Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

Sergio Vessella

Three cylinder inequalities and unique continuation properties for parabolic equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **13** (2002), n.2, p. 107–120.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_2002_9_13_2_107_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 2002.

Rend. Mat. Acc. Lincei
s. 9, v. 13:107-120 (2002)

Equazioni a derivate parziali. — *Three cylinder inequalities and unique continuation properties for parabolic equations.* Nota di SERGIO VESSELLA, presentata (*) dal Socio M. Primicerio.

ABSTRACT. — We prove the following unique continuation property. Let u be a solution of a second order linear parabolic equation and S a segment parallel to the t-axis. If u has a zero of order faster than any non constant and time independent polynomial at each point of S then u vanishes in each point, (x, t'), such that the plane $t = t'$ has a non empty intersection with S.

KEY WORDS: Continuation of solutions; Stability estimates; Ill-posed Problem.

RIASSUNTO. — *Disuguaglianze dei tre cilindri e proprietà di continuazione unica per equazioni paraboliche.* Dimostriamo la seguente proprietà di continuazione unica. Sia u una soluzione di un'equazione parabolica lineare del secondo ordine e S un segmento parallelo all'asse t. Se u ha uno zero di ordine maggiore di qualsiasi polinomio non costante e indipendente dal tempo allora u si annulla in ogni punto, (x, t'), tale che il piano $t = t'$ intersechi S.

1. INTRODUCTION

Let T be a positive number and D a domain in $\mathbb{R}^n$, $n \geq 2$. Let $A(x,t) = \{a^{ij}(x,t)\}_{i,j=1}^n$ be a non analytic matrix valued function. Assume A is symmetric and satisfies an uniformly ellipticity condition in $D \times (-T, T)$. Let us consider the parabolic operator

$$L(\cdot) = \operatorname{div}(A(x,t)\nabla \cdot) - \frac{\partial \cdot}{\partial t}. \tag{1.1}$$

Let u be a weak solution to the equation

$$L(u) + b(x,t)\cdot \nabla u + c(x,t)u = 0\,, \quad \text{in } D\times(-T,T)\,, \tag{1.2}$$

where b is a bounded vector valued function in $D \times (-T, T)$ and c is a bounded function in $D \times (-T, T)$. Denote by B_r the n-dimensional ball of radius r centered in 0. We are interested in two types of unique continuation properties.

(*a*) *Unique continuation in the interior.* Let $D = B_1$. We will prove, under suitable assumptions on A, that

$$\int_{-T}^{T}\int_{B_r} u^2\,dxdt = O(r^\nu), \text{ as } r \to 0, \text{ for every } \nu \in \mathbb{N}, \text{ implies } u \equiv 0. \tag{1.3}$$

(*b*) *Unique continuation at the boundary.* Let φ be a sufficiently smooth function in $\mathbb{R}^{n-1}$ satisfying $\varphi(0) = 0$. Let $D = \{x = (x', x_n) \in B_1 | \varphi(x') < x_n\}$. Set $\Gamma = \operatorname{graph}(\varphi) \cap B_1$.

(*) Nella seduta dell'8 febbraio 2002.

Assume either $u = 0$ on $\Gamma \times (-T, T)$ or $A\nabla u \cdot \mathbf{n} = 0$ on $\Gamma \times (-T, T)$ (here $\mathbf{n}$ denotes the outer unit normal to graph(φ)). We will prove, under suitable assumptions on A that,

$$\int_{-T}^{T} \int_{B_r \cap D} u^2 \, dxdt = O(r^\nu), \text{ as } r \to 0, \text{ for every } \nu \in \mathbb{N}, \text{ implies } u \equiv 0. \tag{1.4}$$

If A does not dependent on time and is Lipschitz continuous, stronger versions of the properties (a) and (b) are known. In this case, Lin in [9], has proved that if $\int_{B_r} u^2(x, 0)dx = O(r^\nu)$, for every $\nu \in \mathbb{N}$, then $u(\cdot, 0) \equiv 0$. Similar results, concerning the continuation at the boundary have been proved in [1, 3, 4]. In [3, 4] three cylinder inequalities, with optimal exponent, are proved and applied to find sharp stability estimates for inverse problems with unknown boundaries.

On the other side, in the case of A time dependent, weaker forms than property (a) are known. If $A \in C^{2,2}$ then Lees and Protter in [8] have proved that if $\int_{-T}^{T} \int_{B_r} u^2 \, dxdt = O(e^{-r^{-\nu}})$, for every $\nu \in \mathbb{N}$, then $u \equiv 0$. In [6] a three cylinder inequality is proved assuming that $A \in C^{3,1}$. The three cylinder inequality proved in [6] is a not simple consequence of the Carleman estimate proved in [5] with the same hypothesis on A. Because of the not uniformity in u of the above mentioned three cylinder inequality, the best we can get is: if $\int_{-T}^{T} \int_{B_r} u^2 \, dxdt = o(e^{-C|\log r|^3})$, where C is a positive constant, then $u \equiv 0$. A similar three cylinder inequality, in the hypothesis $A \in C^{3,1}$, for semilinear parabolic equations has been proved in [10].

This paper is organized as follows: in Section 2 we prove property (a) for $L = \Delta - q_0 \frac{\partial}{\partial t}$. This is the prototype for more general results; namely the properties (a) and (b) (in the case $u = 0$ on $\Gamma \times (-T, T)$) with the hypotheses $A \in C^{2,1}$, $\varphi \in C^{1+\alpha}$, where $\alpha \in (0, 1]$. The exact statement of this results are given in Section 3 (Theorems 3 and 4) and proved in details in [11]. A crucial step in our proofs has been the application of the transformation used by Hörmander, [7, Section 3] to prove strong unique continuation for second order elliptic equations.

2. The case $L = \Delta - q_0 \dfrac{\partial}{\partial t}$

For any $r > 0$ and $t_0 > 0$ denote by $Q_r^{t_0}$ the cylinder $B_r \times (-t_0, t_0)$. Theorems 1 and 2, below, are based on the following notations and hypotheses. Given the positive numbers λ, Λ, R_0, T, with $\lambda \geq 1$ and let q_0 be a given function on $Q_{R_0}^T$, assume that

$$\lambda^{-1} \leq q_0(x, t) \leq \lambda, \text{ if } (x, t) \in Q_{R_0}^T, \tag{2.1}$$

$$R_0 |\nabla q_0| + T \left| \frac{\partial q_0}{\partial t} \right| \leq \Lambda, \text{ a.e. in } Q_{R_0}^T. \tag{2.2}$$

By L we denote the following parabolic operator

$$(Lu)(x, t) = \Delta u(x, t) - q_0(x, t) \frac{\partial u}{\partial t}(x, t), \text{ if } (x, t) \in Q_{R_0}^T.$$

For any $x \in \mathbb{R}^n \setminus \{0\}$ we denote by $(\rho, \omega) \in (0, +\infty) \times S^{n-1}$ (S^{n-1} being the unity sphere of $\mathbb{R}^n$) the polar coordinates of x, with $\rho = |x|$, $\omega = \frac{x}{|x|}$. By ∂_{ω_i}, $i \in \{1, \dots, n\}$, we denote the operator of derivation on the sphere, that is $(\partial_{\omega_i}\phi)(\omega) = \frac{\partial}{\partial x_i}(\phi(\frac{x}{|x|}))_{|x=\omega}$, where $\omega \in S^{n-1}$ and ϕ is a function differentiable on S^{n-1}. We denote by Δ_ω the Laplace-Beltrami operator in the unit sphere, $\Delta_\omega = \sum_{i=1}^n \partial^2_{\omega_i}$. We denote by ∂_ω and $\nabla_{\rho,\omega}$, respectively, the vector operators $(\partial_{\omega_1}, \dots, \partial_{\omega_n})$ and $(\frac{\partial}{\partial\rho}, \rho^{-1}\partial_{\omega_1}, \dots, \rho^{-1}\partial_{\omega_n})$. Moreover, we denote by $\mathcal{L}$ the operator L expressed in polar coordinates, namely

$$(2.4)\quad (\mathcal{L}\widetilde{u})(\rho, \omega, t) = (Lu)(\rho\omega, t) = \left(\frac{\partial^2 \widetilde{u}}{\partial\rho^2} + \frac{n-1}{\rho}\frac{\partial\widetilde{u}}{\partial\rho} + \frac{1}{\rho^2}\Delta_\omega\widetilde{u} - \widetilde{q}_0\frac{\partial\widetilde{u}}{\partial t}\right)(\rho, \omega, t),$$

for $(\rho, \omega, t) \in (0, R_0) \times S^{n-1} \times (-T, T)$, where $\widetilde{u}(\rho, \omega, t) = u(\rho\omega, t)$ and $\widetilde{q}_0(\rho, \omega, t) = q_0(\rho\omega, t)$. We shall fix the space dimension $n \geq 2$ throughout the paper, therefore we shall omit the dependence of various quantities on n. We denote by the letter C the positive constants. The value of the constants may change from line to line, but we specify their dependence everywhere they appear. Sometimes, for any variable s, we shall write u_s instead of $\frac{\partial u}{\partial s}$ and u_{ss} instead of $\frac{\partial^2 u}{\partial s^2}$.

Theorem 1. *Let* (2.1), (2.2) *be satisfied. Let* $k(y) = y + e^y$ *and* $\chi(\rho) = k^{-1}(\log\frac{\rho}{R_0})$. *There exist positive constants* θ, $\theta < 1$, C *and* C_1, θ *depends on* λ *and* Λ *only,* C *is an absolute constant,* C_1 *depends on* λ, Λ *and* $R_0^2 T^{-1}$ *only, such that*

$$(2.5)\quad \begin{aligned} &\int_{-T}^{T}\int_0^{R_0}\int_{S^{n-1}} (\tau\rho^2|\nabla_{\rho,\omega}\widetilde{u}|^2 + \tau^3\widetilde{u}^2)e^{(-2\tau+1)\chi(\rho)}\rho^{-1}\,d\omega\,d\rho\,dt \leq \\ &\leq C\int_{-T}^{T}\int_0^{R_0}\int_{S^{n-1}} (\mathcal{L}\widetilde{u})^2 e^{-2\tau\chi(\rho)}\rho^3\,d\omega\,d\rho\,dt, \end{aligned}$$

for every $u \in C_0^\infty(Q^T_{\theta R_o} \setminus \{0\} \times (-T, T))$ *and* $\tau \geq C_1$.

Proof. Let u be a function in $C_0^\infty(Q^T_{R_o} \setminus \{0\} \times (-T, T))$. Introducing $z = \log\frac{\rho}{R_0}$ as a new coordinates instead of ρ, by (2.4) we have

$$(\mathcal{L}\widetilde{u})(R_0 e^z, \omega, t) = \left(\frac{e^{-2z}}{R_0^2}\left(\frac{\partial^2\widetilde{u}_1}{\partial z^2} + (n-2)\frac{\partial\widetilde{u}_1}{\partial z} + \Delta_\omega\widetilde{u}_1\right) - \widetilde{q}_1\frac{\partial\widetilde{u}_1}{\partial t}\right)(z, \omega, t),$$

for $(z, \omega, t) \in (-\infty, 0) \times S^{n-1} \times (-T, T)$, where $\widetilde{u}_1(z, \omega, t) = \widetilde{u}(R_0 e^z, \omega, t)$ and $\widetilde{q}_1(z, \omega, t) = \widetilde{q}_0(R_0 e^z, \omega, t)$.

Now we introduce the transformation (see [7, Section 3]) $z = k(y)$, where $k(y) = y + e^y$. Setting

$$a(y) = (n-2)(1 + e^y) - \frac{e^y}{(1 + e^y)},$$

$$(2.6)\quad q(y, \omega, t) = R_0^2 e^{2k(y)}(1 + e^y)^2\widetilde{q}_1(k(y), \omega, t),$$

and, defining the operator P, $P = \frac{\partial^2}{\partial y^2} + a(y)\frac{\partial}{\partial y} + (1+e^y)^2\Delta_\omega - q\frac{\partial}{\partial t}$, we get

$$(\mathcal{L}\widetilde{u})(R_0 e^{k(y)}, \omega, t) = \frac{e^{-2k(y)}}{R_0^2(1+e^y)^2}(P\widetilde{u}_2)(y, \omega, t),$$

for $(y, \omega, t) \in (-\infty, y_0) \times S^{n-1} \times (-T, T)$, where y_0 is such that $k(y_0) = 0$ and $\widetilde{u}_2(y, \omega, t) = \widetilde{u}_1(k(y), \omega, t)$.

Set $\widetilde{u}_2 = e^{\tau y} v$, $P_\tau(v) = e^{-\tau y} P(e^{\tau y} v)$,

$$B_0 = \tau^2 + a(y)\tau, \quad B_1 = 2\tau + a(y), \tag{2.7}$$

$$P_\tau^{(1)}(v) = B_0 v + \frac{\partial^2 v}{\partial y^2} + (1+e^y)^2\Delta_\omega v,$$

$$P_\tau^{(2)}(v) = B_1\frac{\partial v}{\partial y} - q\frac{\partial v}{\partial t},$$

we have $P_\tau(v) = P_\tau^{(1)}(v) + P_\tau^{(2)}(v)$.

Denoting by $\int(.)$ the integral $\int_{-T}^{T}\int_{-\infty}^{y_0}\int_{S^{n-1}}(.)d\omega d\rho dt$ we have

$$\int (P_\tau(v))^2 = 2\int P_\tau^{(1)}(v)P_\tau^{(2)}(v) + \int (P_\tau^{(1)}(v))^2 + \int (P_\tau^{(2)}(v))^2. \tag{2.8}$$

Examine the integrals at the right hand side of (2.8).
We have

$$\begin{aligned} 2\int P_\tau^{(1)}(v)P_\tau^{(2)}(v) &= 2\int (1+e^y)^2 B_1 \Delta_\omega v v_y - 2\int (1+e^y)^2 q\Delta_\omega v v_t + \\ &+ \int (B_0 B_1 (v^2)_y - B_0 q(v^2)_t + B_1(v_y^2)_y - 2(qv_t v_y)_y + q(v_y^2)_t + 2q_y v_t v_y). \end{aligned} \tag{2.9}$$

By the symmetry of the operator Δ_ω and the anti-symmetry of the operator $\frac{\partial}{\partial y}$ we obtain

$$2\int (1+e^y)^2 B_1 \Delta_\omega v v_y = -\int ((1+e^y)^2 B_1)_y v \Delta_\omega v,$$

therefore

$$2\int (1+e^y)^2 B_1 \Delta_\omega v v_y = \int ((1+e^y)^2 B_1)_y |\partial_\omega v|^2 \geq 2\tau \int |\partial_\omega v|^2 e^y, \quad \text{if } \tau \geq \frac{3}{2}.$$

Moreover, this inequality and integrations by parts in the second and third integral at the right hand side of (2.9) give

$$\begin{aligned} 2\int P_\tau^{(1)}(v)P_\tau^{(2)}(v) &\geq 2\tau\int |\partial_\omega v|^2 e^y + \int (1+e^y)^2 (2(\partial_\omega q \cdot \partial_\omega v)v_t - |\partial_\omega v|^2 q_t) + \\ &+ \int ((-(B_0 B_1)_y + (B_0 q)_t) v^2 - (a'(y) + q_t)(v_y)^2 + 2q_y v_t v_y), \qquad \text{if } \tau \geq \frac{3}{2}. \end{aligned}$$

This inequality and (2.8) give

$$\begin{aligned}\int (P_\tau(v))^2 \geq 2\tau \int |\partial_\omega v|^2 e^y - C_1 \Lambda R_0^2 \int |\partial_\omega v||v_t| e^{3y} - C_2 \int |\partial_\omega v|^2 e^{2y} + \\ + \int (P_\tau^{(1)}(v))^2 - C_2 \tau^2 \int v^2 e^y + \int H(v_y, v_t; y, \tau),\end{aligned} \tag{2.10}$$

where C_1 is an absolute constant, C_2 depends on Λ and $R_0^2 T^{-1}$ only and $H(\xi, \eta; y, \tau)$ is the following quadratic form in the variables ξ and η

$$H(\xi, \eta; y, \tau) = 2q_y \xi\eta - Ce^y \xi^2 + (B_1 \xi - q\eta)^2, \tag{2.11}$$

where C depends on Λ and $R_0^2 T^{-1}$ only.

Now, we prove that

$$H(\xi, \eta; y, \tau) \geq \frac{\tau}{2}\xi^2 + \frac{R_0^4 e^{4y}}{2\lambda^2 \tau}\eta^2, \text{ if } (\xi, \eta) \in \mathbb{R}^2, \ y \leq -C_1 \text{ and } \tau \geq C_2, \tag{2.12}$$

where C_1 depends on λ and Λ only and C_2 depends on λ, Λ and $R_0^2 T^{-1}$ only. First observe that (2.6) gives

$$\left| \frac{q_y}{q} - 2 \right| \leq (4 + 2e\lambda\Lambda) e^y, \quad \text{if } y \leq y_0. \tag{2.13}$$

The second of (2.7), (2.11) and (2.13) give

$$\begin{aligned}H(\xi, \eta; y, \tau) = \frac{qq_y}{B_1}\left(2 - \frac{1}{B_1}\frac{q_y}{q}\right)\eta^2 - Ce^y\xi^2 + \left(B_1\xi - \left(q - \frac{q_y}{B_1}\right)\eta\right)^2 \geq \\ \geq \frac{R_0^4 e^{4y}}{\lambda^2 \tau}\eta^2 - Ce^y \xi^2, \text{ if } (\xi, \eta) \in \mathbb{R}^2, \ y \leq -C_1 \text{ and } \tau \geq C_2,\end{aligned} \tag{2.14}$$

where C depends on Λ and $R_0^2 T^{-1}$ only, C_1 depends on λ and Λ only and C_2 depends on λ, Λ and $R_0^2 T^{-1}$ only. Similarly we get

$$H(\xi, \eta; y, \tau) \geq 2\tau\xi^2, \text{ if } (\xi, \eta) \in \mathbb{R}^2, \ y \leq -C_1 \text{ and } \tau \geq C_2, \tag{2.15}$$

where C_1 depends on λ and Λ only and C_2 depends on λ, Λ and $R_0^2 T^{-1}$ only. Summing (2.14) and (2.15) we obtain (2.12). (2.12) yields

$$\int H(v_y, v_t; y, \tau) \geq \frac{\tau}{2}\int v_y^2 + \frac{R_0^4}{2\lambda^2\tau}\int v_t^2 e^{4y}, \text{ if } v \in C_0^\infty(\mathcal{O}_{C_1}) \text{ and } \tau \geq C_2, \tag{2.16}$$

here and in the sequel, for any positive number b, we use the notation

$$\mathcal{O}_b = (-\infty, -b) \times S^{n-1} \times (-T, T),$$

C_1 and C_2 are constants, C_1 depends on λ and Λ only, C_2 depends on λ, Λ and $R_0^2 T^{-1}$ only.

Now, we examine the integral $\int (P_\tau^{(1)}(v))^2$. Let δ be a positive number that we shall choose later. We obtain

$$\int (P_\tau^{(1)}(v))^2 = \int (P_\tau^{(1)}(v) - \delta\tau v e^y + \delta\tau v e^y)^2 \geq 2\delta\tau \int (P_\tau^{(1)}(v) - \delta\tau v e^y) v e^y =$$

$$= 2\delta\tau \int \left(\left(B_0 - \delta\tau e^y + \frac{1}{2} \right) v^2 - v_y^2 - (1 + e^y)|\partial_\omega v|^2 \right) e^y.$$

By the last inequality, (2.10) and (2.16) we obtain

$$\int (P_\tau(v))^2 \geq \tau \int \left(2 - 2\delta(1 + e^y) - \frac{C_1 e^y}{\tau} \right) |\partial_\omega v|^2 e^y + \frac{R_0^4}{2\lambda^2\tau} \int v_t^2 e^{4y} +$$

$$(2.17) \qquad + \int (2\delta\tau(B_0 - 2\delta\tau e^y) - C_2\tau^2) v^2 e^y + \tau \int \left(\frac{1}{2} - 2\delta e^y \right) v_y^2 -$$

$$- C\Lambda R_0^2 \int |\partial_\omega v||v_t| e^{3y}, \text{ if } v \in C_0^\infty(\mathcal{O}_{C_2}) \text{ and } \tau \geq C_3,$$

where C is an absolute constant, C_1 depends on Λ and $R_0^2 T^{-1}$ only, C_2 depends on λ and Λ only and C_3 depends on λ, Λ and $R_0^2 T^{-1}$ only. Observe that in the right hand side of (2.17) the coefficient of v^2 is of order three in τ. Now, by the inequality

$$C\Lambda R_0^2 |\partial_\omega v||v_t| e^{3y} \leq \frac{R_0^4}{2\lambda^2\tau} v_t^2 e^{4y} + \frac{C^2\lambda^2\Lambda^2\tau}{2} |\partial_\omega v|^2 e^{2y}$$

choosing $\delta = \frac{1}{8}$, we have by (2.17)

$$(2.18) \quad \int |P_\tau(v)|^2 \geq \int \left(\frac{\tau}{2}|\partial_\omega v|^2 + \frac{\tau}{4} v_y^2 + \frac{\tau^3}{8} v^2 \right) e^y, \text{ if } v \in C_0^\infty(\mathcal{O}_{C_1}) \text{ and } \tau \geq C_2,$$

where C_1 depends on λ and Λ only and C_2 depends on λ, Λ and $R_0^2 T^{-1}$ only. Changing the variables and setting $\theta = e^{k(-C_1)}$, (2.18) easily gives (2.5). □

Theorem 2. *Let M be a nonnegative number. Let $u \in H^{2,1}(Q_{R_0}^T)$ satisfy*

$$(2.19) \qquad |Lu| \leq M(R_0^{-1}|\nabla u| + R_0^{-2}|u|) \quad in \quad Q_{R_0}^T.$$

The following propositions hold true.

a) For every r_0, $r_0 \in (0, \frac{\theta R_0}{3})$, r such that $0 < r_0 < r < \theta R_0$ and $t_0 \in (0, T)$ we have

$$(2.20) \qquad \|u\|_{L^2(Q_r^{T-t_0})} \leq C \left(\left(\left(\frac{R_0}{r_0}\right)^{n/2} \|u\|_{L^2(Q_{2r_0}^T)} \right)^{\delta_{r_0}} \left(\|u\|_{L^2(Q_{\theta R_0}^T)} \right)^{1-\delta_{r_0}} + \right.$$

$$\left. + \left(\frac{R_0}{r_0}\right)^{n/2} e^{C(\chi(\theta R_0/2) - \chi(r_0))} \|u\|_{L^2(Q_{2r_0}^T)} \right),$$

where χ is defined in Theorem 1, θ, C, C_1 are positive constants, θ, $\theta < 1$, depends on λ and

Λ only, C depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1} only and δ_{r_0} is given by

$$\delta_{r_0} = \frac{\chi(\theta R_0/2) - \chi(r)}{\chi(\theta R_0/2) - \chi(r_0)}. \tag{2.21}$$

b) If u satisfies the inequality (2.19) *and*

$$\|u\|_{L^2(Q_s^T)} = O(s^\nu) \quad \textit{as } s \to 0, \textit{for every } \nu \in \mathbb{N},$$

then $u \equiv 0$.

PROOF. Denote by $R_1 = \theta R_0$, where θ is defined in Theorem 1. Let $\zeta \in C_0^2(Q_{R_1}^T \setminus \{0\} \times \times(-T, T))$. Let $\{\zeta_j\}$ be a sequence in $C_0^\infty(Q_{R_1}^T \setminus \{0\} \times (-T, T))$ that converges to ζ in C^2. Let $\{u_j\}$ be a sequence in $C^\infty(Q_{R_0}^T \times (-T, T))$ that converges to u in $\mathrm{H}^{2,1}(Q_{R_0}^T)$. Applying the inequality (2.5) to the functions $u_j\zeta_j$ and passing to the limit we obtain

$$\int_{Q_{R_0}^T} (\tau\rho^2|\nabla_{\rho,\omega}(\widetilde{u\zeta})|^2 + \tau^3(\widetilde{u\zeta}^2)e^{(-2\tau+1)\chi(\rho)}\rho^{-1} \leq C\int_{Q_{R_0}^T} (\mathcal{L}(\widetilde{u\zeta}))^2 e^{-2\tau\chi(\rho)}\rho^3\,, \tag{2.22}$$

$$\text{if } \tau \geq C_1\,,$$

where C is an absolute constant and C_1 depends on λ, Λ and $R_0^2T^{-1}$ only.

The main effort of this proof consists in the construction of a cut-off function ζ that allows us to deduce from (2.19) and (2.22) the inequality (2.20). We choose $\zeta(x, t)$ of the type $f(|x|)\varphi(t)$, where $f \in C_0^2((0, \frac{3}{4}R_1))$ is equal to 1 in $[\frac{3}{2}r_0, \frac{R_1}{2}]$, where $r_0 \in (0, \frac{R_1}{3})$ and f is equal to 0 in $[0, r_0] \cup [\frac{R_1}{2}, \frac{3}{4}R_1]$. Moreover $|f'| \leq \frac{C}{r_0}$, $|f''| \leq \frac{C}{r_0^2}$ in $[r_0, \frac{3}{2}r_0]$ and $|f'| \leq \frac{C}{R_1}$, $|f''| \leq \frac{C}{R_1^2}$ in $[\frac{R_1}{2}, \frac{3}{4}R_1]$, where C is an absolute constant. For a fixed $t_0 \in (0, T)$, the function $\varphi \in C_0^2((-T, T))$ is equal to 1 in $[-T + t_0, T - t_0]$ and is equal to 0 in $[-T, -T + \frac{t_0}{2}] \cup [T - \frac{t_0}{2}, T]$. φ shall be choosen later in $(-T + \frac{t_0}{2}, -T + t_0] \cup [T - t_0, T - \frac{t_0}{2})$.

Now, denote by

$$K_1' = \left\{(x, t) \in \mathbb{R}^{n+1} \middle| \frac{3}{2}r_0 \leq |x| \leq \frac{R_1}{2},\ t \in \left[-T + \frac{t_0}{2}, -T + t_0\right]\right\},$$

$$K_1'' = \left\{(x, t) \in \mathbb{R}^{n+1} \middle| \frac{3}{2}r_0 \leq |x| \leq \frac{R_1}{2},\ t \in \left[T - t_0, T - \frac{t_0}{2}\right]\right\},$$

$$K_2 = \left\{(x, t) \in \mathbb{R}^{n+1} \middle| r_0 \leq |x| \leq \frac{3}{2}r_0,\ t \in \left[-T + \frac{t_0}{2}, T - \frac{t_0}{2}\right]\right\},$$

$$K_3 = \left\{(x, t) \in \mathbb{R}^{n+1} \middle| \frac{R_1}{2} \leq |x| \leq \frac{3}{4}R_1,\ t \in \left[-T + \frac{t_0}{2}, T - \frac{t_0}{2}\right]\right\},$$

$$K_4 = \left\{(x, t) \in \mathbb{R}^{n+1} \middle| \frac{3}{2}r_0 \leq |x| \leq \frac{R_1}{2},\ t \in \left[-T + t_0, T - t_0\right]\right\}.$$

Further, set $K_1 = K_1' \cup K_1''$. With $Q_{R_0}^T \setminus \bigcup_{i=1}^4 K_i$ and K_i, $i = 1, 2, 3, 4$, we have partitioned the cylinder $Q_{R_0}^T$ in five regions. Observe that in $Q_{R_0}^T \setminus \bigcup_{i=1}^4 K_i$ we have $\zeta u \equiv 0$ and, in K_4, we have $\zeta u \equiv u$. Splitting the integrals of the inequality (2.22) on the partition defined above we get

$$\begin{aligned}(2.23)\qquad &\int_{K_4} (\tau\rho^2|\nabla_{\rho,\omega}(\widetilde{u})|^2 + \tau^3(\widetilde{u})^2)e^{(-2\tau+1)\chi(\rho)}\rho^{-1} \leq \\ &\leq J_1 + J_2 + CM^2R_0^{-2}\int_{K_4}(|\nabla_{\rho,\omega}(\widetilde{u})|^2 + R_0^{-2}\widetilde{u}^2)e^{-2\tau\chi(\rho)}\rho^3, \ \text{ if } \tau \geq C_1,\end{aligned}$$

where

$$J_1 = -\int_{K_1}(\tau\rho^2|\nabla_{\rho,\omega}(\widetilde{u\zeta})|^2 + \tau^3(\widetilde{u\zeta}^2))\rho^{-1}e^{(-2\tau+1)\chi(\rho)} + C\int_{K_1}(\widetilde{\zeta}\,\mathcal{L}\widetilde{u} - q\widetilde{\zeta}_t\widetilde{u})^2\rho^3e^{-2\tau\chi(\rho)},$$

$$J_2 = C\int_{K_2\cup K_3}|\mathcal{L}(\widetilde{u\zeta})|^2\rho^3e^{-2\tau\chi(\rho)},$$

where C is an absolute constant and C_1 depends on λ, Λ and $R_0^2T^{-1}$ only. Observe that

$$(2.24)\qquad \log\frac{\rho}{eR_0} \leq \chi(\rho) \leq \log\frac{\rho}{R_0}, \ \text{ if } \rho \in (0, R_0).$$

If τ is sufficiently large then the integral on the left hand side of (2.23) dominates the last integral on the right hand side. So we get

$$(2.25)\qquad \frac{1}{2R_0}\int_{K_4}(\tau\rho^2|\nabla_{\rho,\omega}(\widetilde{u})|^2 + \tau^3(\widetilde{u})^2)e^{-2\tau\chi(\rho)} \leq J_1 + J_2, \ \text{ if } \tau \geq C,$$

where C depends on λ, Λ, M and $R_0^2T^{-1}$.

Now we examine J_1. Using (2.19) and, setting

$$(2.26)\qquad \Psi(\rho, t; \tau) = R_0^{-1}\varphi^2(t)\left(CM^2R_0^{-3}\rho^3 + C\lambda^2\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2R_0\rho^3 - \tau^3e^{-1}\right),$$

we get

$$J_1 \leq \int_{K_1}\Psi(\rho, t; \tau)\widetilde{u}^2e^{-2\tau\chi(\rho)} + \int_{K_1}\left(CM^2\frac{\rho}{R_0} - \tau\right)|\nabla_{\rho,\omega}(\widetilde{u})|^2\frac{\rho^2}{R_0}e^{-2\tau\chi(\rho)},$$

where C is an absolute constant. By the last inequality we have

$$(2.27)\qquad J_1 \leq \int_{K_1}\Psi(\rho, t; \tau)\widetilde{u}^2e^{-2\tau\chi(\rho)}, \ \text{ if } \tau \geq C,$$

C depends on M only.

Denote by T_1 and T_2, respectively, the numbers $T_1 = T - \frac{t_0}{2}$, $T_2 = T - t_0$. Denote by γ a positive number that we pick later, let us choose φ as an even function such that

$$\varphi(t) = \exp - \left(\frac{T^{\gamma}(T_2 + t)^4}{(T_1 + t)^{\gamma}(T_1 - T_2)^4} \right), \quad \text{if } t \in (-T_1, -T_2]. \tag{2.28}$$

We have in K_1'

$$\Psi(\rho, t; \tau) \le \tau^3 (R_0 e)^{-1} \varphi^2(t) \left(-\frac{1}{2} + \frac{C_1(\gamma+1)^2 T^{2\gamma}}{(T_1 + t)^{2(\gamma+1)}} \frac{R_0 \rho^3}{\tau^3} \right), \quad \text{if } \tau \ge C, \tag{2.29}$$

where C_1 depends on λ only and C depends on M only.

Denote by

$$K_{1,\tau}' = \left\{ (x, t) \in K_1' \,\middle|\, -\frac{1}{2} + \frac{C_1(\gamma+1)^2 T^{2\gamma}}{(T_1 + t)^{2(\gamma+1)}} \frac{R_0 \rho^3}{\tau^3} \ge 0 \right\},$$

where C_1 is the same constant appearing in the right hand side of (2.29). Setting $m_\gamma = \max_{s \in (0,1]} s^{-2(1+\gamma)} e^{-s^{-\gamma}(1-s)^4}$, (2.29) gives

$$\int_{K_1'} \Psi(\rho, t; \tau) \widetilde{u}^2 e^{-2\tau\chi(\rho)} \le \frac{C_2(1+\gamma)^2 m_\gamma T^{2\gamma}}{t_0^{2(\gamma+1)}} \int_{K_{1,\tau}'} \varphi(t) \rho^3 \widetilde{u}^2 e^{-2\tau\chi(\rho)}, \quad \text{if } \tau \ge C,$$

where C depends on M only and C_2 depends on λ only.

$$\frac{T_1 + t}{T} \le \left(\frac{2C_1(\gamma+1)^2 R_0 \rho^3}{\tau^3 T^2} \right)^{\frac{1}{2(\gamma+1)}}, \quad \text{in } K_{1,\tau}'.$$

Therefore, by (2.24) and (2.28), we obtain

$$\begin{aligned} &\varphi(t)\rho^3 e^{-2\tau\chi(\rho)} \le \\ &\le \frac{\rho^{n-1}}{(eR_0)^{n-4}} \exp\left(- \left(\frac{\tau^3 T^2}{2C_1(\gamma+1)^2 R_0 \rho^3} \right)^{\frac{\gamma}{2(\gamma+1)}} + (2\tau + n - 4)\log\frac{R_0 e}{\rho} \right), \quad \text{in } K_{1,\tau}'. \end{aligned} \tag{2.30}$$

Arguing in the same way for the region K_1'' and picking $\gamma = 3$ we get

$$J_1 \le \frac{C}{R_0^{n-1}} \int_{K_1} \rho^{n-1} \widetilde{u}^2, \text{ if } \tau \ge C_1, \tag{2.31}$$

where C depends on λ, $R_0^2 T^{-1}$ and Tt_0^{-1}only and C_1 depends on λ, $R_0^2 T^{-1}$ and Tt_0^{-1} only.

By (2.19) we have

$$\begin{aligned} J_2 \le &\frac{Ce^{-2\tau\chi(r_0)}}{r_0^n} \int_{K_2} (r_0^2 |\nabla_{\rho,\omega}(\widetilde{u})|^2 + \widetilde{u}^2)\rho^{n-1} + \\ &+ \frac{Ce^{-2\tau\chi(R_1/2)}}{R_1^n} \int_{K_3} (R_0^2 |\nabla_{\rho,\omega}(\widetilde{u})|^2 + \widetilde{u}^2)\rho^{n-1}, \end{aligned} \tag{2.32}$$

where C depends on λ, M, $R_0^2 T^{-1}$ and Tt_0^{-1} only.

Let $r \in (\frac{3}{2}r_0, \frac{R_1}{2})$ and denote by $K_4^{(r)}$ the region $\{(x,t) \in K_4 | |x| \leq r.\}$. By (2.25), (2.31) and (2.32) we get

$$\tau^3 e^{-2\tau\chi(r)} \int_{K_4^{(r)}} \widetilde{u}^2 \rho^{n-1} \leq \tau^3 \int_{K_4} \widetilde{u}^2 e^{-2\tau\chi(\rho)} \rho^{n-1} \leq$$

$$\leq C\left(e^{-2\tau\chi(r_0)}\left(\frac{R_0}{r_0}\right)^n \int_{K_2} (r_0^2|\nabla_{\rho,\omega}(\widetilde{u})|^2 + \widetilde{u}^2)\rho^{n-1} + \int_{K_1} \widetilde{u}^2\rho^{n-1}\right) + \tag{2.33}$$

$$+ Ce^{-2\tau\chi(R_1/2)} \int_{K_3} (R_0^2|\nabla_{\rho,\omega}(\widetilde{u})|^2 + \widetilde{u}^2)\rho^{n-1}, \text{ if } \tau \geq C_1,$$

where C depends on λ, M, $R_0^2T^{-1}$ and Tt_0^{-1} only and C_1 depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1} only. Now, let us estimate from above the right hand side of (2.33) using the following standard estimate

$$\int_{K_2\cup K_3} |\nabla_{\rho,\omega}(\widetilde{u})|^2\rho^{n-1} \leq C\left(r_0^{-2}\int_{Q_{2r_0}^T} \widetilde{u}^2\rho^{n-1} + R_0^{-2}\int_{Q_{R_1}^T\setminus Q_{R_1/2}^T} \widetilde{u}^2\rho^{n-1}\right),$$

where C depends on λ, Λ, M and $R_0^2t_0^{-1}$. By (2.33) we have

$$\int_{Q_r^{T-t_0}} u^2\,dxdt \leq$$

$$\leq C\left(e^{2\tau(\chi(r)-\chi(r_0))}\left(\frac{R_0}{r_0}\right)^n \int_{Q_{2r_0}^T} u^2\,dxdt + e^{2\tau(\chi(r)-\chi(\frac{R_1}{2}))}\int_{Q_{R_1}^T} u^2\,dxdt\right), \text{ if } \tau \geq C_1, \tag{2.34}$$

where C depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1} only, C_1 depends on λ, Λ, M and $R_0^2T^{-1}$.

Denote by

$$\tau_0 = \frac{-1}{2(\chi(\frac{R_1}{2}) - \chi(r_0))}\log\left(\frac{(R_0r_0^{-1})^n\int_{Q_{2r_0}^T} u^2\,dxdt}{\int_{Q_{R_1}^T} u^2\,dxdt}\right).$$

If $\tau_0 \geq C_1$ then, choosing in (2.35) $\tau = \tau_0$, we obtain

$$\|u\|_{L^2(Q_r^{T-t_0})} \leq C\left(\left(\frac{R_0}{r_0}\right)^{n/2}\|u\|_{L^2(Q_{2r_0}^T)}\right)^{\delta_{r_0}}(\|u\|_{L^2(Q_{\theta R_0}^T)})^{1-\delta_{r_0}}, \tag{2.35}$$

where

$$\delta_{r_0} = \frac{\chi(R_1/2) - \chi(r)}{\chi(R_1/2) - \chi(r_0)}$$

and C depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1} only and C_1 depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1}only. If $\tau_0 < C_1$ then (2.34) gives trivially

$$\|u\|_{L^2(Q_r^{T-t_0})} \leq e^{C_1(\chi(R_1/2)-\chi(r_0))}\left(\frac{R_0}{r_0}\right)^{n/2}\|u\|_{L^2(Q_{2r_0}^T)},$$

where C_1 depends on λ, Λ, M, $R_0^2T^{-1}$ and Tt_0^{-1}only. By the last inequality and (2.35) we obtain (2.20).

Now, let us prove the proposition *b)* by contradiction. Assume that

$$\|u\|_{L^2(Q_s^T)} = O(s^\nu)\,, \text{ as } s \to 0\,, \text{ for every } \nu \in \mathbb{N}\,. \tag{2.36}$$

If u were not identically equal to zero in $Q_{R_1}^T$ we can normalize it, hence we assume

$$\|u\|_{L^2(Q_{R_1}^T)} = 1.$$

Let us fix $r \in (0\,, \frac{R_1}{2})$ and $t_0 \in (0\,, T)$, by (2.20) and (2.36) we obtain

$$\|u\|_{L^2(Q_r^{T-t_0})} \leq C(E_\nu s^{\nu-\frac{n}{2}})^{\delta_s} + Ce^{C_1(\chi(\frac{R1}{2})-\chi(s))}R_0^{\frac{n}{2}}s^{\nu-\frac{n}{2}}\,, \text{if } s \in \left(0\,,\frac{r}{2}\right) \text{ and } \nu \in \mathbb{N},$$

where E_ν is a sequence, C and C_1 are constants. Passing to the limit as $s \to 0$, the last inequality gives

$$\|u\|_{L^2(Q_r^{T-t_0})} \leq Ce^{-(\nu-\frac{n}{2})(\chi(R_1/2)-\chi(r))}\,, \text{ for every } \nu \in \mathbb{N},$$

passing to the limit as $\nu \to \infty$, we obtain $u \equiv 0$ in $Q_r^{T-t_0}$. By iteration we get $u \equiv 0$ in $Q_{R_1}^T$ contradicting the hypothesis. □

3. The case $L(\cdot) = \mathrm{div}(A(x\,,t)\nabla\cdot) - \dfrac{\partial\cdot}{\partial t}$

Now we state the results proved in [11]. A sketch of the proofs is contained in Remark 1. Denote by

$$C^{1,1}(\overline{Q_{R_0}^T}) = \left\{f \in C^0(\overline{Q_{R_0}^T})\Big|\,\frac{\partial f}{\partial x^i},\ \frac{\partial f}{\partial t} \in C^0(\overline{Q_{R_0}^T}),\ i = 1\,,\dots,\,n\right\}\,,$$

$$C^{2,1}(\overline{Q_{R_0}^T}) = \left\{f \in C^{1,1}(\overline{Q_{R_0}^T})\Big|\,\frac{\partial^2 f}{\partial x^i\partial x^j}\,,\ \frac{\partial^2 f}{\partial x^i\partial t} \in C^0(\overline{Q_{R_0}^T}),\ i\,,j = 1\,,\dots,\,n\right\}.$$

Assume that $q_0 \in C^{1,1}(\overline{Q_{R_0}^T})$ and let A be a $n\times n$ symmetric matrix whose entries are in $C^{2,1}(\overline{Q_{R_0}^T})$. Further, assume that $\lambda^{-1} \leq q_0(x\,,t) \leq \lambda$, if $(x\,,t) \in Q_{R_o}^T$ and $\lambda^{-1}|\xi|^2 \leq$ $\leq A(x\,,t)\xi\cdot\xi \leq \lambda|\xi|^2$, if $\xi \in \mathbb{R}^n$ and $(x\,,t) \in Q_{R_o}^T$. Let $\varepsilon \in (0\,,1)$, set $k_\varepsilon(y) = y + e^{\varepsilon y}$ and $\chi_\varepsilon(\rho) = k_\varepsilon^{-1}(\log\frac{\rho}{R_0})$.

Theorem 3. *Let L be the following operator*

$$(Lu)(x\,,t) = \left(\mathrm{div}(A\nabla u) - q_0\frac{\partial u}{\partial t}\right)(x\,,t),\ \textit{if}\ (x\,,t) \in Q_{R_o}^T\,,$$

The following propositions hold true.

a) If $A(0\,,t) = I$ then there exist two positive constants $\theta \in (0\,,1)$ and τ_0 depending on λ, $R_0^2T^{-1}$, the $C^{1,1}$ norm of q_0 and the $C^{2,1}$ norm of A such that if $u \in C_0^\infty(Q_{\theta R_o}^T \setminus \{0\} \times$

$\times(-T, T))$ *and* $\tau \geq \tau_0$ *then*

$$\int_{Q^T_{R_o}} (\tau|x||\nabla u|^2 + \tau^3|x|^{-1}u^2)|x|^{1-n}e^{(-2\tau+\varepsilon)\chi_\varepsilon(|x|)}dxdt \leq C\int_{Q^T_{R_o}} |Lu|^2|x|^{4-n}e^{-2\tau\chi_\varepsilon(|x|)}dxdt,$$

where C *depends on* ε *and* λ *only.*

b) Let M *be a nonnegative number. If* u *is a function in* $H^{2,1}(Q^T_{R_o})$ *satisfying*

$$|Lu| \leq M(R_0^{-1}|\nabla u| + R_0^{-2}|u|) \quad in\ Q^T_{R_0},$$

and

$$\|u\|_{L^2(Q^T_s)} = O(s^\nu) \quad as\ s \to 0, for\ every\ \nu \in \mathbb{N},$$

then $u \equiv 0$ *in* $Q^T_{R_0}$.

In the next theorem we use the following notation and hypotheses. Denote by B'_{R_0} the $(n-1)$- dimensional ball of radius R_0 centered in 0. For a number $\alpha \in (0, 1]$ let $\varphi \in C^{1+\alpha}(\overline{B'_{R_0}})$, i.e. $\varphi \in C^1(\overline{B'_{R_0}})$ such that

$$\sup_{x', y' \in B'_{R_0}, x' \neq y'} \frac{|\nabla\varphi(x') - \nabla\varphi(y')|}{|x' - y'|^\alpha} < \infty,$$

assume that $\varphi(0)=0$. Set $D^T_{R_0} = \{(x,t) \in Q^T_{R_0} | \varphi(x') < x_n\}$ and $\Gamma^T_{R_0} = \{(x,t) \in Q^T_{R_o} | \varphi(x') = x_n\}$.

THEOREM 4. *Let* L *be the following operator*

$$(Lu)(x, t) = \left(\mathrm{div}(A\nabla u) - q_0\frac{\partial u}{\partial t}\right)(x, t), \ if\ (x, t) \in D^T_{R_o}.$$

Let $\varepsilon \in (0, \alpha)$, *the following propositions hold true.*

a) If $A(0, t) = I$ *then there exist two positive constants* $\theta \in (0, 1)$ *and* τ_0 *depending on* ε, λ, $R_0^2T^{-1}$, *the* $C^{1,1}$ *norm of* q_0, *the* $C^{2,1}$ *norm of* A *and the* $C^{1+\alpha}$ *norm of* φ *such that: if* $u \in C^{1,1}(\overline{D^T_{R_o}}) \cap C^{2,1}(D^T_{R_o})$, $u = 0$ *on* $\Gamma^T_{R_o}$, $\zeta \in C^2_0(Q^T_{\theta R_o} \setminus \{0\} \times (-T, T))$ *and* $\tau \geq \tau_0$ *then*

$$\int_{D^T_{R_o}} (\tau|x||\nabla(u\zeta)|^2 + \tau^3|x|^{-1}(u\zeta)^2)|x|^{1-n}e^{(-2\tau+\varepsilon)\chi_\varepsilon(|x|)}dxdt \leq$$

$$\leq C\int_{D^T_{R_o}} |L(u\zeta)|^2|x|^{4-n}e^{-2\tau\chi_\varepsilon(|x|)}dxdt,$$

where C *depends on* ε *and* λ *only.*

b) Let M *be a nonnegative number. If* u *is a function in* $C^{1,1}(\overline{D^T_{R_o}}) \cap C^{2,1}(D^T_{R_o})$ *satisfying*

$$u = 0, \ on\ \Gamma^T_{R_o},$$

$$|Lu| \leq M(R_0^{-1}|\nabla u| + R_0^{-2}|u|), \quad in\ D^T_{R_0}$$

and

$$\|u\|_{L^2(D_s^T)} = O(s^\nu) \quad \text{as } s \to 0, \text{ for every } \nu \in \mathbb{N},$$

then $u \equiv 0$ *in* $D_{R_0}^T$.

REMARK 1. To prove Theorems 3 and 4 we preliminary write the elliptic part of the operator L in the Laplace-Beltrami form. Namely

$$\frac{\partial}{\partial x^i}\left(a^{ij}(x,t)\frac{\partial\cdot}{\partial x^j}\right) = \frac{1}{\sqrt{g(x,t)}}\frac{\partial}{\partial x^i}\left(\sqrt{g(x,t)}g^{ij}(x,t)\frac{\partial\cdot}{\partial x^j}\right),$$

where (if $n \geq 3$)

$$g^{ij}(x,t) = (\det A(x,t))^{\frac{1}{2-n}} a^{ij}(x,t), \quad i,j \in \{1,\dots,n\}$$

and $g(x,t) = \det(\{g_{ij}(x,t)\}_{i,j=1}^n)$, the matrix $\{g_{ij}(x,t)\}_{i,j=1}^n$ is the inverse of $\{g^{ij}(x,t)\}_{i,j=1}^n$. Then we transform the operator L in polar coordinates. To this aim we have adapted to a time dependent metric tensor the strategy of Aronszajn *et al.* [2]. Then we prove a Carleman estimate and a three cylinder inequality with optimal exponent, thus we get the property *of unique continuation in the interior*. The above mentioned transformation turns out to be a particular useful tool in the proof of the property *of unique continuation at the boundary*. To prove the just mentioned property we preliminarly transform the graph(φ) by means of the transformation found in [1, Section 2]. Setting $\widetilde{\varphi}$ the transformed graph, in a second step we observe that the set $\{x|\ x_n > \widetilde{\varphi}(x')\}$ is starshaped in the geometry induced by a distance conformal to $g_{ij}(x,t)dx^i dx^j$, for every $t \in (-T,T)$. The above mentioned transformations allow us to prove a Carleman estimate, a three cylinder inequality with optimal exponent and the property of unique continuation at the boundary.

This work was partially supported by MURST, grant number MM01111258.
This paper is dedicated to the memory of my friend Fabio Bardelli.

REFERENCES

[1] V. ADOLFSSON - L. ESCAURIAZA, $C^{1,\alpha}$ *domains and unique continuation at the boundary*. Comm. Pure Appl. Math., L, 1997, 935-969.

[2] N. ARONSZAJN - A. KRZYWICKI - J. SZARSKI, *A unique continuation theorem for exterior differential forms on Riemannian manifold*. Ark. for Matematik, 4, (34), 1962, 417-453.

[3] B. CANUTO - E. ROSSET - S. VESSELLA, *Quantitative estimates of unique continuation for parabolic equations and inverse-initial boundary value problems with unknown boundaries*. Transactions of AMS, to appear.

[4] B. CANUTO - E. ROSSET - S. VESSELLA, *A stability result in the localization of cavities in a thermic conducting medium*. Preprint n. 59, 2001, Laboratoire de Mathématiques Appliquées, Université de Versailles.

[5] DÈ-YUAN' LI, *An inequality for the parabolic operator*. Sci. Sinica, 12, 1963, 1425-1467.

[6] R.JA. GLAGOLEVA, *Some properties of solutions of a linear second order parabolic equation*. Math. USSR-Sbornik, 3 (1), 1967, 41-67.

[7] L. HÖRMANDER, *Uniqueness theorem for second order elliptic differential equations*. Comm. Part. Diff. Equations, 8 (1), 1983, 21-64.

[8] M. LEES - M.H. PROTTER, *Unique continuation for parabolic equations*. Duke Math. J., 28, 1961, 369-382.

[9] F.H. Lin, *A uniqueness theorem for parabolic equations*. Comm. Pure Appl. Math., XLIII, 1990, 127-136.

[10] A.A. Varin, *Three-Cylinder theorem for certain class of semilinear parabolic equations*. Mat. Zametki, 51, (1), 1992, 32-41.

[11] S. Vessella, *Carleman estimates, optimal three cylinder inequality and unique continuation properties for solutions to parabolic equations*. Quaderno DiMaD, novembre 2001, 1-12.

Pervenuta il 5 ottobre 2001,
in forma definitiva il 7 gennaio 2002.

DiMaD - Dipartimento di Matematica per le Decisioni
Università degli Studi di Firenze
Via C. Lombroso, 6/17 - 50134 Firenze
sergio.vessella@dmd.unifi.it