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Morse index and blow-up points of solutions of some nonlinear problems

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Equazioni a derivate parziali. — *Morse index and blow-up points of solutions of some nonlinear problems.* Nota (*) di KHALIL EL MEHDI e FILOMENA PACELLA, presentata dal Socio A. Ambrosetti.

ABSTRACT. — In this *Note* we consider the following problem

$$\begin{cases} -\Delta u = N(N-2)u^{p_\varepsilon} - \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth starshaped domain in $\mathbb{R}^N$, $N \geq 3$, $p_\varepsilon = \frac{N+2}{N-2} - \varepsilon$, $\varepsilon > 0$, and $\lambda \geq 0$. We prove that if u_ε is a solution of Morse index $m > 0$ than u_ε cannot have more than m maximum points in Ω for ε sufficiently small. Moreover if Ω is convex we prove that any solution of index one has only one critical point and the level sets are starshaped for ε sufficiently small.

KEY WORDS: Elliptic problem; Morse index; Blow-up analysis.

RIASSUNTO. — *Indice di Morse e punti di massimo di soluzioni di alcuni problemi non lineari.* Si consideri il seguente problema

$$\begin{cases} -\Delta u = N(N-2)u^{p_\varepsilon} - \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{su } \partial\Omega \end{cases}$$

dove Ω è un dominio regolare limitato e stellato in $\mathbb{R}^N$, $N \geq 3$, $p_\varepsilon = \frac{N+2}{N-2} - \varepsilon$, $\varepsilon > 0$, e $\lambda \geq 0$. Si dimostra che se u_ε è una soluzione di indice di Morse $m > 0$ allora u_ε non può avere più di m punti di massimo in Ω, se ε è sufficientemente piccolo. Inoltre, se Ω è convesso si dimostra che ogni soluzione di indice di Morse 1 ha un unico punto critico e gli insiemi di livello sono stellati, se ε è sufficientemente piccolo.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let us consider the following problem

$$\begin{cases} -\Delta u = N(N-2)u^{p_\varepsilon} - \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ starshaped with respect to the origin, $N \geq 3$, $p_\varepsilon = \frac{N+2}{N-2} - \varepsilon$, $\varepsilon > 0$, and $\lambda \geq 0$.

We recall that if $\varepsilon = 0$, p_ε becomes $\frac{N+2}{N-2}$ which is equal to $2^* - 1$, where 2^* is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$. It is well known that for $\varepsilon = 0$, Problem (1.1) does not have any solution in starshaped domains, because of the Pohozaev's identity [6]. Hence the L^∞-norm of the solutions of (1.1) must blow-up as $\varepsilon \to 0$.

(*) Pervenuta in forma definitiva all'Accademia il 7 novembre 2001.

In this *Note* we prove two results for positive solutions of (1.1) which give a relation between the Morse index of the solution and the number of the maximum points. We recall that the Morse index of a solution u_ε of (1.1) is the number of negative eigenvalues of the linearized operator L_ε at u_ε, that is $L_\varepsilon = -\Delta - N(N-2)p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} - \lambda$.

Our first result is the following

THEOREM 1.1. *Let u_ε be a solution of (1.1) of Morse index m. Then for ε sufficiently small, u_ε cannot have more than m points where it achieves its maximum.*

A somehow similar result, in the case $\lambda = 0$, has been proved by Bahri-Li-Rey [1] in a different way.

Our second result deals with solution of (1.1) of Morse index one in convex domains.

THEOREM 1.2. *Let Ω be convex and u_ε be a solution of (1.1) of Morse index one. Then for ε sufficiently small u_ε has only one critical point x_ε where u_ε achieves its maximum. Moreover the level sets of u_ε are starshaped with respect to x_ε.*

This theorem extends to solutions of index one and to the case $\lambda > 0$ a recent result of Grossi-Molle [4] proved for the case $\lambda = 0$ and for solutions whose energy tends to the best Sobolev constant S, as $\varepsilon \to 0$. In doing so we also simplify the proof of [4] for the case $\lambda = 0$. However the main idea of the proof is taken from [4]. Finally in [4] the same kind of result is proved also for «single peak solutions» of some other subcritical problems. It is easy to see that our result for solutions of index one applies also to these problems.

2. PROOF OF THEOREM 1.1

PROOF. Let x_ε^1 be a maximum point for u_ε in Ω, that is $u_\varepsilon(x_\varepsilon^1) = ||u_\varepsilon||_\infty$. Since Ω is starshaped we know that $||u_\varepsilon||_\infty \to +\infty$. Then we set

$$\Omega_\varepsilon = ||u_\varepsilon||_\infty^{1/\alpha_\varepsilon}(\Omega - x_\varepsilon^1)$$

where $\alpha_\varepsilon = \frac{2}{p_\varepsilon - 1}$ and define, in Ω_ε, the function

$$v_\varepsilon(y) = \frac{1}{||u_\varepsilon||_\infty} u_\varepsilon\left(x_\varepsilon^1 + \frac{y}{||u_\varepsilon||_\infty^{1/\alpha_\varepsilon}}\right)$$

which satisfies

$$(2.1)\qquad \begin{cases} -\Delta v_\varepsilon = N(N-2)v_\varepsilon^{p_\varepsilon} - \dfrac{\lambda}{||u_\varepsilon||_\infty^{p_\varepsilon - 1}} v_\varepsilon & \text{in } \Omega_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \\ v_\varepsilon(0) = 1\,, \text{and } 0 < v_\varepsilon \le 1 & \text{in } \Omega_\varepsilon\,. \end{cases}$$

Using standard elliptic estimates we have that v_ε converges to U in C^2_{loc} where U is a positive solution of the equation

$$(2.2)\qquad -\Delta u = N(N-2)u^{\frac{N+2}{N-2}} \qquad \text{in } A$$

where A is either $\mathbb{R}^N_+$ or $\mathbb{R}^N$. Since, by a result of Gidas-Spruck [2] (2.2) has no solutions in $\mathbb{R}^N_+$, the only possibility is that $A = \mathbb{R}^N$ and hence U is the unique positive solution of (2.2) in $\mathbb{R}^N$, with $U(0) = 1$, thus

$$U(y) = \frac{1}{(1+|y|^2)^{\frac{(N-2)}{2}}}\,.$$

Now, let us consider the function

$$\varphi^1_\varepsilon(x) = (x - x^1_\varepsilon)\cdot \nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x)\,.$$

It is easy to check that φ^1_ε satisfies

$$-\Delta\varphi^1_\varepsilon - N(N-2)p_\varepsilon u_\varepsilon^{p_\varepsilon - 1}\varphi^1_\varepsilon + \lambda\varphi^1_\varepsilon = -2\lambda u_\varepsilon \le 0 \text{ in } \Omega. \tag{2.3}$$

We also have $\varphi^1_\varepsilon(x^1_\varepsilon) > 0$, because $\nabla u_\varepsilon(x^1_\varepsilon) = 0$.

Let us denote by B_1 the ball $B(x^1_\varepsilon\,, \frac{\bar{R}}{||u_\varepsilon||_\infty^{1/\alpha_\varepsilon}})$, with $\bar{R} > 1$. We claim that

$$\varphi^1_\varepsilon(x) < 0 \quad \text{for any} \quad x \in \partial B_1 \tag{2.4}$$

for ε sufficiently small.

In fact, by the convergence of v_ε to U and some easy computations we have that, for every $R > 0$

$$y\nabla v_\varepsilon(y) + \alpha_\varepsilon v_\varepsilon(y) \to \frac{N-2}{2}\,\frac{1-|y|^2}{(1+|y|^2)^{\frac{N}{2}}} \tag{2.5}$$

as $\varepsilon \to 0$, uniformly on $\partial B(0\,, R)$.

If $R > 1$, from (2.5) we derive that for ε sufficiently small

$$y\nabla v_\varepsilon(y) + \alpha_\varepsilon v_\varepsilon(y) < 0 \quad \text{on } \partial B(0\,, R)\,.$$

From this we get (2.4) which together with $\varphi^1_\varepsilon(x^1_\varepsilon) > 0$ imply that there exists a nodal region D^1_ε in B_1 where φ^1_ε is positive.

Multiplying (2.3) by φ^1_ε and integrating over D^1_ε we get that the quadratic form corresponding to the linearized operator L_ε computed in φ^1_ε is less or equal than zero in D^1_ε. This implies that the first eigenvalue of L_ε is less or equal than zero in $D^1_\varepsilon \subset B_1$. Moreover, using the strict concavity of the limit function U and the C^2_{loc} convergence of v_ε to U, it is easy to prove that u_ε does not have any other critical point in B_1 other than x^1_ε, for ε sufficiently small and, actually u_ε is strictly concave and $(x - x^1_\varepsilon)\nabla u_\varepsilon < 0$ in $B_1 \setminus \{0\}$.

Now, if u_ε has another maximum point x^2_ε in $\Omega \setminus B_1$, repeating exactly the same blow-up procedure, around x^2_ε, and considering the function $\varphi^2_\varepsilon(x) = (x - x^2_\varepsilon)\nabla u_\varepsilon +$ $+ \alpha_\varepsilon u_\varepsilon(x)$, we get another ball $B_2 = B(x^2_\varepsilon\,, \frac{\bar{R}}{||u_\varepsilon||_\infty^{1/\alpha_\varepsilon}})$ and inside it a nodal region D^2_ε of φ^2_ε where the first eigenvalue of L_ε is less or equal than zero. Since in B_2 the solution u_ε has the same concavity properties as in B_1 and, in particular, $(x - x^2_\varepsilon)\nabla u_\varepsilon < 0$ in $B_2 \setminus \{x^2_\varepsilon\}$, we deduce that B_1 and B_2 are disjoints.

Iterating this procedure we have that, for ε sufficiently small, if u_ε has q maximum points, there exist q disjoints balls B_1, $B_2\,, \ldots\,, B_q$ each of one containing a region

D^i_ε, $i = 1, \ldots, q$, where the first eigenvalue of the linearized operator is less or equal than zero. If the solution has index m, using the variational characterisation of the $(m+1)$-th eigenvalue, we get that q cannot be bigger than m. □

REMARK 2.1. If $x^1_\varepsilon, x^2_\varepsilon, \ldots, x^q_\varepsilon$ are just local maximum points of u_ε with the property that $u_\varepsilon(x^i_\varepsilon) \to +\infty$ as $\varepsilon \to 0$ then by a result of Schoen [8] (see also [5. 7] for the same kinds of ideas) we have that for ε small enough $|x^i_\varepsilon - x^j_\varepsilon| \geq d > 0$ and hence, since $|x^i_\varepsilon - x^j_\varepsilon|^2(u_\varepsilon(x^i_\varepsilon)u_\varepsilon(x^j_\varepsilon))^{1/\alpha_\varepsilon} \to +\infty$, our proof applies showing that if the index is m then cannot exist more than m such points, for ε sufficiently small.

3. PROOF OF THEOREM 1.2

PROOF. Let u_ε be a solution of index one. By the previous theorem we know that, for ε sufficiently small, u_ε has only one maximum point, say $x^1_\varepsilon \in \Omega$.

Let us consider, as before, the function

$$\varphi^1_\varepsilon(x) = (x - x^1_\varepsilon).\nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x).$$

In the previous proof we have shown that there exists a nodal region D^1_ε in $B_1 = B(x^1_\varepsilon, \frac{\bar{R}}{||u_\varepsilon||_\infty^{1/\alpha_\varepsilon}})$, $\bar{R} > 1$, where the first eigenvalue of the linearized operator L_ε at u_ε is less or equal than zero. Since u_ε is of index one, we know that the second eigenvalue $\lambda_2(L_\varepsilon, \Omega) \geq 0$. Hence, by the variational charecterisation of the second eigenvalue, we have that the first eigenvalue of L_ε in $\Omega \setminus B_1$ is positive.

On the other side, since Ω is convex and so starshaped with respect to x^1_ε, using the Hopf's Lemma (see [3]), we have that

$$(x - x^1_\varepsilon).\nabla u_\varepsilon(x) < 0, \quad \text{for any } x \in \partial\Omega.$$

Hence, $\varphi^1_\varepsilon(x) < 0$ on $\partial\Omega$ because $u_\varepsilon = 0$ on $\partial\Omega$. Thus there cannot exist any point $\bar{x}$ in $\Omega \setminus B_1$ where $(\bar{x} - x^1_\varepsilon).\nabla u_\varepsilon(\bar{x}) \geq 0$ otherwise we would have $\varphi^1_\varepsilon(\bar{x}) > 0$ and hence there would exist a connected component C_ε in $\Omega \setminus B_1$, $C_\varepsilon \cap \partial\Omega = \emptyset$, where $\varphi^1_\varepsilon > 0$. Then using (2.3) we would have that the first eigenvalue of L_ε in C_ε would be less or equal than zero against the fact the first eigenvalue of L_ε in $\Omega \setminus B_1$ is positive. This proves that $(x - x^1_\varepsilon).\nabla u_\varepsilon(x) < 0$ in $\Omega \setminus B_1$ and so also in $\Omega \setminus \{x^1_\varepsilon\}$ because of what we showed in the proof of Theorem 1.1. Therefore the level sets of u_ε, for ε sufficiently small, are strictly starshaped with respect to x^1_ε and, in particular, u_ε does not have any critical point other than x^1_ε. □

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