambda,4)} |U_\tau(\cdot,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(2+\lambda,4)} |U(\cdot,\tau)|_{C^{2+\alpha}(J_\tau)} \le \\
&\le C\Bigg\{ \sup_{\tau\in(2+\frac{\lambda}{2},4)} |F_\epsilon(\cdot,\tau)|_{C^\alpha(J_\tau)}\left(1+\left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \epsilon^{\frac{\alpha}{2}}[\widetilde{F}_\epsilon]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2},4)} + \\
&+ \sup_{\tau\in(2+\frac{\lambda}{2},4)} |G_\epsilon(\tau)|\left(1+\left(\frac{\epsilon}{\lambda}\right)^{\frac{1+\alpha}{2}}\right) + \epsilon^{\frac{1+\alpha}{2}}[G_\epsilon]^{(\frac{1+\alpha}{2})}_{(2+\frac{\lambda}{2},4)} + \\
&+ \sup_{\tau\in(2+\frac{\lambda}{2},4)} |H_\epsilon(\tau)|\left(1+\left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \epsilon^{\frac{\alpha}{2}}[H_\epsilon]^{(\frac{\alpha}{2})}_{(2+\frac{\lambda}{2},4)} + \\
&+ \epsilon\left(\delta_4 \sup_{\tau\in(2+\frac{\lambda}{2},4)} [U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{5}{4}-\frac{\alpha}{2}} \sup_{\tau\in(2+\frac{\lambda}{2},4)} \| U \|_{L_2(J_\tau)}\right) + \\
&+ \lambda^{\frac{\frac{\alpha}{2}}{1-\frac{\alpha}{2}}}\left(\delta_4 \sup_{\tau\in(2+\frac{\lambda}{2},4)} [U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{\frac{5}{4}+\frac{\alpha}{2}}{1-\alpha^2/4}} \sup_{\tau\in(2+\frac{\lambda}{2},4)} \| U \|_{L_2(J_\tau)}\right)\left(1+\left(\frac{\epsilon}{\lambda}\right)^{\frac{\alpha}{2}}\right) + \\
&+ \left(1+\frac{\epsilon}{\lambda}\right)\Bigg\{\delta_1\epsilon\Bigg[\sup_{\tau\in(2+\frac{\lambda}{2},4)} |U_\tau(\cdot,\tau)| + C \sup_{\tau\in(2+\frac{\lambda}{2},4)} ([U]^{(\frac{\alpha}{2})}_{J_\tau} + \| U \|_{L_2(J_\tau)})\Bigg] + \\
&\qquad + c(\delta_1)\delta_4\Bigg[\sup_{\tau\in(2+\frac{\lambda}{2},4)} [U]^{(2+\alpha)}_{J_\tau} + c(\delta_4)\lambda^{-\frac{\frac{5}{4}+\frac{\alpha}{2}}{1-\alpha^2/4}} \sup_{\tau\in(2+\frac{\lambda}{2},4)} \| U \|_{L_2(J_\tau)}\Bigg]\Bigg\}\Bigg\}.
\end{aligned} \tag{27}$$

Setting

$$f(\lambda) = \lambda^{\frac{\frac{5}{4}+\frac{\alpha}{2}}{1-\alpha^2/4}}\left(\epsilon \sup_{\tau\in(2+\lambda,4)} |U_\tau(\cdot,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(2+\frac{\lambda}{2},4)} |U(\cdot,\tau)|_{C^{(2+\alpha)}}(J_\tau)\right), \tag{28}$$

we realize that (27) implies that

$$f(\lambda) \le \frac{1}{2}f\left(\frac{\lambda}{2}\right) + K + C \sup_{\tau\in(2,4)} \| U \|_{L_2(J_\tau)} =: \frac{1}{2}f\left(\frac{\lambda}{2}\right) + \overline{K} \tag{29}$$

where K dominates the sum of all terms not including U, *i.e.*

$$\begin{aligned}
K = C\Bigg(&\sup_{\tau\in(2,4)} |F_\epsilon|_{C^\alpha(J_\tau)} + \epsilon^{\frac{\alpha}{2}}[\widetilde{F}_\epsilon]^{(\frac{\alpha}{2})}_{(2,4)} + \sup_{\tau\in(2,4)} |G_\epsilon| + \epsilon^{\frac{1+\alpha}{2}}[G_\epsilon]^{(\frac{1+\alpha}{2})}_{(2,4)} + \\
&+ \sup_{\tau\in(2,4)} |H_\epsilon| + \epsilon^{\frac{\alpha}{2}}[H_\epsilon]^{(\frac{\alpha}{2})}_{(2,4)}\Bigg).
\end{aligned} \tag{30}$$

Indeed it suffices to choose δ_1 small enough and after that δ_4 small enough in order to get the coefficient $\frac{1}{2}$ in front of $f(\frac{\lambda}{2})$. Of course the constants denoted by C include the dependence on δ_1, δ_4.

Recursive application of (29) lead to the inequality

$$f(\lambda) \leq \frac{1}{2^n} f\left(\frac{\lambda}{2^n}\right) + \overline{K} \sum_{i=0}^{n-1} \frac{1}{2^i} \quad , \quad \forall \quad n > 1$$

and eventually to

$$f(\lambda) \leq 2\overline{K}. \tag{31}$$

Setting $\lambda = 1$ in (31) we conclude that

$$\epsilon \sup_{\tau \in (3,4)} |U_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau \in (3,4)} |U(\cdot\,,\tau)|_{C^\alpha(J_\tau)} \leq C\left(K + \sup_{\tau \in (2,4)} \| U \|_{L_2(J_\tau)}\right). \tag{32}$$

Now we use (7)-(9) to get

$$\epsilon^{1+\frac{\alpha}{2}} [U_\tau]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}} \leq \epsilon^{\frac{\alpha}{2}} [U_{xx}]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}} + \epsilon^{\frac{\alpha}{2}} [F_\epsilon]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}}, \tag{33}$$

where

$$[f]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}} = \sup_{(x,\tau)\in\Omega_{3,4}} \sup_{h\in(0,4-\tau)} h^{-\frac{\alpha}{2}} |f(x,\tau+h) - f(x,\tau)|,$$

and

$$[\widetilde{U}_\tau]^{(\frac{1+\alpha}{2})}_{(3,4)} \leq [H_\epsilon]^{(\frac{1+\alpha}{2})}_{(3,4)} + [\widetilde{U}_x]^{(\frac{1+\alpha}{2})}_{(3,4)}, \tag{34}$$

$$\sup |\widetilde{U}_\tau| \leq \sup |H_\epsilon| + b \sup |\widetilde{U}_x|. \tag{35}$$

We can estimate the norms of U_{xx}, $\widetilde{U}_x$ appearing in (33), (34) by means of interpolation inequalities as follows

$$\epsilon^{\frac{\alpha}{2}} [U_{xx}]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}} \leq C\left(\epsilon \sup_{\tau\in(3,4)} |U_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(3,4)} |U(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)}\right) \tag{36}$$

$$\epsilon^{\frac{1+\alpha}{2}} [\widetilde{U}_x]^{(\frac{1+\alpha}{2})}_{(3,4)} \leq C\left(\epsilon \sup_{\tau\in(3,4)} |U_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(3,4)} |U(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)}\right). \tag{37}$$

In this way we can put all the estimates together and write

$$\begin{aligned} &\epsilon \sup_{\tau\in(3,4)} |U_\tau(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \sup_{\tau\in(3,4)} |U(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)} + \\ &\qquad + \epsilon^{1+\frac{\alpha}{2}} [U_\tau]^{(\frac{\alpha}{2})}_{t,\Omega_{3,4}} + \epsilon^{\frac{1+\alpha}{2}} [\widetilde{U}_\tau]^{(\frac{1+\alpha}{2})}_{(3,4)} + \sup_{\tau\in(3,4)} |\widetilde{U}_\tau| \leq \\ &\leq C\Bigg(\sup_{\tau\in(2,4)} |F_\epsilon(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \epsilon^{\frac{\alpha}{2}} [F_\epsilon]^{(\frac{\alpha}{2})}_{t,\Omega_{2,4}} + \\ &\qquad + \sup_{\tau\in(2,4)} |G_\epsilon(\tau)| + \epsilon^{\frac{1+\alpha}{2}} [G_\epsilon]^{(\frac{1+\alpha}{2})}_{(2,4)} + \sup_{\tau\in(2,4)} |H_\epsilon| + \\ &\qquad + \epsilon^{\frac{1+\alpha}{2}} [H_\epsilon]^{(\frac{1+\alpha}{2})}_{(2,4)} \Bigg) + C \sup_{\tau\in(2,4)} \| U(\cdot\,,\tau) \|_{L_2(J_\tau)}. \end{aligned} \tag{38}$$

Returning to the original variables we deduce

$$(39)\quad \begin{aligned} &\| u \|_{\widehat{C}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_{\frac{3\tau}{4},\tau})} + \| \widetilde{u}_t \|_{\widehat{C}^{\frac{1+\alpha}{2}}(\frac{3\tau}{4},\tau)} \le \\ &\le C\Big(\| f \|_{\widehat{C}^{\alpha,\frac{\alpha}{2}}(\Omega_{\frac{\tau}{2},\tau})} + \| g \|_{\widehat{C}^{\frac{1+\alpha}{2}}(\frac{\tau}{2},\tau)} + \| h \|_{\widehat{C}^{\frac{1+\alpha}{2}}(\frac{\tau}{2},\tau)} + \tau^{-\frac{5}{2}-\alpha} \sup \| u \|_{L_2(J_\tau)} \Big). \end{aligned}$$

4. Conclusion of the Proof of Theorem 1.1

We turn our attention to $\| u \|_{L^2(J_\tau)}$.

Using (1)-(3) we can derive the equality

$$(40)\quad \frac{1}{2}\frac{d}{dt}\left(\int_{J(t)} u^2\,dx + b^{-1}\widetilde{u}^2 \right) + \int_{J(t)} u_x^2\,dx = b^{-1}h\widetilde{u} - gu(0,t) + \int_{J(t)} fu\,dx + \frac{k}{2}\widetilde{u}^2,$$

in which we replace

$$u(0,t) \quad \text{by} \quad \widetilde{u}(t) - \int_0^k u_x(x,t)\,dx$$

and

$$\int_{J(t)} fu\,dx \quad \text{by} \quad \widetilde{u}(t)\int_{J(t)} f(x,t)\,dx - \int_{J(t)} f(x,t)\left(\int_x^{kt} u_\xi(\xi,t)\,d\xi \right) dx.$$

Using the estimate

$$\left| \int_x^{kt} u_\xi\,d\xi \right| \le \int_0^{kt} \left| u_x \right| dx \le \sqrt{kt}\left(\int_{J(t)} u_x^2\,dx \right)^{\frac{1}{2}},$$

we obtain the differential inequality

$$(41)\quad \frac{d}{dt}\left(\int_{J(t)} u^2\,dx + b^{-1}\widetilde{u}^2 \right) + \int_{J(t)} u_x^2\,dx \le |b^{-1}h + I_f - g|\,|\widetilde{u}| + \frac{k}{2}\widetilde{u}^2 + C(|g|^2 + I_{|f|}^2)t,$$

with $I_f = \int_{J(t)} f\,dx$.

We can use (41) as a Gronwall type inequality for $sup_{\tau<t}\widetilde{u}^2$ (after time integration), arriving at the estimate

$$(42)\quad \sup_{\tau<t} |\widetilde{u}(\tau)| \le Ct \sup_{\tau<t}(|g| + |h| + I_{|f|}).$$

We only need to replace

$$\int_0^t \left| b^{-1}h + I_f - g \right| |\widetilde{u}|\,d\tau \quad \text{by} \quad \frac{1}{2c}\sup_{\tau<t}\widetilde{u}^2 + \frac{c}{2}\left(\int_0^t \left| b^{-1}h + I_f - g \right| d\tau \right)^2.$$

Let us now consider the quantity

$$(43)\qquad t^{-2}\int_{J(t)} u^2\,dx = \int_{J(t)} \left[t^{-1}\tilde{u}(t) - t^{-1}\int_x^{kt} u_\xi\,d\xi \right]^2 dx.$$

Using the inequality $(a-b)^2 \le 2(a^2+b^2)$ and

$$t^{-1}\left|\int_x^{kt} u_\xi\,d\xi\right| \le \sqrt{\frac{k}{t}}\left(\int_{J(t)} u_x^2\,dx\right)^{\frac{1}{2}}$$

we obtain

$$(44)\qquad \frac{1}{2}t^{-2}\int_{J(t)} u^2\,dx \le \frac{k}{t}\tilde{u}^2 + k^2\int_{J(t)} u_x^2\,dx.$$

Thanks to (44), by adding $\frac{1}{kt}\tilde{u}^2$ to (41) we can derive an inequality not containing $\int_{J(t)} u_x^2\,dx$, in which we can make use of (42), thus obtaining

$$(45)\qquad \frac{d}{dt}\int_{J(t)} u^2\,dx + \frac{1}{2k^2t^2}\int_{J(t)} u^2\,dx \le C\mu(t) - 2b^{-1}\tilde{u}\tilde{u}_t$$

with

$$(46)\qquad \mu(t) = \sup_{\tau<t}(g^2 + h^2 + I_{|f|}^2)t.$$

Now we need an estimate of $\tilde{u}_t$.

From (1)-(3) we find immediately

$$\frac{d}{dt}I_u = \frac{d}{dt}\int_{J(t)} u\,dx = I_f + b^{-1}h - g + k\tilde{u} - b^{-1}\tilde{u}_t\,,$$

i.e.

$$(47)\qquad b^{-1}\tilde{u}_t = -\frac{d}{dt}I_u + I_f + b^{-1}h - g + k\tilde{u}\,,$$

implying

$$\begin{aligned} b^{-1}\tilde{u}\tilde{u}_t &= -\frac{d}{dt}(\tilde{u}I_u) + \tilde{u}_t I_u + (I_f + b^{-1}h - g + k\tilde{u})\tilde{u} = \\ &= -\frac{d}{dt}(\tilde{u}I_u) + bI_u\left(-\frac{d}{dt}I_u + I_f - b^{-1}h - g + k\tilde{u}\right) + (I_f + b^{-1}h - g + k\tilde{u})\tilde{u}\,, \end{aligned}$$

which can be read as

$$(48)\qquad 2b^{-1}\tilde{u}\tilde{u}_t = -\frac{d}{dt}(2\tilde{u}I_u + bI_u^2) + 2(bI_u + \tilde{u})(I_f + b^{-1}h - g + k\tilde{u}).$$

Thus, going back to (45) we obtain

$$\frac{d}{dt}\left(\int_{J(t)} u^2\,dx - 2\tilde{u}I_u - bI_u^2\right) + \frac{1}{2k^2t^2}\int_{J(t)} u^2\,dx \leq \tag{49}$$
$$\leq C\mu(t) + 2(b|I_u| + |\tilde{u}|)|I_f + b^{-1}h - g + k\tilde{u}|.$$

Since $I_u^2 \leq kt \int_{J(t)} u^2\,dx$, we have

$$2|\tilde{u}|\,|I_u| \leq \frac{1}{2}\int_{J(t)} u^2\,dx + 2kt\tilde{u}^2 \leq \frac{1}{2}\int_{J(t)} u^2\,dx + Ct^2\mu(t)\,, \tag{50}$$

where we have used (42) and the definition (46) of $\mu(t)$. At this point (49) leads to

$$\frac{d}{dt}\left(\int_{J(t)} u^2\,dx - 2\tilde{u}I_u - bI_u^2\right) + \frac{C}{t^2}\left(\int_{J(t)} u^2\,dx - 2\tilde{u}I_u - bI_u^2\right) \leq C\mu(t). \tag{51}$$

Indeed we know how to dominate the added term $t^{-2}(2\tilde{u}I_u + bI_u^2)$ in terms of a fraction of $t^{-2}\int_{J(t)} u^2 dx$ and of $C\mu(t)$. Moreover, on the r.h.s. of (49) we have again terms like $\tilde{u}^2$ and $2|\tilde{u}|\,|I_u|$, while we may eliminate the linear terms in $|I_u|\,, |\tilde{u}|$ by using $(b|I_u| + |\tilde{u}|)\,|I_f + b^{-1}h + g| \leq \frac{1}{2t}(b|I_u| + |\tilde{u}|)^2 + C\mu(t)$. Multiplying by $e^{-C/t}$ we can integrate, obtaining

$$\int_{J(t)} u^2\,dx - 2\tilde{u}I_u - bI_u^2 \leq Ce^{C/t}\int_0^t \mu(t)e^{-C/\tau}\,d\tau \leq \tag{52}$$
$$\leq Ce^{C/t}\sup_{\tau<t}(\mu(\tau)\tau^2)\int_0^t e^{-C/\tau}\frac{d\tau}{\tau^2} = \sup_{\tau<t}(\mu(\tau)\tau^2)\,,$$

which brings us very close to our final estimate, since it gives

$$\int_{J(t)} u^2\,dx \leq \sup_{\tau<t}(\mu(\tau)\tau^2) + \sqrt{kt}\left(\int_{J(t)} u^2\,dx\right)^{\frac{1}{2}}(2|\tilde{u}| + b|I_u|)\,, \tag{53}$$

and in turn

$$\int_{J(t)} u^2\,dx \leq C\left\{\sup_{\tau<t}(\mu(\tau)\tau^2) + t(|\tilde{u}^2| + I_u^2)\right\}. \tag{54}$$

Once again we recall that $|\tilde{u}|$ is estimated by (42). It remains to estimate I_u. Integrating (47) we get

$$I_u = -b^{-1}\tilde{u} + \int_0^t (I_f + b^{-1}h - g)\,dx + k\int_0^t \tilde{u}\,d\tau\,, \tag{55}$$

which, together with (42) and (54), gives

$$\| u \|^2_{L_2(J(t))} \le C \sup (|g| + |h| + I_{|f|})^2 t^3. \tag{56}$$

Now, in order to conclude the proof of the theorem, all we have to do is to recall (39) and to evaluate

$$\begin{aligned} t^{-\frac{5}{2}-\alpha} \| u \|_{L_2(J(t))} &\le Ct^{-1-\alpha} \sup (|g| + |h| + I_{|f|}) \le \\ &\le C\{\sup \tau^{-1-\alpha} |g(\tau)| + \sup \tau^{-1-\alpha} |h(\tau)| + \sup_{J(\tau)} \tau^{-\alpha} |f(x,\tau)|\}. \end{aligned} \tag{57}$$ □

The theorem just proved can be slightly refined by replacing the spaces $\widehat{C}^{l,l/2}(\Omega_T)$, $\widehat{C}^l(0, T)$ with the spaces $\widehat{C}^{l,l/2}_\beta(\Omega_T)$, $\widehat{C}^l_\beta(0, T)$ whose respective norms are

$$\| u \|_{\widehat{C}^{l,l/2}_\beta(\Omega_T)} = \sup_{\tau \in (0,T)} \tau^\beta [u]^{(l,l/2)}_{\Omega_{\tau/2,\tau}} + \sup_{\Omega_T} t^{-l+\beta} |u(x,t)| ,$$

$$\| u \|_{\widehat{C}^l_\beta(0,T)} = \sup_{\tau \in (0,T)} \tau^\beta [u]^{(l)}_{(\tau/2,\tau)} + \sup_{t \in (0,T)} t^{-2l+\beta} |u(t)|$$

with $\beta > 0$.

Appendix

Auxiliary estimates for a linear problem in a trapezoidal domain

Here we derive the estimate we have used for problem (13)-(16)

Theorem A.2. *Let $V(x, t)$ solve the problem*

$$\epsilon V_t - V_{xx} = F(x,t), \qquad 0 < x < kt,\ 1 < t < 4, \tag{A.1}$$

$$V(x,1) = 0, \qquad 0 < x < k, \tag{A.2}$$

$$V_x(0,t) = G(t), \qquad 1 < t < 4, \tag{A.3}$$

$$\widetilde{V}_t(t) + bV_x(kt,t) = H(t), \qquad 1 < t < 4, \tag{A.4}$$

where $\widetilde{V}(t) = V(kt, t)$. Suppose that $F \in C^{\alpha, \frac{\alpha}{2}}$, $G \in C^{\frac{1+\alpha}{2}}$, $H \in C^{\frac{\alpha}{2}}$ and that

$$F(x,1) = 0, \qquad 0 < x < k, \tag{A.5}$$

$$G(1) = H(1) = 0. \tag{A.6}$$

Then we have

$$\begin{aligned} \epsilon \sup_{\tau \in (1,t)} |V_\tau(\cdot,\tau)|_{C^\alpha(J_\tau)} &+ \sup_{\tau \in (1,t)} |V(\cdot,\tau)|_{C^{2+\alpha}(J_\tau)} \le \\ &\le c\Bigg(\sup_{\tau \in (1,t)} |F(\cdot,\tau)|_{C^\alpha(J_\tau)} + \epsilon^{\frac{\alpha}{2}} [\widetilde{F}]^{(\frac{\alpha}{2})}_{(1,t)} + \sup_{\tau \in (1,t)} |G(\tau)| + \\ &\quad + \epsilon^{\frac{1+\alpha}{2}} [G]^{(\frac{1+\alpha}{2})}_{(1,t)} + \sup_{\tau \in (1,t)} |H(\tau)| + \epsilon^{\frac{\alpha}{2}} [H]^{(\frac{\alpha}{2})}_{(1,t)} \Bigg), \end{aligned} \tag{A.7}$$

where $\widetilde{F}(t) = F(k(t)\,,\,t)$, J_τ *denotes the interval* $(0\,,\,k\tau)$, *and* c *is a constant independent of the data.*

The philosophy of the proof is based upon the observation that while interior estimates are standard, the basic contribution comes from the behaviour near the lateral boundaries. Also we can say that if we make transformations letting lower order terms appear in the equation, they can be absorbed in the source term, since it will be easy to eliminate them from the right-hand side of the final estimate (A.7).

In view of this remark we may confine ourselves to studying the following pair of problems in the half strip $x > 0$, $1 < t < 4$, thanks to a localization procedure, utilizing multiplication by suitable cut-off functions.

Problem 1.

$$\epsilon V_{1t} - V_{1xx} = F_1(x, t)\,, \quad x > 0\,,\ 1 < t < 4\,, \tag{A.8}$$

$$V_1(x, 1) = 0\,, \quad x > 0\,, \tag{A.9}$$

$$V_{1x}(0\,,\,t) = G(t)\,, \quad 1 < t < 4. \tag{A.10}$$

Problem 2.

$$\epsilon V_{2t} - V_{2xx} = F_2(x, t)\,, \quad x > 0\,,\ 1 < t < 4\,, \tag{A.11}$$

$$V_2(x, 1) = 0\,, \quad x > 0\,, \tag{A.12}$$

$$V_{2t}(0\,,\,t) - bV_{2x}(0\,,\,t) = H(t)\,, \quad 1 < t < 4. \tag{A.13}$$

The functions $F_1\,,\,F_2$ may be supposed to have the same regularity as F in (A.1). Problem 2 comes actually from mapping the domain $x < kt$, $1 < t < 4$ into $x > 0$, $1 < t < 4$ by means of a linear transformation which generates a convective term, incorporated in the same term F_2.

The first problem is standard and for it we have the estimate

$$\begin{aligned} &\epsilon \sup_{\tau\in(1\,,\,t)} [V_{1\tau}(\cdot\,,\,\tau)]^{(\alpha)}_{\mathbf{R}_+} + \sup_{\tau\in(1\,,\,t)} [V_1(\cdot\,,\,\tau)]^{(2+\alpha)}_{\mathbf{R}_+} \leq \\ &\qquad \leq c\left(\sup_{\tau\in(1\,,\,t)} [F_1(\cdot\,,\,\tau)]^{(\alpha)}_{\mathbf{R}_+} + \epsilon^{\frac{1+\alpha}{2}} [G]^{(\frac{1+\alpha}{2})}_{(1\,,\,t)} \right), \quad 1 < t < 4. \end{aligned} \tag{A.14}$$

Coming to Problem 2, let us represent its solution in the form

$$V_2 = W + W_1 \tag{A.15}$$

where W, W_1 solve.

Problem 3.

$$\epsilon W_{1t} - W_{1xx} = F_2(x, t)\,, \quad x > 0\,,\ 1 < t < 4\,, \tag{A.16}$$

$$W_1(x, 1) = 0\,, \quad x > 0\,, \tag{A.17}$$

$$W_1(0\,,\,t) = 0\,, \quad 1 < t < 4. \tag{A.18}$$

PROBLEM 4.

$$\epsilon W_t - W_{xx} = 0\,, \quad x > 0\,, \quad 1 < t < 4\,, \tag{A.19}$$

$$W(x,1) = 0\,, \quad x > 0\,, \tag{A.20}$$

$$W_t(0,t) - bW_x(0,t) = H(t) + bW_{1x}(0,t) \equiv H'(t)\,, \quad 1 < t < 4. \tag{A.21}$$

Again we have a standard estimate for Problem 3 (see [6]):

$$\begin{aligned}\epsilon \sup_{\tau\in(1,t)} [W_{1\tau}(\cdot,\tau)]^{(\alpha)}_{\mathbf{R}_+} + \sup_{\tau\in(1,t)} [W_1(\cdot,\tau)]^{(2+\alpha)}_{\mathbf{R}_+} \le \\ \le c\left(\sup_{\tau\in(1,t)} [F_2(\cdot,\tau)]^{(\alpha)}_{\mathbf{R}_+} + \epsilon^{\frac{\alpha}{2}} [F_2(0,\tau)]^{(\frac{\alpha}{2})}_{(1,t)}\right).\end{aligned} \tag{A.22}$$

The solution of Problem 4 has an explicit representation:

$$W(x,t) = \int_1^t G_\epsilon(x, t-\tau) H'(\tau)\, d\tau\,, \tag{A.23}$$

where

$$G_\epsilon(x,t) = -2\int_0^t \Gamma_{\epsilon x}(x+bu, t-u)\, du\,, \tag{A.24}$$

with

$$\Gamma_\epsilon(x,t) = \frac{1}{\epsilon}\Gamma(x, \frac{t}{\epsilon}) = \frac{1}{2\sqrt{\pi t\epsilon}} e^{-\epsilon x^2/(4t)}\,, \tag{A.25}$$

so that $-2\Gamma_{\epsilon x}(x,t) = \frac{x\sqrt{\epsilon}}{2\sqrt{\pi}t^{\frac{3}{2}}} e^{-\epsilon x^2/(4t)}$.

The kernel G_ϵ is constructed in such a way that it behaves as a double layer potential for the boundary operator $\frac{\partial}{\partial t} - b\frac{\partial}{\partial x}$. Indeed

$$\left(\frac{\partial}{\partial t} - b\frac{\partial}{\partial x}\right) G_\epsilon = 2\int_0^t \frac{\partial}{\partial u}\Gamma_{\epsilon x}(x+bu, b-u)\, du = -2\Gamma_{\epsilon x}(x,t)\,,$$

so that the function $W(x,t)$ defined by (A.23) satisfies condition (A.21) and is a solution of Problem 4.

We can now evaluate W_{xx} by observing that

$$\begin{aligned} G_{\epsilon xx} &= -2\epsilon\int_0^t \Gamma_{\epsilon xt}(x+bu, t-u)\, du = \\ &= 2\epsilon\left\{\int_0^t \frac{\partial}{\partial u}\Gamma_{\epsilon x}(x+bu, t-u)\, du - b\int_0^t \Gamma_{\epsilon xx}(x+bu, t-u)\, du\right\} = -2\epsilon\Gamma_{\epsilon x}(x,t) + \epsilon b G_{\epsilon x}\end{aligned}$$

and that

$$G_{\epsilon x} = -2\epsilon\Gamma_\epsilon + \epsilon b G_\epsilon$$

by similar calculations, so that

$$G_{\epsilon xx} = -2\epsilon\left[\Gamma_{\epsilon x} + \epsilon b\left(\Gamma_\epsilon - \frac{b}{2}G_\epsilon\right)\right],$$

which finally gives

$$W_{xx} = -2\epsilon\int_1^t \Gamma_{\epsilon x}(x, t-\tau)H'(\tau)\,d\tau + \epsilon^2 b\int_1^t(-2\Gamma_\epsilon(x, t-\tau) + bG_\epsilon(x, t-\tau))H'(\tau)\,d\tau.$$

From (A.26) we derive the estimate

$$\begin{aligned}[W_{xx}(\cdot\,, t)]^{(\alpha)}_{\mathbf{R}_+} &\le c\left(\epsilon^{1+\frac{\alpha}{2}}[H']^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup_{\tau\in(1\,,t)}|H'(\tau)|\right) \le \\ &\le c\left(\epsilon^{1+\frac{\alpha}{2}}[H]^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup|H|\right) + \\ &\quad + c\left(\epsilon^{1+\frac{\alpha}{2}}[W_{1x}(0\,,\cdot)]^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup_{\tau\in(1\,,t)}|W_{1x}(0\,,\tau)|\right).\end{aligned} \tag{A.27}$$

Noting that

$$\epsilon^{1+\frac{\alpha}{2}}[W_{1x}(0\,,\cdot)]^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup_{\tau\in(1\,,t)}|W_{1x}(0\,,\tau)| \le c\epsilon^{\frac{1}{2}}\epsilon^{\frac{1+\alpha}{2}}[W_{1x}(0\,,\cdot)]^{(\frac{1+\alpha}{2})}_{(1\,,t)}$$

and using the interpolation inequality

$$\epsilon^{\frac{1+\alpha}{2}}[W_{1x}(0\,,\cdot)]^{(\frac{1+\alpha}{2})}_{(1\,,t)} \le c\left(\epsilon\sup_{\tau\in(1\,,t)}[W_{1\tau}(\cdot\,,\tau)]^{(\alpha)}_{\mathbf{R}_+} + \sup_{\tau\in(1\,,t)}[W_1(\cdot\,,\tau)]^{(2+\alpha)}_{\mathbf{R}_+}\right),$$

from (A.27) we obtain

$$\begin{aligned}\epsilon\sup_{\tau\in(1\,,t)}[W_\tau(\cdot\,,\tau)]^{(\alpha)}_{\mathbf{R}_+} &+ \sup_{\tau\in(1\,,t)}[W(\cdot\,,\tau)]^{(2+\alpha)}_{\mathbf{R}_+} \le \\ &\le c\left(\epsilon^{1+\frac{\alpha}{2}}[H]^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup_{\tau\in(1\,,t)}|H(\tau)|\right) + \\ &\quad + c\epsilon^{\frac{1}{2}}\left(\epsilon\sup_{\tau\in(1\,,t)}[W_{1\tau}(\cdot\,,\tau)]^{(\alpha)}_{\mathbf{R}_+} + 2\sup_{\tau\in(1\,,t)}[W_1(\cdot\,,\tau)]^{(2+\alpha)}_{\mathbf{R}_+}\right).\end{aligned} \tag{A.28}$$

Putting together (A.14), (A.22), (A.28), we deduce that the solution $V(x, t)$ of (A.1)-(A.4) satisfies

$$\begin{aligned}\epsilon\sup_{\tau\in(1\,,t)}|V_t(\cdot\,,\tau)|_{C^\alpha(J_\tau)} &+ \sup_{\tau\in(1\,,t)}|V(\cdot\,,\tau)|_{C^{2+\alpha}(J_\tau)} \le \\ &\le c\left(\sup_{\tau\in(1\,,t)}|F(\cdot\,,\tau)|_{C^\alpha(J_\tau)} + \epsilon^{\frac{\alpha}{2}}[\widetilde{F}]^{(\frac{\alpha}{2})}_{(1\,,t)} + \right. \\ &\quad \left. + \epsilon^{\frac{1+\alpha}{2}}[G]^{(\frac{1+\alpha}{2})}_{(1\,,t)} + \epsilon^{1+\frac{\alpha}{2}}[H]^{(\frac{\alpha}{2})}_{(1\,,t)} + \epsilon^{\frac{3+\alpha}{2}}\sup|H(\tau)|\right) + \\ &\quad + c\sum_{j=0}^{2}\sup_{\tau\in(1\,,t)}\sup_{x\in J_\tau}|D^j_x V(x\,,\tau)|,\end{aligned} \tag{A.29}$$

where the presence of the last sum is the consequence of the localization procedure. The latter term can be replaced by an estimate of the L_2-norm of V, thanks to the interpolation inequality

$$\sum_{j=0}^{2} \sup_{\tau\in(1,t)} \sup_{x\in J_\tau} |D_x^j V(x,\tau)| \le \kappa \sup_{\tau\in(1,t)} [V(\cdot,\tau)]_{J_\tau}^{(2+\alpha)} + c(\kappa) \sup_{\tau\in(1,t)} \| V(\cdot,\tau) \|_{L_2(J_\tau)}, \tag{A.30}$$

κ being a positive constant which can be taken as small as desired.

Thus it remains to estimate the L_2-norm of V. Using the identity

$$\epsilon V V_t + V_x^2 = (V_x V)_x + FV$$

we can write

$$\frac{1}{2}\frac{d}{dt}\left(\epsilon \int_{J(t)} V^2\,dx + b^{-1}\widetilde{V}^2\right) + \int_{J(t)} V_x^2\,dx = (b^{-1}H - G)\widetilde{V} + \frac{1}{2}k\epsilon\widetilde{V}^2 + G\int_0^{kt} V_x\,dx + \int_{J(t)} FV\,dx,$$

and adding and subtracting $I_F\widetilde{V}$, with $I_F = \int_{J(t)} F\,dx$, we can rewrite the right-hand side as follows

$$(b^{-1}H - G + I_F)\widetilde{V} + \frac{1}{2}k\epsilon\widetilde{V}^2 - G\int_0^{kt} V_x dx - \int_{J(t)} F\,dx \int_x^{kt} V_\xi(\xi,\tau)\,d\xi. \tag{A.31}$$

This allows to derive the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\epsilon \int_{J(t)} V^2 dx + b^{-1}\widetilde{V}^2\right) + \frac{1}{2}\int_{J(t)} V_x^2 dx \le \\ &\le \frac{1}{2}k\epsilon|\widetilde{V}|^2 + |b^{-1}H - G + I_F|\,|\widetilde{V}| + c(G^2 + I_{|F|}^2), \end{aligned} \tag{A.32}$$

yielding in particular

$$\sup_{\tau\in(1,t)} |\widetilde{V}(\tau)| \le c \sup_{\tau\in(1,t)} (|G| + |H| + |I_F|). \tag{A.33}$$

At this point (A.32) provides the desired L_2 estimate, if we can estimate $\widetilde{V}_t$. To this end we start from the equality

$$\epsilon\frac{d}{dt}I_V = \epsilon\frac{d}{dt}\int_{J(t)} V\,dx = I_F - G + k\epsilon\widetilde{V} + b^{-1}(H - \widetilde{V}_t) \tag{A.34}$$

and we multiply both sides by $\widetilde{V}$, obtaining

$$b^{-1}\widetilde{V}\widetilde{V}_t = -\epsilon\widetilde{V}\frac{d}{dt}I_V + \widetilde{V}(I_F - G + k\epsilon\widetilde{V} + b^{-1}H).$$

We note that

$$\widetilde{V}\frac{d}{dt}I_V = \frac{d}{dt}(\widetilde{V}I_V) - I_V\widetilde{V}_t$$

and we use once more (A.34), so to get

$$b^{-1}\widetilde{V}\widetilde{V}_t = -\epsilon\frac{d}{dt}\left(\widetilde{V}I_V + \frac{b}{2}I_V^2\right) + \widetilde{V}(I_F - G + k\epsilon\widetilde{V} + b^{-1}H) + \tag{A.35}$$
$$+ b\epsilon I_V(I_F + b^{-1}H - G + k\epsilon\widetilde{V}).$$

Putting this expression in (A.31), using estimate (A.32) and defining

$$\mu_0(t) = \sup_{\tau\in(1,t)} (|G| + |H| + I_{|F|})^2, \tag{A.36}$$

we get the inequality

$$\epsilon\frac{d}{dt}\left(\int\limits_{J(t)} V^2\,dx - 2\widetilde{V}I_V - b\epsilon I_V^2\right) + c\int\limits_{J(t)} V^2\,dx \le \tag{A.37}$$
$$\le c\mu_0(t) + 2(\epsilon|I_V| + |\widetilde{V}|)(I_{|F|} + b^{-1}|H| + |G| + k\epsilon|\widetilde{V}|).$$

Setting

$$Z(t) = \int\limits_{J(t)} V^2\,dx - 2\widetilde{V}I_V - \epsilon I_V^2, \tag{A.38}$$

for ϵ small enough and using once more (A.32), we can write

$$\epsilon\frac{dZ}{dt} + cZ \le c\mu_0 \quad , \quad 1 < t < 4, \tag{A.39}$$

and remembering that $Z(1) = 0$, we conclude that

$$Z(t) \le \frac{c}{\epsilon}\sup_{\tau\in(1,t)}\mu_0(\tau)\int\limits_1^t e^{-(c/\epsilon)(t-\tau)}\,d\tau \le c\sup_{\tau\in(1,t)}\mu_0(\tau). \tag{A.40}$$

Going back to (A.38), recalling again (A.32) and for ϵ small enough we obtain an algebraic inequality, which finally leads to

$$\| V \|_{L_2(J(t))} \le C\sup_{\tau\in(1,t)}\left(|H(\tau)| + |G(\tau)| + \int\limits_{J(t)} |F|\,dx\right). \tag{A.41}$$

At this point, remembering (A.29), (A.30), we obtain the desired estimate (A.7). Hence the theorem is proved, actually in a slightly stronger form, since the coefficient of $[H]^{(\alpha)}_{(1,t)}$ turns out to be $\epsilon^{1+\frac{\alpha}{2}}$ instead of $\epsilon^{\frac{\alpha}{2}}$.

Acknowledgements

Work partially supported by the Italian MURST Project «Free Boundary Problems ... ».

References

[1] E. Comparini - R. Gianni - P. Mannucci, *A filtration problem in a composite porous material with two free boundaries*. Advances in Math. Sc. and Appl., 11, 2001.

[2] A. Fasano, *A one-dimensional flow problem in porous media with hydrophile granules*. Math. Meth. Appl. Sci., 22, 1999, 605-617.

[3] A. Fasano, *Some two-scale processes involving parabolic equations*. In: I. Athanasopoulos - G. Makrakis - J. Rodrigues (eds.), *Free Boundary Problems, Theory and Applications*. CRC Press, 1999, 39-50.

[4] A. Fasano, *Porous media with hydrophile granules*. In: A. Fasano (ed.), *Complex Flows in Industrial Processes*. MSSET Birkhäuser, Chapt. 10, 2000, 307-332.

[5] A. Fasano - V.A. Solonnikov, *On one-dimensional unsaturated flow in a porous medium with hydrophile grains*. Functional Differential Equations, 8, 2001, 195-224.

[6] O.A. Ladyzhenskaya - V.A. Solonnikov - N.N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*. Trans. Math. Monographs, vol. 23, American Mathematical Society, 1988.

Pervenuta l'8 giugno 2001,
in forma definitiva il 15 ottobre 2001.

A. Fasano:
Dipartimento di Matematica «U. Dini»
Università degli Studi di Firenze
Viale Morgagni, 67/A - 50134 Firenze
fasano@math.unifi.it

V. Solonnikov:
V.A. Steklov Institute of Mathematics
(S. Petersburg Department)
Fontanka, 27 - 191011 S. Petersburg (Russia)
solonnik@pdmi.ras.ru