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Meccanica dei solidi. — *Shakedown theorems in poroplastic dynamics.* Nota (*) di
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Abstract. — The constitutive model assumed in this *Note* is poroplastic two-phase (solid-fluid) with full saturation and stable in Drucker's sense. A solid or structure of this material is considered, subjected to dynamic external actions, in particular periodic or intermittent, in a small deformation regime. A sufficient condition and a necessary one are established, by a «static» approach, for shakedown (or adaptation), namely for boundedness in time of the cumulative dissipated energy.

Key words: Shakedown; Poroplasticity; Dynamics.

Riassunto. — *Teoremi di adattamento in dinamica poroplastica.* Il modello costitutivo assunto in questa *Nota* è poroplastico bifase (solido-fluido) a saturazione totale e stabile nel senso di Drucker. Un solido o struttura di questo materiale è considerato soggetto ad azioni esterne dinamiche, in particolare periodiche o intermittenti, in regime di piccole deformazioni. Si dimostrano, in base ad un approccio «statico», una condizione sufficiente e una necessaria per l'adattamento (o «shakedown»), inteso come caratterizzato da limitatezza nel tempo dell'energia dissipata cumulativa.

1. Introduction

Coupled problems concerning multiphase media, in particular poroelasticity and poroplasticity problems, arise in various areas of modern technology, such as environmental, geotechnical, bio- and dam engineering. The mechanical theory underlying the time-marching linear and nonlinear computational methods, which are widely used nowadays to solve coupled solid-fluid problems, stems from the pioneering works on poroplasticity by Terzaghi, Fillunger and Biot and is at present the subject of comprehensive treatises, see *e.g.* [6, 13].

«Direct» methods are procedures apt to provide practically useful information on the nonholonomic response of a solid or structure to external actions without following its evolution in time (as for elastoplasticity, see surveys *e.g.* in [10, 12, 16]). These methods are still at their infancy in poroplasticity. Ultimate limit state analysis of earth dams and of masonry and concrete dams in the presence of diffused cracking (see *e.g.* [8]) motivated recent investigations on direct methods in poroplasticity of fully saturated systems, [2, 3, 19].

The contributions presented in this *Note* are generalizations of the shakedown theorems established in [2, 3] for quasi-static piecewise-linearized poroplasticity. The present results can also be interpreted as counterparts in hardening poroplasticity to Ceradini's theorems [1], which in turn had extended to dynamics classical Melan's theorems of plasticity. The poroplastic counterparts to the kinematic theorems of dynamic shakedown in poroplasticity, established in [5], will be presented elsewhere.

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The novel theorems proven and commented on in what follows concern the dynamics of two-phase solids and structures, the material behaviour of which is described by a hardening non-linear poroplastic model expounded in detail *e.g.* in [6]. The following hypotheses are adopted: infinitesimal deformation (linear kinematics); full saturation of the skeleton by the liquid; classical linear filtration law according to Darcy (constant permeability matrix); material stability in Drucker's sense. Some of these hypotheses may be unrealistic and too restrictive in certain engineering situations and, hopefully, will be relaxed in future developments.

2. Problem formulation

Reference is made in what follows to a poroplastic solid which occupies the volume V enclosed by the boundary S, which is assumed to be smooth, *i.e.* endowed with a unique outward normal n_i everywhere, in a Cartesian reference system x_i, $(i = 1\,,2\,,3)$. Dots will denote time derivatives, the summation convention is adopted for repeated indices.

The field equations can be formulated as follows over the domain V:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$\sigma_{ji,j} - (1-n)\rho^s \ddot{u}_i - n\rho^f \ddot{U}_i + b_i = 0 \tag{2}$$

$$s_i = n(U_i - u_i)\,, \quad q_i = \dot{s}_i\,, \quad \dot{\zeta} = -q_{i,i} \tag{3}$$

$$\dot{s}_i = k_{ij}\pi_j\,, \quad \pi_i = -p_{,i} - \rho^f \ddot{U}_i + b_i^f\,. \tag{4}$$

The above employed symbols have the following meaning: ε_{ij} denotes the strain tensor of the solid skeleton; u_i its (absolute) displacement vector; σ_{ij} the total stress tensor; U_i represents the displacement vector of the fluid moving in the pores; s_i the effective relative displacements of the fluid with respect to the skeleton; ζ means fluid content (fluid volume per unit volume of the bulk medium or «mixture»); q_i is the flux vector; p the fluid pressure; π_i denotes the driving force for the fluid filtration motion; n the porosity (volume of voids per unit volume of mixture); ρ^s and ρ^f are mass densities of the skeleton material and the fluid, respectively; b_i and b_i^f represent given body forces per unit volume of mixture and fluid, respectively (if only gravity acceleration g_i, as in most cases, then: $b_i = \rho g_i$ and $b_i^f = \rho^f g_i$, where $\rho = \rho^s(1-n) + \rho^f n$); k_{ij} indicates the permeability tensor, herein assumed to be constant in time.

Equation (1) expresses geometric compatibility of the skeleton; eq. (2) dynamic equilibrium of the mixture; eq. (3*c*) enforces fluid mass conservation; eqs. (4) formulate the (linear) Darcy's filtration law generalized to dynamics.

It is worth noting that the unknown pressure p and fluid content ζ have to be interpreted as addends to respective reference values. These addends are not sign-constrained in the present problem formulation (clearly, absolute values of these variables are so and should be dealt with as such in the more general mechanical framework of partial saturation poroplasticity, see *e.g.* [13]).

The material behaviour is assumed to be associative poroplastic isotropic with generally nonlinear hardening (see *e.g.* [6, 13], and [4, 15] for the counterparts in plasticity) and, hence, is governed by the following constitutive relation set, where $\alpha = 1, \ldots, y$, $\beta = 1, \ldots, \bar{y}$ (y and $\bar{y}$ being the numbers of the yield modes and of the internal variables, respectively):

$$\sigma_{ij} + mp\delta_{ij} = D_{ijrs}\varepsilon^e_{rs}, \qquad p = -Mm\varepsilon^e_{ii} + M\zeta^e \tag{5}$$

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}, \qquad \zeta = \zeta^e + \zeta^p \tag{6}$$

$$\varphi_\alpha(\sigma_{ij}, p, \chi_\beta) \leq 0 \tag{7}$$

$$\dot{\varepsilon}^p_{ij} = \frac{\partial\varphi_\alpha}{\partial\sigma_{ij}}\dot{\lambda}_\alpha, \qquad \dot{\zeta}^p = \frac{\partial\varphi_\alpha}{\partial p}\dot{\lambda}_\alpha \tag{8}$$

$$\dot{\eta}_\beta = -\frac{\partial\varphi_\alpha}{\partial\chi_\beta}\dot{\lambda}_\alpha, \qquad \chi_\beta = \frac{\partial\mathcal{H}}{\partial\eta_\beta} \tag{9}$$

$$\dot{\lambda}_\alpha \geq 0, \qquad \varphi_\alpha\dot{\lambda}_\alpha = 0. \tag{10}$$

In the above relationships: m and M are material parameters of isotropic poroelasticity; D_{ijrs} represents the elastic stiffness tensor of the porous skeleton endowed with the usual symmetries; superscripts e and p denote the poroelastic and the irreversible «plastic» part, respectively, of both strains and fluid content; φ_α indicates yield function and plastic potential as well, of the α-th yielding mode ($\alpha = 1, \ldots, y$); χ_β and η_β are static and kinematic internal variables, respectively ($\beta = 1, \ldots, \bar{y}$); $\mathcal{H}$ denotes Helmholz potential of «free» energy locked-in by re-arrangements at the material microscale.

Mechanically interpreted, the above relations have the following meanings: eqs. (5) describe (coupled) isotropic poroelasticity according to Biot, with four material parameters; eqs. (6) postulate additivity of poro-elastic and poroplastic (*i.e.* irreversible, path-dependent) strains and fluid content; inequalities (7) define the current poroelastic domain in the space of static variables σ_{ij} and p (or the fixed domain in the «augmented space» in the space of all static variables, see [4]); eqs. (8) and (9*a*) express the flow rules; eqs. (9*b*) establish a link between kinematic and static internal variables; finally, together with eqs. (7), eqs. (10) formulate Prager's consistency rule through the complementarity relationship, customary in plasticity and damage mechanics.

The evolution of the poroplastic solid considered is fully described by the combination of the above relationships and of the boundary and initial conditions specified below (with $S_u \cup S_t = S$, $S_u \cap S_t = \{\emptyset\}$; $S_p \cup S_q = S$, $S_p \cap S_q = \{\emptyset\}$):

$$u_i = \hat{u}_i\,, \quad \text{on } S_u; \qquad \sigma_{ji} n_j = \hat{t}_i\,, \quad \text{on } S_t \tag{11}$$

$$p = \hat{p}\,, \quad \text{on } S_p; \qquad q_i n_i = \hat{q}\,, \quad \text{on } S_q \tag{12}$$

$$\begin{cases} u_i = \hat{u}_i^0 \\ \dot{u}_i = \hat{\dot{u}}_i^0 \end{cases}, \quad \begin{cases} s_i = \hat{s}_i^0 \\ \dot{s}_i = \hat{\dot{s}}_i^0 \end{cases} \quad \text{and} \quad p = \hat{p}^0\,, \quad \text{in} \quad V \quad \text{at} \quad \tau = 0\,. \tag{13}$$

Like in the classical theory of plasticity, see *e.g.* [10-12], shakedown or adaptation of the poroplastic solid under a time-history of variable-repeated external actions means boundedness of the total «plastic» dissipated energy, cumulative in space (over V) and time (over $0 \leq \tau' \leq \tau$), namely:

$$\lim_{\tau \to \infty} \int_0^\tau \int_V \dot{\mathcal{D}}(x_i\,, \tau') dV\, d\tau' < +\infty\,. \tag{14}$$

Here $\dot{\mathcal{D}}$ represents the dissipation rate per unit volume. It is required to be nonnegative by thermodynamics, namely (account taken of eq. (9*b*), $\dot{\mathcal{V}} = \sigma_{ij}\dot{\varepsilon}_{ij}^e + p\dot{\zeta}^e + \dot{\mathcal{H}}$ being the rate of the total free energy variation per unit volume):

$$\dot{\mathcal{D}}(\tau) = \sigma_{ij}\dot{\varepsilon}_{ij} + p\dot{\zeta} - \dot{\mathcal{V}} = \sigma_{ij}\dot{\varepsilon}_{ij}^p + p\dot{\zeta}^p - \chi_\beta \dot{\eta}_\beta \geq 0\,. \tag{15}$$

Drucker's postulate of material stability in plasticity (*e.g.* [10-12]) preserves its unifying classification role in poroplasticity [6] and reads:

$$(\sigma_{ij} - \sigma'_{ij})\dot{\varepsilon}_{ij}^p + (p - p')\dot{\zeta}^p \geq 0 \tag{16}$$

where primed symbols mark static quantities that, at the considered instant, are meant to merely comply with the yield inequalities (7), while the other symbols refer to quantities (σ_{ij}, p and $\dot{\varepsilon}_{ij}^p$, $\dot{\zeta}^p$) which correspond to each other through the constitutive model in the actual process at time instant τ. Inequality (16) is here assumed to hold always (*i.e.* for any current internal variables η_β) and, like in classical plasticity, implies convexity of the current poroelastic domain, defined by eq. (7), and normality, eqs. (8). Inequality (16) might also be shown to entail «stability in the small» (positiveness of the second order work performed by an external agency for the kinematic perturbations it generates), namely:

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} + \dot{p}\dot{\zeta} \geq 0\,. \tag{17}$$

By algebraic manipulations of the constitutive relations (5)-(10) reduced to rates, it would be easy to show that eq. (17) implies the convexity of the internal potential $\mathcal{H}$ (*i.e.* «non-softening» behaviour in the jargon of plasticity).

The main objective pursued in what follows is to establish, for physical and engineering situations susceptible of the above description, sufficient and necessary conditions for shakedown in the sense of eq. (14). The final purpose of these conditions is to provide a basis for methods apt to achieve, without solving the nonlinear i.b.v. problem formulated by eqs. (1)-(13), practically important information about the «safety margin» of the system with respect to the ultimate limit-state characterized by lack of shakedown (or inadaptation).

3. A sufficient and a necessary condition for shakedown

First the following statement will be proven below.

THEOREM I. *The poroplastic solid will shakedown, in the sense of eq.* (14), *under the given history of dynamic external actions, if a scalar* $\xi > 1$ *exists such that the yield conditions eqs.* (7), *after a finite time interval* $[0, \tau^*]$, *are fulfilled by the fictitious linear poroelastic dynamic response* (σ_{ij}^E, p^E), *amplified by* ξ *and superposed to a self-equilibrated stress field* $\xi\sigma_{ij}^S(x_r)$ *constant in time, in the presence of static internal variables* $\xi\breve{\chi}_\beta(x_r)$ *constant in time*:

$$\varphi_\alpha\big(\xi\sigma_{ij}^E(x_r, \tau) + \xi\sigma_{ij}^S(x_r), \xi p^E(x_r, \tau), \xi\breve{\chi}_\beta(x_r)\big) \leq 0,$$
(18)
$$\alpha = 1, \dots, y, \qquad \beta = 1, \dots, \bar{y}, \qquad \forall x_r \in V, \qquad \forall \tau \geq \tau^*.$$

Before the formal proof of the above statement, a specification of the fields involved appears to be suitable.

The fictitious poroelastic response of the two-phase solid to the given history of external actions, $\sigma_{ij}^E(x_r, \tau)$ and $p^E(x_r, \tau)$, is governed by eqs. (1)-(5) and (11)-(13), *i.e.* for ε_{ij}^p and ζ^p set identically equal to zero.

The set of (time independent) self-stresses $\sigma_{ij}^S(x_r)$ consists of all stress fields complying with homogeneous equilibrium equations, namely with eq. (2) over V in the absence of inertia and body forces and with eq. (11*b*) on S_t in the absence of tractions. Any of these fictitious self-stress states is conceived to be concomitant with a pressure field which satisfies time-independent and homogeneous boundary conditions on S_p and homogeneous initial conditions on V, and with flux field which satisfies time-independent and homogeneous boundary conditions on S_q. Therefore, in the presence of those states, both fields identically vanish, *i.e.* $p^S(x_r) = 0$ and $q_i^S(x_r) = 0$ at any time, in view of the homogeneous field equations of mass conservation and filtration, eqs. (3) and (4), concerning them.

The set of variables $\breve{\chi}_\beta(x_r)$ mentioned in the statement consists of all static internal variables which can be generated through the constitutive relationships (9*b*) by (unconstrained) kinematic internal variables $\breve{\eta}_\beta(x_r)$. Depending on the definition of Helmholz energy function $\mathcal{H}$, the dimensionality of $\breve{\chi}_\beta$ is equal or less than that of $\breve{\eta}_\beta$.

PROOF. Consider the following stress and pressure fields arising from the superposition of the fictitious poroelastic and self equilibrated fields referred to in the statement and specified in the above remarks:

$$\overline{\sigma}_{ij}(x_r, \tau) \equiv \sigma_{ij}^E(x_r, \tau) + \sigma_{ij}^S(x_r) \qquad \overline{p}(x_r, \tau) \equiv p^E(x_r, \tau). \tag{19}$$

The bars in the above symbols will be adopted for concomitant similarly generated fields of other quantities, when needed.

Let Δ denote the difference between the actual response field of the poroplastic system at time τ and its fictitious counterpart (barred) defined by eq. (19) at the same τ. Let us consider the following overall energy quantities (non-negative in view of the meaning of their integrands), generated as functionals of the above defined differences (and, hence, similarly marked by Δ):

$$\Delta\dot{\mathcal{D}}^f(\Delta q_i) \equiv \int_V [k_{rs}]^{-1} \Delta q_r \Delta q_s dV \geq 0 \tag{20}$$

$$\begin{aligned} &\Delta\mathcal{E}(\Delta\varepsilon^e_{ij}, \Delta\zeta^e) \equiv \\ &\equiv \frac{1}{2}\int_V \left[D_{ijrs}\Delta\varepsilon^e_{ij}\Delta\varepsilon^e_{rs} + Mm^2 \cdot (\Delta\varepsilon^e_{rr})^2 - 2mM\Delta\varepsilon^e_{rr}\Delta\zeta^e + M\cdot(\Delta\zeta^e)^2\right] dV \geq 0 \end{aligned} \tag{21}$$

$$\Delta\mathcal{K}(\Delta\dot{u}_s, \Delta\dot{U}_s) \equiv \frac{1}{2}\int_V (1-n)\rho^S \Delta\dot{u}_r\Delta\dot{u}_r + n\rho^f \Delta\dot{U}_r\Delta\dot{U}_r\, dV \geq 0\,. \tag{22}$$

Let $\breve{\eta}_\beta(x_s)$ and $\breve{\chi}_\beta(x_s)$ be fictitious conjugate internal variables constant in time and corresponding to each other through eqs. (9b). We define as follows another energy functional, $\mathcal{L}^s$, which turns out to be nonnegative in view of the convexity and differentiability of Helmholz potential $\mathcal{H}$:

$$\mathcal{L}^s(\eta_\beta, \breve{\eta}_\beta) \equiv \int_V [\mathcal{H}(\eta_\beta) - \mathcal{H}(\breve{\eta}_\beta) - \breve{\chi}_\beta \cdot (\eta_\beta - \breve{\eta}_\beta)] dV \geq 0\,. \tag{23}$$

The time derivative of the above energy functional reads:

$$\dot{\mathcal{L}}^s(\dot{\eta}_\beta, \eta_\beta, \breve{\eta}_\beta) = \int_V [\dot{\mathcal{H}}(\eta_\beta) - \breve{\chi}_\beta\dot{\eta}_\beta] dV = \int_V (\chi_\beta - \breve{\chi}_\beta)\dot{\eta}_\beta dV\,. \tag{24}$$

The differences $\Delta\sigma_{ij}$ and Δp, in view of their definitions given with reference to eq. (19), are associated with vanishing external actions, namely: body forces in V, boundary tractions on S_t, boundary pressure on S_p, displacements on S_u and fluxes on S_q. This fact becomes clear if the actual plastic strains ε^p_{ij} and the permanent fluid content ζ^p in their (unknown) time histories are conceived as external actions and the linearity of the poroelastic dynamic governing equations is noticed and, consequently, superposition of effects is applied. In quasi-static plasticity, similar standpoint is historically attributed to G. Colonnetti. It is worth noting that a field ζ^p constant in time does not give rise to self-stresses nor pressure in the steady-state response to it, as it can be seen from the governing relation set of poroplasticity in Section 2.

As a consequence of the above remarks on the difference fields denoted by Δ, the virtual work principle generalized to coupled problems (see *e.g.* [6]) provides the equation:

$$\begin{aligned} &\int_V (\Delta\sigma_{ij}\Delta\dot{\varepsilon}_{ij} + \Delta p\Delta\dot{\zeta}) dV = \\ &= -\int_V \left[(1-n)\rho^s\Delta\ddot{u}_r\Delta\dot{u}_r + n\rho^f\Delta\ddot{U}_r\Delta\dot{U}_r\right] dV - \int_V [k_{rs}]^{-1}\Delta q_r\Delta q_s dV\,. \end{aligned} \tag{25}$$

Account taken of the additivity hypothesis eq. (6) and, once again, of the meaning of field differences marked by Δ, through time integration and making use of the energy functionals defined by eqs. (20)-(22), the above virtual work equation yields:

$$\int_{\tau^*}^{\tau}\int_V (\Delta\sigma_{ij}\Delta\dot{\varepsilon}^p_{ij} + \Delta p \Delta\dot{\zeta}^p)dV\, d\tau' - \left[\Delta\mathcal{E}(\tau^*) + \Delta\mathcal{K}(\tau^*) + \Delta\mathcal{D}^f(\tau^*)\right] = \tag{26}$$

$$= -\left[\Delta\mathcal{E}(\tau) + \Delta\mathcal{K}(\tau) + \Delta\mathcal{D}^f(\tau)\right] \leq 0$$

where the inequality follows from the above noted nonnegativeness of each addend in the r.h.s. of the equality (26). It is worth noting that the last addend (at difference from the two others) represents an energy which depends on the whole time history of the flux field.

In the fictitious (barred symbols) response, eqs. (19), there are no rates of plastic strains and of the irreversible changes of fluid content (*i.e.* $\dot{\bar{\varepsilon}}^p_{ij} = 0$, $\dot{\bar{\zeta}}^p = 0$). Therefore, the inequality (26) can be re-written in the form:

$$\int_{\tau^*}^{\tau}\int_V (\Delta\sigma_{ij}\dot{\varepsilon}^p_{ij} + \Delta p \dot{\zeta}^p)dV\, d\tau' \leq \left[\Delta\mathcal{E}(\tau^*) + \Delta\mathcal{K}(\tau^*) + \Delta\mathcal{D}^f(\tau^*)\right]. \tag{27}$$

By subtracting side by side from the above inequality (27) the equation (24), integrated in time, and dropping $-\mathcal{L}^s(\tau)$ in view of its negativeness from the r.h.s. of the resulting inequality, we can write it as follows:

$$\int_{\tau^*}^{\tau}\int_V \left[\Delta\sigma_{ij}\dot{\varepsilon}^p_{ij} + \Delta p \dot{\zeta}^p - (\chi_\beta - \breve{\chi}_\beta)\dot{\eta}_\beta\right]dV\, d\tau' \leq \tag{28}$$

$$\leq \left[\Delta\mathcal{E}(\tau^*) + \Delta\mathcal{K}(\tau^*) + \Delta\mathcal{D}^f(\tau^*) + \mathcal{L}^s(\tau^*)\right].$$

Consider now the yield inequalities (18) and denote by $\widetilde{\varphi}_\alpha$ the yield functions on their l.h.s. Clearly, the stated conditions (18) and the constitutive sign constraints (10*a*) imply the inequality:

$$\dot{\lambda}_\alpha \widetilde{\varphi}_\alpha \leq 0\,. \tag{29}$$

The following further inequality flows from eqs. (29) and (10*b*):

$$(\widetilde{\varphi}_\alpha - \varphi_\alpha)\dot{\lambda}_\alpha \leq 0\,. \tag{30}$$

Taking into account this inequality, the convexity and smoothness of yield functions φ_α and the flow rules eqs. (8) and (9*a*), we can write:

$$(\widetilde{\varphi}_\alpha - \varphi_\alpha)\dot{\lambda}_\alpha \geq (\xi\,\bar{\sigma}_{ij} - \sigma_{ij})\cdot\frac{\partial\varphi_\alpha}{\partial\sigma_{ij}}(\sigma_{ij}, p, \chi_\beta)\dot{\lambda}_\alpha + (\xi\,\bar{p} - p)\cdot\frac{\partial\varphi_\alpha}{\partial p}(\sigma_{ij}, p, \chi_\beta)\dot{\lambda}_\alpha + \tag{31}$$

$$+ (\xi\breve{\chi}_\beta - \chi_\beta)\cdot\frac{\partial\varphi_\alpha}{\partial\chi_\beta}(\sigma_{ij}, p, \chi_\beta)\dot{\lambda}_\alpha = (\xi\,\bar{\sigma}_{ij} - \sigma_{ij})\dot{\varepsilon}^p_{ij} + (\xi\,\bar{p} - p)\dot{\zeta}^p - (\xi\breve{\chi}_\beta - \chi_\beta)\dot{\eta}_\beta.$$

Equations (30) and (31) imply:

$$(\xi\,\bar{\sigma}_{ij} - \sigma_{ij})\dot{\varepsilon}^p_{ij} + (\xi\,\bar{p} - p)\dot{\zeta}^p - (\xi\breve{\chi}_\beta - \chi_\beta)\dot{\eta}_\beta \leq 0\,. \tag{32}$$

This inequality, if combined with the second expression of dissipation rate in eq. (15) amplified by ξ, through trivial manipulations, leads to:

$$(33) \qquad \frac{(\xi-1)}{\xi}\dot{\mathcal{D}} \leq (\sigma_{ij} - \overline{\sigma}_{ij})\dot{\varepsilon}^p_{ij} + (p - \overline{p})\dot{\zeta}^p - (\chi_\beta - \breve{\chi}_\beta)\dot{\eta}_\beta \,.$$

Now, let eq. (33) be integrated in space and time from a finite instant τ^*:

$$(34) \qquad \frac{(\xi-1)}{\xi}\int_{\tau^*}^{\tau}\int_V \dot{\mathcal{D}}(\tau')dV\,d\tau' \leq \int_{\tau^*}^{\tau}\int_V \left[(\sigma_{ij} - \overline{\sigma}_{ij})\dot{\varepsilon}^p_{ij} + (p - \overline{p})\dot{\zeta}^p - (\chi_\beta - \breve{\chi}_\beta)\dot{\eta}_\beta\right] dV\,d\tau.$$

Account taken of inequality (28), the following final inequality is obtained from eq. (34):

$$(35) \qquad \int_{\tau^*}^{\tau}\int_V \dot{\mathcal{D}}(\tau')dV\,d\tau' \leq \frac{\xi}{(\xi-1)}\left[\Delta\mathcal{E}(\tau^*) + \Delta\mathcal{K}(\tau^*) + \Delta\mathcal{D}^f(\tau^*) + \mathcal{L}^s(\tau^*)\right].$$

The r.h.s. of inequality (35) contains finite quantities only, since $\tau^* < \tau \leq \infty$. Therefore the cumulative dissipated energy turns out to be bounded, *i.e.* shakedown does occur under the conditions specified by Statement I, in particular by eq. (18).

THEOREM II. *If a poroplastic solid shakes down after a finite time (say τ^{**}) under a given history of external actions, then a self-equilibrium time-constant stress field $\sigma^S_{ij}(x_r)$ and a time-constant internal variable field $\chi_\beta(x_r)$ exist such that these self-stresses superposed to the fictitious poroelastic dynamic response to the external actions in the presence of those internal variables, the yield inequalities eqs. (7) after a time interval $[0\,,\tau^{**}]$ are fulfilled everywhere, namely:*

$$(36) \qquad \begin{gathered} \varphi_\alpha\left(\sigma^E_{ij}(x_r\,,\tau) + \sigma^S_{ij}(x_r)\,,\, p^E(x_r\,,\tau)\,,\, \chi_\beta(x_r)\right) \leq 0\,, \\ \alpha = 1,\ldots,y\,, \quad \beta = 1,\ldots,\overline{y}\,, \quad \forall x_r \in V\,, \quad \forall \tau \geq \tau^*\,. \end{gathered}$$

The above proposition can be proven by paraphrasing the proof of classical Melan's «second statement» given by W. Koiter in [11] («nearly self-evident proof» in his words).

Shakedown is clearly impossible if no time-constant distributions of residual stresses σ^S_{ij} and of internal variables χ_β exist, on which the stresses and pressures due to a poroelastic response to the external loads may be superimposed without violation of the yield condition after a time τ^{**}. This remark directly implies the above Theorem II.

Statements I and II together represent the generalization to dynamic poroplasticity of Melan's theorems of quasi-static elastoplasticity.

4. ON APPLICATIONS

In the engineering situations which are amenable to the mathematical model formulated in Section 2 and involve variable-repeated dynamic loads, the following crucial question often arises: what is the «safety factor» s with respect to inadaptation, *i.e.* the critical value s of the load factor μ such that for $\mu < s$ shakedown occurs, whereas for $\mu > s$ it does not? For $\mu > s$ the cumulative dissipated energy grows unboundedly in the simulation, so that, in practical terms, either incremental collapse or alternating plastic yielding is expected to eventually lead to structural failure.

Often, a distinction is made between external actions which are accurately *a priori* known and constant in time («dead loads») and those («live loads») expected to possibly growth in accidental heavy service conditions: then, the load factor μ is applied to the latter category only.

Due to the linearity of poroelastic i.b.v. problems, the response (σ_{ij}^{E}, p^{E}) can be split in two addends related to dead loads and live loads separately, the latter one alone being multiplied by μ.

The factor ξ, which shows up in the sufficient condition alone, under the only requirement to be larger than one, can be dropped in view of a classical stability argument, see *e.g.* [7, 10].

Because of the homogeneous nature of the self-equilibrium equations characterizing the variables $\sigma_{ij}^{S}(x_r)$, clearly, they can appear in inequalities (18) without the factor ξ.

Thus, as a consequence of the sufficient (I) and the necessary (II) shakedown conditions established in what precedes, the above defined safety factor μ can be computed by solving the optimization problem:

$$(37) \qquad s = \max_{\mu, \tau^*, \sigma_{ij}^{S}, \breve{\chi}_\beta} \{\mu\}, \quad \text{subject to the following inequalities:}$$

$$\varphi_\alpha\big(\mu\sigma_{ij}^{E}(x_r, \tau) + \sigma_{ij}^{S}(x_r), p^{E}(x_r, \tau), \breve{\chi}_\beta(x_r)\big) \le 0,$$

$$\alpha = 1, \dots, y, \quad \beta = 1, \dots, \overline{y}, \quad \forall x_r \in V, \quad \forall \tau \ge \tau^*.$$

The path of reasoning, which is beyond the present purposes, leading from Theorems I and II to the formulation (37) of shakedown analysis, is similar to the traditional one, which was given mathematical rigour in [7] and was followed in elastoplasticity (see *e.g.* [12]), in elastic-plastic dynamics [1, 5] and in quasi-static poroplasticity [2, 3].

When discretization is performed by means of finite element modelling in space (see *e.g.* [14]) and a convex envelope is preliminarily and efficiently computed everywhere in V for the poroelastic response (σ_{ij}^{E}, p^{E}), the above constrained optimization becomes a problem in convex nonlinear programming, if the time τ^* is no longer a variable.

Such computationally advantageous simplification is readily be achievable in the following practically meaningful situations.

(*i*) Whatever the loading history may be, if merely a conservative lower bound $s^* \le s$ is sought, then, clearly, τ^* can be fixed *a priori* on the basis of engineering judgement.

(*ii*) If the external actions are reasonably assumed to vary periodically in time, then $\tau^* = 0$ can be set in eq. (37). In fact, under this hypothesis the poroelastic response (σ_{ij}^{E}, p^{E}) consists of a periodic cyclic addend and a transient one. Since the latter addend is damped off and becomes negligible after a finite time, it is immaterial for shakedown analysis, which, therefore, can be carried out by assuming initial conditions apt to ensure periodicity starting from $\tau = 0$. A similar circumstance was pointed out in [9] for dynamic plasticity.

(*iii*) The dynamic excitation in some cases (typically in seismic engineering) can be interpreted as a sequence of known loading histories occurring in an unknown order and separated by time intervals with the same steady-state, statical regime equal to the initial one assumed before the first loading event. The safety factor s with respect to such intermittent external actions can be proven to result from eq. (37) when $\tau^* = 0$ is assumed in it, by the arguments similar to those adopted in [17, 18] for the special case of dynamic plasticity.

5. Conclusions

With respect to the shakedown theory in traditional (inviscid) plasticity, the novelty of the theoretical results presented in what precedes rests on the combination of two kinds of time-dependence: the one implicit in poroplasticity, expressed by time-derivatives of the first order; the dynamic one entailing second order time-derivatives.

For such combination, sufficient and necessary shakedown conditions have been established herein by a «static» approach, in the spirit of classical Melan's theorems. These conditions provide, in a number of engineering situations interpretable by dynamic poroplasticity, a basis for computing the safety factor with respect to inadaptation failure by means of inequality constrained maximization, namely «directly», avoiding costly step-by-step procedures.

Computational issues and relaxations of the present restrictive assumptions (in primis: Drucker's stability postulate, linearity of filtration laws and full saturation) are subjects of current research.

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